Abstract

In this paper we introduce the notion of \((\lambda, \mu)\)-fuzzy ternary subsemigroup (respectively left ideal, right ideal, lateral ideal, bi-ideal, interior ideal, quasi-ideal) in an ordered ternary semigroups is introduce. We have shown that \(f\) is a \((\lambda, \mu)\)-fuzzy ternary subsemigroup (respectively left ideal, right ideal, lateral ideal, bi-ideal, interior ideal, quasi-ideal) of an ordered ternary semigroup \(T\) if and only if the upper level cut \(U(f; t)\) is a ternary subsemigroup (respectively left ideal, right ideal, lateral ideal, bi-ideal, interior ideal, quasi-ideal) of \(T\).

Keywords: \((\lambda, \mu)\)-fuzzy ternary subsemigroups, \((\lambda, \mu)\)-fuzzy ideals, \((\lambda, \mu)\)-fuzzy bi-ideal, \((\lambda, \mu)\)-fuzzy generalised bi-ideal, \((\lambda, \mu)\)-fuzzy interior ideal, \((\lambda, \mu)\)-fuzzy quasi-ideal, \((\lambda, \mu)\)-characteristic function.

1 Introduction

In 1932, Lehmer introduced the concept of ternary semigroup \([6]\). The algebraic structures of ternary semigroups were also studied by some authors, for example, Sioson studied ideals in ternary semigroups \([13]\). Dixit and Dewan studied quasi-ideals and bi-ideals in ternary semigroups \([2]\), Kar and Maity studied congruences of ternary semigroups \([4]\) and Iampan studied minimal and maximal lateral ideals of ternary semigroups \([3]\). The concept of a fuzzy subset was introduced by Zadeh \([15]\), provides a natural framework for generalizing some of the notions of classical algebraic structures. In 1971, Rosenfeld \([11]\) defined the concept of fuzzy group. Since then many papers have been published in the field of fuzzy algebra. Several researchers conducted the researches on the generalizations of the notions of fuzzy subsets with huge applications in computer, logics and many branches of pure and applied mathematics. Fuzzy semigroups have been first considered by Kuroki \([5]\). Chinram and Saelee \([1]\), studied the concept of fuzzy ideals and fuzzy filters of ordered ternary semigroups. Naveed Yahoob etal. \([10]\) studied the concepts of roughness and fuzziness in ordered ternary semigroups. S.Lekkoksung \([7, 8]\) studied fuzzy interior ideals and some properties of \(\Omega\)-fuzzy ideals in ordered ternary semigroups. In this paper we introduce the notion of \((\lambda, \mu)\)-fuzzy ternary subsemigroup (respectively left ideal, right ideal, lateral ideal, bi-ideal, interior ideal, quasi-ideal) in an ordered ternary semigroups which can be regarded as the generalization of fuzzy ternary subsemigroup (respectively left ideal, right ideal, lateral ideal, bi-ideal, interior ideal, quasi-ideal) in ordered ternary semigroups. We prove that \(f\) is a \((\lambda, \mu)\)-fuzzy ternary subsemigroup (respectively left ideal, right ideal, lateral ideal, bi-ideal, interior ideal, quasi-ideal) of an ordered ternary semigroup \(T\) if and only if the upper level cut \(U(f; t)\) is a ternary subsemigroup (respectively left ideal, right ideal, lateral ideal, bi-ideal, interior ideal, quasi-ideal) of \(T\).

2 Preliminaries

\*Corresponding author.

E-mail address: anitha81t@gmail.com (T. Anitha).
Definition 2.1. A non-empty set $T$ is called a ternary semigroup if there exists a ternary operation $T \times T \times T \to T$, written as $(a, b, c) \to abc$ satisfying the following identity, for any $a, b, c, d, e \in T$, $((abc)de)=(a(bcd)e)=(ab(cde))$.

Any semigroup can be reduced to a ternary semigroup. However, Banach showed that a ternary semigroup does not necessarily reduce to a semigroup by this example. Let $T = \{-i, 0, i\}$ be a ternary semigroup while $T$ is not a semigroup under the multiplication over complex numbers. Los showed that every ternary semigroup can be embedded in a semigroup \[^9\].

Definition 2.2. An ordered ternary semigroup we mean a structure $(T; \cdot; \leq)$ in which

1. $(T; \cdot)$ is a ternary semigroup,
2. $(T; \leq)$ is a poset,
3. $a \leq (ab \cdot c)$ for all $a, b, c \in T$.

Let $T$ be an ordered ternary semigroup. For any non-empty subsets, $A, B$ and $C$ of $T$, $ABC = \{abc / a \in A, b \in B$ and $c \in C\}$. For $A \subseteq T$, denote $(A) = \{t \in T / t \leq h$ for some $h \in A\}$.

Definition 2.3. A non-empty subset $A$ of $T$ is called a ternary subsemigroup of $T$ if $(A) \subseteq A$ and $AAA \subseteq A$.

Definition 2.4. A non-empty subset $A$ of $T$ is called a left ideal of $T$ if $(A) \subseteq A$ and $TTA \subseteq A$, a lateral ideal of $T$ if $(A) \subseteq A$ and $TAT \subseteq A$ and a right ideal of $T$ if $(A) \subseteq A$ and $ATT \subseteq A$. If $A$ is both a left and right ideal then $A$ is called a two-sided ideal of $T$. If $A$ is a left, right and lateral ideal of $T$, then $A$ is called an ideal of $T$.

Definition 2.5. A ternary subsemigroup $B$ of $T$ is called a bi-ideal of $T$ if $(B) \subseteq B$ and $BTB \subseteq B$.

Definition 2.6. A non-empty subset $A$ of $T$ is called a generalized bi-ideal of $T$ if $(A) \subseteq A$ and $ATATA \subseteq A$.

Clearly, every bi-ideal of $T$ is a generalized bi-ideal but not the converse.

Definition 2.7. A non-empty subset $A$ of $T$ is called an interior ideal of $T$ if $(A) \subseteq A$ and $TTATT \subseteq A$.

Definition 2.8. A non-empty subset $Q$ of $T$ is called a quasi-ideal of $T$ if $(Q) \subseteq Q$, $QTQ \cap TQT \cap TTQ \subseteq Q$ and $QTT \cap TTQTT \cap TTQ \subseteq Q$.

Definition 2.9. An ordered ternary semigroup $T$ is said to be regular if for each $a \in T$, there exists $x \in T$ such that $a \leq axa$.

Let $X$ be a non-empty set. A map $f : X \to [0, 1]$ is called a fuzzy subset in $X$.

Definition 2.10. Let $f$, $g$ and $h$ be any three fuzzy subsets of an ordered ternary semigroup $T$. Then $f \cap g$, $f \cup g$, $f \circ g \circ h$ are fuzzy subsets of $T$ defined by

\[
(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \land g(x),
\]

\[
(f \cup g)(x) = \max\{f(x), g(x)\} = f(x) \lor g(x), \text{ for all } x \in T.
\]

For $x \in T$, we define $A_x = \{(u, v, w) \in T \times T \times T / x \leq uvw\}$. The product of $f \circ g \circ h$ is defined by for all $x \in T$,

\[
(f \circ g \circ h)(x) = \begin{cases}
\vee_{(u,v,w)\in A_x} \min\{f(u), g(v), h(w)\} & , \text{ if } A_x \neq \phi, \\
0 & , \text{ if } A_x = \phi.
\end{cases}
\]

Definition 2.11. Let $X$ be a non-empty set and let $f$ be a fuzzy subset of $X$. Let $0 \leq t \leq 1$. Then the set $U(f; t) = \{x \in X / f(x) \geq t\}$ is called the upper level cut of $X$ with respect to $f$.

Definition 2.12. Let $f$ be a fuzzy subset of an ordered ternary semigroup $T$. Then $f$ is called a fuzzy ternary subsemigroup of $T$ if

1. $x \leq y$ implies $f(x) \geq f(y)$
2. $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$.

Definition 2.13. A fuzzy subset $f$ of an ordered ternary semigroup $T$ is called a fuzzy ideal of $T$ if

(i) $x \leq y$ implies $f(x) \geq f(y)$
(ii) $f(xyz) \geq f(x)$
(iii) $f(xyz) \geq f(z)$ and
(iv) $f(xyz) \geq f(y)$ for all $x, y, z \in T$. 

A fuzzy subset \( f \) with conditions (i) and (ii) is called a fuzzy right ideal of \( T \). If \( f \) satisfies (i) and (iii), then it is called a fuzzy left ideal of \( T \). If \( f \) satisfies (i) and (iv), then it is called a fuzzy lateral ideal of \( T \). Also if \( f \) satisfies (i), (ii) and (iii), then it is called a fuzzy two sided ideal of \( T \). It is clear that \( f \) is a fuzzy ideal of an ordered ternary semigroup \( T \) if and only if \( f(xyz) \geq \max\{f(x), f(y), f(z)\} \) for all \( x, y, z \in T \).

**Definition 3.14.** A fuzzy ternary subsemigroup \( f \) of an ordered ternary semigroup \( T \) is called a fuzzy bi-ideal of \( T \) if \( f(xuyz) \geq \min\{f(x), f(y), f(z)\} \) for all \( x, y, u, v \in T \).

**Definition 3.15.** A fuzzy subset \( f \) of an ordered ternary semigroup \( T \) is called a fuzzy generalised bi-ideal of \( T \) if
1. \( x \leq y \) implies \( f(x) \geq f(y) \)
2. \( f(xuyz) \geq \min\{f(x), f(y), f(z)\} \) for all \( x, y, z, u, v \in T \).

**Definition 3.16.** A fuzzy subset \( f \) of an ordered ternary semigroup \( T \) is called a fuzzy interior ideal of \( T \) if
1. \( x \leq y \) implies \( f(x) \geq f(y) \)
2. \( f(rsatu) \geq f(u) \) for all \( r, s, a, t, u \in T \).

**Definition 3.17.** A fuzzy subset \( f \) of an ordered ternary semigroup \( T \) is called a fuzzy quasi-ideal of \( T \) if it satisfies for all \( x, y \in T \),
1. \( x \leq y \) implies \( f(x) \geq f(y) \)
2. \( f(xyz) \geq \lambda \geq f(y) \land \mu \)
3. \( f(xyz) \land \lambda \geq f(x) \land f(y) \land f(z) \land \mu \) for all \( x, y, z \in T \).

**Theorem 3.1.** Let \( f \) be a fuzzy subset of a fuzzy termary semigroup \( T \). Then \( f \) is a \((\lambda, \mu)\)-fuzzy ternary subsemigroup of \( T \) if and only if \( U(f; t) \) is a termary subsemigroup of \( T \) for all \( t \in [\lambda, \mu] \) whenever non-empty.

**Proof.** Let \( f \) be a \((\lambda, \mu)\)-fuzzy ternary subsemigroup of \( T \) and \( t \in [\lambda, \mu] \). Let \( y \in U(f; t) \) and let \( x \in T \) such that \( x \leq y \). Thus \( f(x) \lor \lambda \geq f(y) \lor \mu \geq t \). Hence \( x \in U(f; t) \). Next, let \( x, y, z \in U(f; t) \). Then \( f(x) \lor t \geq f(y) \lor t \geq t \), \( f(x) \lor t \geq f(z) \lor t \). Now \( f(xyz) \lor \lambda \geq f(x) \lor f(y) \lor f(z) \lor \mu \). Thus \( f(xyz) \lor \lambda \geq t \). Hence \( xyz \in U(f; t) \). Hence \( U(f; t) \) is a ternary subsemigroup of \( T \).

Conversely, let \( U(f; t) \) be a ternary subsemigroup of \( T \) for all \( t \in [\lambda, \mu] \). Let \( x, y \in T \) be such that \( x \leq y \). Choose \( t = f(y) \lor \mu \). Thus, \( y \in U(f; t) \), which implies \( x \in U(f; t) \). Then \( f(x) \lor \lambda \geq f(y) \lor \mu \). Suppose if there exist \( x, y, z \in T \) such that \( f(xyz) \lor \lambda \geq t = f(x) \lor f(y) \lor f(z) \lor \mu \) then \( t \in (\lambda, \mu) \) and \( x, y, z \in U(f; t) \). This contradicts to that \( U(f; t) \) is a ternary subsemigroup. Hence \( f(xyz) \lor \lambda \geq f(x) \lor f(y) \lor f(z) \lor \mu \). Therefore \( f \) is a \((\lambda, \mu)\)-fuzzy ternary subsemigroup of \( T \).

**Remark 3.1.** Every fuzzy ternary subsemigroup is a \((\lambda, \mu)\)-fuzzy ternary subsemigroup by taking \( \lambda = 0 \) and \( \mu = 1 \). But the converse need not be true.

**Example 3.1.** Let \( (Z^-, \cdot, \leq) \) be an ordered ternary semigroup. Let \( \mu : Z^- \rightarrow [0, 1] \) defined by

\[
\mu(x) = \begin{cases} 
0.9 & \text{if } x = -3 \\
0.8 & \text{if } x \in (-\infty, -4] \\
0 & \text{otherwise}.
\end{cases}
\]

Clearly, \( \mu \) is a \((0.3, 0.8)\)-fuzzy ternary subsemigroup. But \( \mu \) is not a fuzzy ternary subsemigroup as \( \mu(-4) = 0.8 < \mu(-3) = 0.9 \).
Definition 3.19. A fuzzy subset $f$ of an ordered ternary semigroup $T$ is called a $(\lambda, \mu)$-fuzzy right (resp. left, lateral) ideal of $T$ if
1. $x \leq y$ implies $f(x) \lor \lambda \geq f(y) \land \mu$
2. $f(xyz) \lor \lambda \geq f(x) \land \mu$ (resp. $f(xyz) \lor \lambda \geq f(z) \land \mu$, $f(xyz) \lor \lambda \geq f(y) \land \mu$) for all $x, y, z \in T$.

Definition 3.20. A fuzzy subset $f$ of an ordered ternary semigroup $T$ is called a $(\lambda, \mu)$-fuzzy ideal of $T$ if
1. $x \leq y$ implies $f(x) \lor \lambda \geq f(y) \land \mu$
2. $f(xyz) \lor \lambda \geq [f(x) \lor f(y) \lor f(z)] \land \mu$ for all $x, y, z \in T$.

Definition 3.21. A $(\lambda, \mu)$-fuzzy ternary subsemigroup $f$ of an ordered ternary semigroup $T$ is called a $(\lambda, \mu)$-fuzzy bi-ideal of $T$ if
\[ f(xuyvz) \lor \lambda \geq \min\{f(x), f(y), f(z)\} \land \mu \text{ for all } x, y, z, u, v \in T. \]

Definition 3.22. A fuzzy subset $f$ of an ordered ternary semigroup $T$ is called a $(\lambda, \mu)$-fuzzy generalised bi-ideal of $T$ if
1. $x \leq y$ implies $f(x) \lor \lambda \geq f(y) \land \mu$
2. $f(xuyvz) \lor \lambda \geq \min\{f(x), f(y), f(z)\} \land \mu$ for all $x, y, z, u, v \in T$.

Theorem 3.2. Let $f$ be a fuzzy subset of an ordered ternary semigroup $T$. Then $f$ is a $(\lambda, \mu)$-fuzzy generalised bi-ideal of $T$ if and only if $U(f;t)$ is a generalised bi-ideal of $T$ for all $t \in (\lambda, \mu)$ whenever non-empty.

Proof. Let $f$ be a $(\lambda, \mu)$-fuzzy generalised bi-ideal of $T$ and $t \in (\lambda, \mu)$. Let $y \in U(f;t)$ and let $x \in T$ such that $x \leq y$. Thus $f(x) \lor \lambda \geq f(y) \land \mu \geq t$. Hence $x \in U(f;t)$. Now, let $x, y, z \in U(f;t)$ and $u, v \in T$ then $f(x) \lor \lambda \geq t$, $f(y) \lor \lambda \geq t$, $f(z) \lor \lambda \geq t$. Hence $f(xuyvz) \lor \lambda \geq \min\{f(x), f(y), f(z)\} \land \mu \geq t$. Hence $f(xuyvz) \lor \lambda \geq \min\{f(x), f(y), f(z)\} \land \mu \geq t$. Hence $f(xuyvz) \lor \lambda \geq \min\{f(x), f(y), f(z)\} \land \mu$. Therefore $f$ is a $(\lambda, \mu)$-fuzzy generalised bi-ideal of $T$.

Theorem 3.3. Let $f$ be a fuzzy subset of an ordered ternary semigroup $T$. Then $f$ is a $(\lambda, \mu)$-fuzzy bi-ideal of $T$ if and only if $U(f;t)$ is a generalised bi-ideal of $T$ for all $t \in (\lambda, \mu)$ whenever non-empty.

Proof. Follows from Theorem 3.2 and Theorem 3.2.

Theorem 3.4. Let $f$ be a fuzzy subset of an ordered ternary semigroup $T$. Then $f$ is a $(\lambda, \mu)$-fuzzy right (resp. left, lateral) ideal of $T$ if and only if $U(f;t)$ is a right (resp. left, lateral) ideal of $T$ for all $t \in (\lambda, \mu)$ whenever non-empty.

Proof. The proof follows from Theorem 3.2 and using the method applied in Theorem 3.2.

Remark 3.2. Every fuzzy right (resp. left, lateral) ideal is a $(\lambda, \mu)$-fuzzy right (resp. left, lateral) ideal by taking $\lambda = 0$ and $\mu = 1$. But the converse need not be true.

Example 3.2. Let $(Z^-, \cdot, \leq)$ be an ordered ternary semigroup. Let $\mu : Z^- \rightarrow [0,1]$ defined by
\[
\mu(x) = \begin{cases} 
0.85 & \text{if } x = -2 \\
0.8 & \text{if } x \in (-\infty, -3] \\
0 & \text{otherwise.}
\end{cases}
\]
Clearly $\mu$ is a $(0.3, 0.8)$-fuzzy right ideal. But $\mu$ is not a fuzzy right ideal as $\mu(-8 = -2 \cdot -2) = 0.8 < \mu(-2) = 0.85$.

Theorem 3.5. A non-empty subset $L$ of an ordered ternary semigroup $T$ is a right (resp. left, lateral) ideal of $T$ if and only if the fuzzy subset $f$ defined by
\[
f(x) = \begin{cases} 
t & \text{if } x \in T - L \\
r & \text{if } x \in L,
\end{cases}
\]
where $t \leq r$ and $t, r \in (\lambda, \mu)$, is a $(\lambda, \mu)$-fuzzy right (resp. left, lateral) ideal of $T$. 
Proof. Suppose $L$ is a right ideal of $T$ and let $x, y \in T$ be such that $x \leq y$. If $y \in T - L$, then $f(x) \lor \lambda \geq f(y) \land \mu = t$. If $y \in L$, implies $x \in L$, then $f(x) \lor \lambda = f(y) \land \mu$. Hence $f(x) \lor \lambda \geq f(y) \land \mu$. Now let $x, y, z \in T$. Let $x \in L$. Then $xyz \in L$. Hence $f(xyz) \lor \lambda = f(x) \land \mu$. If $x \not\in L$, then $f(x) \land \mu = t \leq f(xyz) \lor \lambda$. Hence $f$ is a $(\lambda, \mu)$-fuzzy right ideal of $T$. Conversely, assume that $f$ is a $(\lambda, \mu)$-fuzzy right ideal of $T$. Let $x \leq y \in L$. Then $f(x) \lor \lambda \geq f(y) \land \mu = r$, this implies that $f(x) = r$. Thus, $x \in L$. Now, if $x \in L$ and $y, z \in T$, then $f(xyz) \lor \lambda \geq f(x) \land \mu = r$. This implies that $f(xyz) = r$, that is $xyz \in L$. Hence $L$ is a right ideal of $T$. \qed

Corollary 3.1. A non-empty subset $I$ of an ordered ternary semigroup $T$ is a right (lateral, left) ideal of $T$ if and only if the characteristic function of $I$, defined by

$$
\chi^*_I(x) = \begin{cases} 
\lambda & \text{if } x \in T - I \\
\mu & \text{if } x \in I 
\end{cases}
$$

is a $(\lambda, \mu)$-fuzzy right (lateral, left) ideal of $T$.

The following theorems can be proved in a similar way.

Theorem 3.6. A non-empty subset $L$ of an ordered ternary semigroup $T$ is a ternary subsemigroup of $T$ if and only if the fuzzy subset $f$ of $T$ defined by

$$
f(x) = \begin{cases} 
t \text{ if } x \in T - L \\
r \text{ if } x \in L,
\end{cases}
$$

where $t \leq r$ and $r \in (\lambda, \mu)$, is a $(\lambda, \mu)$-fuzzy ternary subsemigroup of $T$.

Theorem 3.7. A non-empty subset $L$ of an ordered ternary semigroup $T$ is a generalised bi-ideal (bi-ideal) of $T$ if and only if the fuzzy subset $f$ of $T$ defined by

$$
f(x) = \begin{cases} 
t \text{ if } x \in T - L \\
r \text{ if } x \in L,
\end{cases}
$$

where $t \leq r$ and $r \in (\lambda, \mu)$, is a $(\lambda, \mu)$-fuzzy generalised bi-ideal (bi-ideal) of $T$.

Corollary 3.2. A non-empty subset $I$ of an ordered ternary semigroup $T$ is a generalised bi-ideal (bi-ideal) of $T$ if and only if the characteristic function $\chi^*_I$ of $I$ is a $(\lambda, \mu)$-fuzzy generalised bi-ideal (bi-ideal) of $T$.

Lemma 3.1. Union of any family of $(\lambda, \mu)$-fuzzy right (left, lateral) ideals of an ordered ternary semigroup $T$ is a $(\lambda, \mu)$-fuzzy right (left, lateral) ideal of $T$.

Proof. Let $\{f_i\}_{i \in I}$ be a family of $(\lambda, \mu)$-fuzzy right ideals of an ordered ternary semigroup $T$. Let $x, y \in T$ be such that $x \leq y$. Then, $\bigcup_{i \in I} f_i(x) \lor \lambda = \bigcup_{i \in I} (f_i(x) \lor \lambda) \geq \bigcup_{i \in I} (f_i(y) \land \mu) = (\bigcup_{i \in I} f_i)(y) \land \mu$. Now, let $x, y, z \in T$. Then, $\bigcup_{i \in I} (f_i)(xyz) \lor \lambda = \bigcup_{i \in I} (f_i)(xyz) \lor \lambda \geq \bigcup_{i \in I} (f_i(x) \land \mu) = (\bigcup_{i \in I} f_i)(x) \land \mu$. Hence $\bigcup_{i \in I} f_i$ is a $(\lambda, \mu)$-fuzzy right ideal of $T$. \qed

Lemma 3.2. Intersection of any family of $(\lambda, \mu)$-fuzzy right (left, lateral) ideals of an ordered ternary semigroup $T$ is a $(\lambda, \mu)$-fuzzy right (left, lateral) ideal of $T$.

Proof. Suppose let $\{f_i\}_{i \in I}$ be a family of $(\lambda, \mu)$-fuzzy right ideals of an ordered ternary semigroup $T$. Let $x, y \in T$ be such that $x \leq y$. Then, $\bigcap_{i \in I} f_i(x) \lor \lambda = \bigcap_{i \in I} (f_i(x) \lor \lambda) \geq \bigcap_{i \in I} (f_i(y) \land \mu) = (\bigcap_{i \in I} f_i)(y) \land \mu$. Now, let $x, y, z \in T$. Then, $\bigcap_{i \in I} (f_i)(xyz) \lor \lambda = \bigcap_{i \in I} (f_i)(xyz) \lor \lambda \geq \bigcap_{i \in I} (f_i(x) \land \mu) = (\bigcap_{i \in I} f_i)(x) \land \mu$. Hence $\bigcap_{i \in I} f_i$ is a $(\lambda, \mu)$-fuzzy right ideal of $T$. \qed
Definition 3.23. Let \( f, g \) and \( h \) be any three fuzzy subsets of an ordered ternary semigroup \( T \). The \((\lambda, \mu)\)-fuzzy product \( f \ast g \ast h \) is defined by for all \( x \in T \),

\[
(f \ast g \ast h)(x) = \begin{cases} 
\bigvee_{(u,v,w) \in A_x} \{\min\{f(u),g(v),h(w),\mu\} \lor \lambda\} & \text{if } A_x \neq \phi, \\
0 & \text{if } A_x = \phi.
\end{cases}
\]

Proposition 3.1. The \((\lambda, \mu)\)-fuzzy product of three \((\lambda, \mu)\)-fuzzy right (left, lateral) ideals of an ordered ternary semigroup \( T \) is again a \((\lambda, \mu)\)-fuzzy right (left, lateral) ideal of \( T \).

Proof. Let \( f, g \) and \( h \) be three \((\lambda, \mu)\)-fuzzy right ideals of \( T \). Let \( x, y \in T \) such that \( x \leq y \). Let \( (u,v,w) \in A_y \) then \( y \leq uvw \). Since \( x \leq y \) implies \( x \leq uvw \), that is \( (u,v,w) \in A_x \). Hence \( A_y \subseteq A_x \). Now

\[
(f \ast g \ast h)(y) \land \mu = \bigvee_{(p,q,r) \in A_y} \{f(p) \land g(q) \land h(r) \land \mu\} \lor \lambda \land \mu
\]

\[
= \bigvee_{(p,q,r) \in A_y} \{f(p) \land g(q) \land h(r) \land \mu\} \land \mu \lor (\lambda \land \mu)
\]

\[
\leq \bigvee_{(p,q,r) \in A_x} \{f(p) \land g(q) \land h(r) \land \mu\} \lor \lambda
\]

\[
= \bigvee_{(p,q,r) \in A_x} \{f(p) \land g(q) \land h(r) \land \mu\} \lor \lambda
\]

Since \( h \) is a \((\lambda, \mu)\)-fuzzy right ideal, \( h(ryz) \lor \lambda \geq h(r) \land \mu \).

\[
(f \ast g \ast h)(x) \land \mu = \bigvee_{(p,q,r) \in A_x} \{f(p) \land g(q) \land h(r) \land \mu\} \lor \lambda \land \mu
\]

\[
= \bigvee_{(p,q,r) \in A_x} \{f(p) \land g(q) \land h(r) \land \mu\} \land \mu \lor (\lambda \land \mu)
\]

\[
= \bigvee_{(p,q,r) \in A_x} \{f(p) \land g(q) \land h(r) \land \mu\} \lor \lambda
\]

\[
\leq \bigvee_{(p,q,r) \in A_{xyz}} \{\{f(p) \land g(q) \land h(ryz) \land \mu\} \lor \lambda\} \lor \lambda
\]

\[
\leq \bigvee_{(p,q,r) \in A_{xyz}} \{f(p) \land g(q) \land h(ryz) \land \mu\} \lor \lambda
\]

\[
= \bigvee_{(p,q,r) \in A_{xyz}} \{f(p) \land g(q) \land h(ryz) \land \mu\} \lor \lambda
\]

If \( A_x = \phi \), then \( (f \ast g \ast h)(x) \land \mu = 0 \leq (f \ast g \ast h)(xyz) \lor \lambda \). Thus \( f \ast g \ast h \) is a \((\lambda, \mu)\)-fuzzy right ideal of \( T \).

4 \((\lambda, \mu)\)-Fuzzy interior ideals

Definition 4.24. A fuzzy subset \( f \) of an ordered ternary semigroup \( T \) is called a \((\lambda, \mu)\)-fuzzy interior ideal of \( T \) if

1. \( x \leq y \) implies \( f(x) \lor \lambda \geq f(y) \land \mu \)
2. \( f(rsatu) \lor \lambda \geq f(a) \land \mu \) for all \( r, s, a, t, u \in T \).

Theorem 4.8. Let \( f \) be a fuzzy subset of an ordered ternary semigroup \( T \). Then \( f \) is a \((\lambda, \mu)\)-fuzzy interior ideal of \( T \) if and only if \( U(f; t) \) is an interior ideal of \( T \) for all \( t \in (\lambda, \mu) \) whenever non-empty.

Proof. The proof is similar to the proof of Theorem 3.2.
Theorem 4.9. A non-empty subset $L$ of an ordered ternary semigroup $T$ is an interior ideal of $T$ if and only if the fuzzy subset $f$ of $T$ defined by

$$f(x) = \begin{cases} t & \text{if } x \in T - L \\ r & \text{if } x \in L, \end{cases}$$

where $t \leq r$ and $t, r \in (\lambda, \mu]$, is a $(\lambda, \mu)$-fuzzy interior ideal of $T$.

Proof. The proof is similar to the proof of Theorem 3.5. \qed

Corollary 4.3. A non-empty subset $I$ of an ordered ternary semigroup $T$ is an interior ideal of $T$ if and only if the characteristic function $\chi_I^T$ of $I$ is a $(\lambda, \mu)$-fuzzy interior ideal of $T$.

Proposition 4.2. Let $T$ be an ordered ternary semigroup and $f$ a $(\lambda, \mu)$-fuzzy two-sided ideal of $T$. Then $f$ is a $(\lambda, \mu)$-fuzzy interior ideal of $T$.

Proof. Let $r, s, a, t, u \in T$. Since $f$ is a $(\lambda, \mu)$-fuzzy left ideal of $T$, we have $f(rsatu) \lor \lambda \geq f(atu) \land \mu$. Since $f$ is a $(\lambda, \mu)$-fuzzy right ideal of $T$, we have $f(atu) \lor \lambda \geq f(a) \land \mu$. Then we have $f(rsatu) \lor \lambda = [f(rsatu) \lor \lambda] \lor \lambda \geq [f(atu) \land \mu] \lor \lambda = [f(atu) \lor \lambda] \land \mu \geq [f(a) \land \mu] \land \mu = f(a) \land \mu$. Thus $f$ is a $(\lambda, \mu)$-fuzzy interior ideal of $T$. \qed

Proposition 4.3. Let $T$ be an ordered ternary semigroup and $f$ a $(\lambda, \mu)$-fuzzy lateral ideal of $T$. Then $f$ is a $(\lambda, \mu)$-fuzzy interior ideal of $T$.

Proof. Let $r, s, a, t, u \in T$. Since $f$ is a $(\lambda, \mu)$-fuzzy lateral ideal of $T$, we have $f(rsatu) \lor \lambda \geq f(satu) \land \mu$. Again since $f$ is a $(\lambda, \mu)$-fuzzy lateral ideal of $T$, we have $f(satu) \lor \lambda \geq f(a) \land \mu$. Then we have $f(rsatu) \lor \lambda = [f(rsatu) \lor \lambda] \lor \lambda \geq [f(satu) \land \mu] \lor \lambda = [f(satu) \lor \lambda] \land \mu \geq [f(a) \land \mu] \land \mu = f(a) \land \mu$. Thus $f$ is a $(\lambda, \mu)$-fuzzy interior ideal of $T$. \qed

Proposition 4.4. Let $T$ be a regular ordered ternary semigroup and $f$ a $(\lambda, \mu)$-fuzzy interior ideal of $T$. Then $f$ is a $(\lambda, \mu)$-fuzzy ideal of $T$.

Proof. Let $x, y, z \in T$. Since $T$ is regular and $x \in T$, there exists $a \in T$ such that $x \leq xax$. Then we have $xyz \leq xaxyz$ which implies $f(xyz) \lor \lambda \geq f(xaxyz) \land \mu$. Since $f$ is a $(\lambda, \mu)$-fuzzy interior ideal of $T$, $f(xaxyz) \lor \lambda \geq f(x) \land \mu$. Now we have $f(xyz) \lor \lambda = [f(xyz) \lor \lambda] \lor \lambda \geq [f(xaxyz) \land \mu] \lor \lambda = [f(xaxyz) \lor \lambda] \land \mu \geq [f(x) \land \mu] \land \mu = f(x) \land \mu$. Thus $f$ is a $(\lambda, \mu)$-fuzzy right ideal of $T$. In similar way we can prove that $f$ is a $(\lambda, \mu)$-fuzzy left ideal of $T$. Thus $f$ is a $(\lambda, \mu)$-fuzzy lateral ideal of $T$. \qed

Theorem 4.10. In regular ordered ternary semigroups the concepts of $(\lambda, \mu)$-fuzzy ideals and $(\lambda, \mu)$-fuzzy interior ideals coincide.

Proof. The proof follows from Proposition 4.3 and Proposition 4.4. \qed

5. $(\lambda, \mu)$-Fuzzy quasi-ideals

Definition 5.25. A fuzzy subset $f$ of an ordered ternary semigroup $T$ is called a $(\lambda, \mu)$-fuzzy quasi-ideal of $T$ if it satisfies for all $x, y \in T$,

1. $x \leq y$ implies $f(x) \lor \lambda \geq f(y) \land \mu$
2. $f(x) \lor \lambda \geq \min\{(f \circ T \circ f)(x), (T \circ f \circ T)(f)(x), (T \circ T \circ f)(x)\} \land \mu$
3. $f(x) \lor \lambda \geq \min\{(f \circ T \circ T)(x), (T \circ T \circ f \circ T)(f)(x), (T \circ T \circ T)(x)\} \land \mu$.

Remark 5.3. Every fuzzy quasi ideal is a $(\lambda, \mu)$-fuzzy quasi ideal of $T$ by taking $\lambda = 0$ and $\mu = 1$. But the converse need not be true.

Theorem 5.11. Let $f$ be a fuzzy subset of an ordered ternary semigroup $T$. Then $f$ is a $(\lambda, \mu)$-fuzzy quasi-ideal of $T$ if and only if $UI(f; t)$ is a quasi-ideal of $T$ for all $t \in (\lambda, \mu]$ whenever non-empty.
Proof. Let \( f \) be a \((\lambda, \mu)\)-fuzzy quasi-ideal of \( T \) and \( t \in (\lambda, \mu) \). Let \( y \in U(f; t) \) and let \( x \in T \) such that \( x \leq y \). Thus \( f(x) \vee \lambda \geq f(y) \land \mu \geq t \). Hence \( x \in U(f; t) \). Let \( x \in (U(f; t)T)T \cap (TU(f;T)t) \cap (TTU(f;t)) \). Then there exist \( a, b, d \in U(f; t) \) and \( s_1, s_2, s_3, s_4, s_5, s_6 \in T \) such that \( x \leq a_1s_2 = s_3s_4 = s_5s_6 \). Thus \( f(a) \geq t, f(b) \geq t, f(d) \geq t \).

Then \( \min\{(f \circ T \circ T)(x), (T \circ f \circ T)(x), (T \circ T \circ f)(x)\} = \min\{\min\{f(a)\}, \min\{f(b)\}, \min\{f(d)\}\} \geq t \).

Now \( \min\{(f \circ T \circ T)(x), (T \circ f \circ T)(x), (T \circ T \circ f)(x)\} \land \mu \geq \min\{t, \mu\} = t \).

Since \( f \) is a \((\lambda, \mu)\)-fuzzy quasi-ideal of \( T \), \( f(x) \lor \lambda \geq \min\{(f \circ T \circ T)(x), (T \circ f \circ T)(x), (T \circ T \circ f)(x)\} \land \mu \geq t > \lambda \). Then \( f(x) \geq t \) and \( x \in U(f; t) \). Hence \( (U(f; t)T)T \cap (TU(f;T)t) \cap (TTU(f;t)) \subseteq U(f; t) \). Next, let \( x \in (U(f; t)T)T \cap (TU(f;T)t) \cap (TTU(f;t)) \). Then there exist \( a, c, d \in U(f; t) \) and \( s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 \in T \) such that \( x \leq a_1s_2 = s_3s_4 = s_5s_6 = s_7s_8 \). Thus \( f(a) \geq t, f(c) \geq t, f(d) \geq t \).

Then \( \min\{(f \circ T \circ T)(x), (T \circ f \circ T \circ T)(x), (T \circ T \circ T \circ f)(x)\} = \min\{\min\{f(a)\}, \min\{f(c)\}, \min\{f(d)\}\} \geq t \).

Now \( \min\{(f \circ T \circ T)(x), (T \circ f \circ T \circ T)(x), (T \circ T \circ T \circ f)(x)\} \land \mu \geq \min\{t, \mu\} = t \).

Since \( f \) is a \((\lambda, \mu)\)-fuzzy quasi-ideal of \( T \), \( f(x) \lor \lambda \geq \min\{(f \circ T \circ T)(x), (T \circ f \circ T \circ T)(x), (T \circ T \circ T \circ f)(x)\} \land \mu \geq t > \lambda \). Then \( f(x) \geq t \) and \( x \in U(f; t) \). Hence \( (U(f; t)T)T \cap (TU(f;T)t) \cap (TTU(f;t)) \subseteq U(f; t) \).

Then \( U(f; t) \) is a quasi-ideal in \( T \).

Conversely let us assume that \( U(f; t) \), \( t \in (\lambda, \mu) \) is a quasi-ideal in \( T \). Let \( x, y \in T \) be such that \( x \leq y \). Choose \( t = f(y) \land \mu \). Thus, \( y \in U(f; t) \), which implies \( x \in U(f; t) \). Then \( f(x) \lor \lambda \geq f(y) \land \mu \). Let \( x \in T \) be such that \( f(x) \lor \lambda \geq f(y) \land \mu \). Now \( f(x) \lor \lambda \geq f(y) \land \mu \). Now let \( x \in T \) be such that \( f(x) \lor \lambda = r \). Then \( f(x) \lor \lambda = f(y) \land \mu \). Hence \( f(x) \lor \lambda = f(y) \land \mu \). Now let \( x \in T \) be such that \( f(x) \lor \lambda = r \). Then \( f(x) \lor \lambda = r \). If \( x \not\in Q \) then \( f(x) \lor \lambda \neq t \). Since \( Q \) is a quasi-ideal, \( x \not\in Q \) implies \( x \not\in (QTT \cap TQT \cap TTQ) \) which implies \( x \not\in QTT, x \not\in QTT \) and \( x \not\in TTQ \). Suppose \( \min\{(f \circ T \circ T)(x), (T \circ f \circ T)(x), (T \circ T \circ f)(x)\} \land \mu = r \) then \((f \circ T \circ T)(x) = r, (T \circ f \circ T)(x) = r, (T \circ T \circ f)(x) = r \) which implies \( \sup\{f(u)\} = r, \sup\{f(v)\} = r \) and \( \sup\{f(w)\} = r \) for some \( t_1, t_2, t_3, t_4, t_5, u, v, w \in T \). Since \( \sup\{f(u)\} = r \) implies \( u \in Q \) that is \( x \leq ul_1t_2 \in QTT \) which is a contradiction to \( x \not\in QTT \). Thus \( \min\{(f \circ T \circ T)(x), (T \circ f \circ T)(x), (T \circ T \circ f)(x)\} \land \mu = t \). Hence \( f \) is a \((\lambda, \mu)\)-fuzzy quasi-ideal of \( T \).

Conversely, assume that \( f \) is a \((\lambda, \mu)\)-fuzzy quasi-ideal of \( T \). Let \( x \leq y \in Q \). Then \( f(x) \lor \lambda \geq f(y) \land \mu = r \), this implies that \( f(x) = r \). Thus, \( x \in Q \). Let \( a \in (QTT \cap TTQ \cap TTQ) \) then \( \min\{(f \circ T \circ T)(a), (T \circ f \circ T)(a), (T \circ T \circ f)(a)\} \land \mu = r \). Since \( f \) is a \((\lambda, \mu)\)-fuzzy quasi-ideal of \( T \), \( \min\{(f \circ T \circ T)(a), (T \circ f \circ T)(a), (T \circ T \circ f)(a)\} \land \mu = r \leq f(a) \lor \lambda \). Which implies \( a \in Q \). Similarly, if \( a \in QTT \cap TTQ \cap TTQ \) implies \( a \in Q \). Thus \( Q \) is a quasi-ideal of \( T \).

\[ \square \]

Corollary 5.4. A non-empty subset \( I \) of an ordered ternary semigroup \( T \) is a quasi-ideal of \( T \) if and only if the characteristic function \( \chi^\ast_I \) of \( I \) is a \((\lambda, \mu)\)-fuzzy quasi-ideal of \( T \).
Theorem 5.13. Let f be a fuzzy subset of an ordered ternary semigroup T. If f is a \((\lambda, \mu)\)-fuzzy lateral (left, right) ideal of T then f is a \((\lambda, \mu)\)-fuzzy quasi-ideal of T.

Proof. Let f be a \((\lambda, \mu)\)-fuzzy lateral (left, right) ideal of T. Let \(x, a, b, c, d, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 \in T\). Then \(f(s_3bs_4) \lor \lambda \geq f(b) \land \mu\). Now \(\min\{f \circ T \circ f\}(x), (T \circ f \circ T\}(x), (T \circ T \circ f)(x)\} \land \mu\)
\[
= \min\left\{ \bigvee_{(a_1, s_2) \in A_x} f(a), T(s_1), T(s_2) \right\}, \bigvee_{(s_3, b, s_4) \in A_x} \min\{T(s_3), f(b), T(s_4)\}, \bigvee_{(s_5, s_6, d) \in A_x} \min\{T(s_5), T(s_6), f(d)\} \right\} \land \mu \]
\[
= \min\left\{ \bigvee_{(a_1, s_2) \in A_x} \{f(a)\}, \bigvee_{(s_3, b, s_4) \in A_x} \{f(b)\}, \bigvee_{(s_5, s_6, d) \in A_x} \{f(d)\} \right\} \land \mu \]
\[
\leq \min\left\{ 1, \bigvee_{(s_3, b, s_4) \in A_x} \{f(b)\} \right\} \land \mu \]
\[
= \bigvee_{(s_3, b, s_4) \in A_x} \{f(b)\} \land \mu \leq \bigvee_{(s_3, s_4, s_5, s_6) \in A_x} \{f(b) \land \mu\} \leq f(s_3bs_4) \lor \lambda \leq f(x) \lor \lambda.
\]
Since f is a \((\lambda, \mu)\)-fuzzy lateral ideal, then \(f(s_3s_4s_5s_6) \lor \lambda \geq f(c) \land \mu\).

Now \(\min\{f \circ T \circ f\}(x), (T \circ f \circ T\}(x), (T \circ T \circ f)(x)\} \land \mu\)
\[
= \min\left\{ \bigvee_{(a_1, s_2) \in A_x} \min\{f(a), T(s_1), T(s_2)\}, \bigvee_{(s_3, s_4, s_5, s_6) \in A_x} \min\{T(s_3), T(s_4), f(c), T(s_5), T(s_6)\}, \bigvee_{(s_7, s_8, d) \in A_x} \min\{T(s_7), T(s_8), f(d)\} \right\} \land \mu \]
\[
= \min\left\{ \bigvee_{(a_1, s_2) \in A_x} \{f(a)\}, \bigvee_{(s_3, s_4, s_5, s_6) \in A_x} \{f(c)\}, \bigvee_{(s_7, s_8, d) \in A_x} \{f(d)\} \right\} \land \mu \]
\[
\leq \min\left\{ 1, \bigvee_{(s_3, s_4, s_5, s_6) \in A_x} \{f(c)\} \right\} \land \mu \]
\[
= \bigvee_{(s_3, s_4, s_5, s_6) \in A_x} \{f(c)\} \land \mu \leq \bigvee_{(s_3, s_4, s_5, s_6) \in A_x} \{f(c) \land \mu\} \leq f(s_3s_4s_5s_6) \lor \lambda \leq f(x) \lor \lambda.\]
Thus f is a \((\lambda, \mu)\)-fuzzy quasi-ideal of T. \(\square\)

Theorem 5.14. Every \((\lambda, \mu)\)-fuzzy quasi-ideal of T is a \((\lambda, \mu)\)-fuzzy bi-ideal of T.

Proof. Let f be a \((\lambda, \mu)\)-fuzzy quasi-ideal of T. Let \(x, y, z, u, v \in T\). Then \(f(xyz) \lor \lambda \geq \min\{f \circ T \circ f\}(xyz), (T \circ f \circ T)(xyz), (T \circ T \circ f)(xyz)\} \land \mu\)
\[
= \min\left\{ \bigvee_{(a, b, c) \in A_{xyz}} \min\{f(a), T(b), T(c)\}, \bigvee_{(p, q, r) \in A_{xyz}} \min\{T(p), f(q), T(r)\}, \bigvee_{(l, m, n) \in A_{xyz}} \min\{T(l), T(m), f(n)\} \right\} \land \mu \]
\[
\geq \min\{f(x), T(y), T(z)\}, \min\{T(x), f(y), T(z)\}, \min\{T(x), T(y), f(z)\}\} \land \mu \]
\[
= \min\{f(x), f(y), f(z)\} \land \mu.
\]
Thus f is a ternary subsemigroup of T. Also \(f(xuyvz) \lor \lambda \geq \min\{f \circ T \circ f\}(xuyvz), (T \circ f \circ f \circ T)(xuyvz), (T \circ f \circ f)(xuyvz)\} \land \mu\)
\[
= \min\left\{ \bigvee_{(a, b, c) \in A_{xuyvz}} \min\{f(a), T(b), T(c)\} \right\},
\]
\[
\left( \bigcup_{(p,q,r,s,t) \in A_{xuyvz}} \min\{T(p), T(q), f(r), T(s), T(t)\} \right) \wedge \mu \\
\left( \bigcup_{(i,j,k) \in A_{xuyvz}} \min\{T(i), T(j), f(k)\} \right) \wedge \mu \\
\geq \min\{\min\{f(x), T(uv), T(z)\}, \min\{T(x), T(u), f(y), T(v), T(z)\}, \min\{T(x), T(uv), f(z)\}\} \wedge \mu \\
= \min\{f(x), f(y), f(z)\} \wedge \mu.
\]

Hence \( f \) is a \((\lambda, \mu)\)-fuzzy bi-ideal of \( T \).

**Corollary 5.5.** Every \((\lambda, \mu)\)-fuzzy left (lateral, right) ideal of an ordered ternary semigroup \( T \) is a \((\lambda, \mu)\)-fuzzy bi-ideal of \( T \).

**References**


**Received:** July 10, 2015; **Accepted:** September 23, 2015

**UNIVERSITY PRESS**

Website: http://www.malayajournal.org/