A Maxmin Advantage Approach to find Critical Path in Project Networks

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Abstract
This paper deals with project networks, where we analyze the problem under uncertain completion times. The maxmin advantage criterion is used to obtain the critical path of the project.

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1 Introduction

Project management concerns the scheduling and control of activities in such a way that the project is completed in minimum time \[6\]. In Critical Path Method some jobs must be completed before the other jobs are started, that is an order of precedence will be followed. \[2\] and \[3\] proposed different methods to identify critical path where the activity durations are fuzzy numbers. \[7\] proposed a new approach based on linear programming for backward pass calculation. In \[4\], a goal programming approach was proposed to solve a project with multiple objectives. \[5\] proposed a backward recursion on fuzzy subtraction, but this method fails to compute floats in all types of networks. \[1\] introduced a new iterative method to find critical path in a project network. In this paper, we have formulated a linear programming approach for critical path problems, where the constraints and objective function uses the right and left ?-cut values. The paper is organized as follows, section 2 consists of some basic ideas, section 3 is on the formulation of the problem, section 4 consists of the proposed method, section 5 deals with a numerical example, section 6 is on biobjective formulation.

Definition 1.1. Trapezoidal Fuzzy Number: A fuzzy number whose membership function is given by

\[ \mu_A(x) = \begin{cases} \frac{x-a}{b-a} , & a \leq x < b \\ 1 , & b \leq x \leq c \\ \frac{d-x}{d-c} , & c < x \leq d \\ 0 ; & \text{otherwise} \end{cases} \]

is called a trapezoidal fuzzy number \( \tilde{A} = (a, b, c, d) \)

Definition 1.2. \(\alpha\)-cut The \(\alpha\)-cut of a trapezoidal fuzzy number \( \tilde{A} = (a, b, c, d) \) is given by \( ^\alpha \tilde{A} = [(b - a)\alpha + a, d - (d - c)\alpha] \)

Definition 1.3. Directed Acyclic Graph A directed acyclic graph is represented by \( G=(N,E) \), where \( N \) is the set of nodes and \( E \) is the set of edges. \( G \) will represent the project and its nodes are identified with the beginning or ending of a subset of tasks and their edges will denote the activities of the project.

Critical Path Problem For all \((i, j) \in E\), if \( w_{ij} \) be the execution time of the activity associated to the arc \((i, j)\) under the

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scenario $S$. Let us assume that for each $(i,j) \in E$, the cartesian product of the set of intervals will form the set of acceptable scenario denoted by $S$. Let $P$ denote the set of directed paths in the graph $G$ with $n$ nodes. The length of the largest path in $P$ usually represents the minimum time for the execution of the whole project $G$, which is obtained by solving

$$\max_{x \in P} \sum_{(i,j) \in E} w_{ij}^x x_{ij}$$

where $x$ is the path which is identified by the matrix with binary entries $x_{ij}$. To find the changing subset of critical tasks, we propose to maximize the minimum advantage over the whole scenario. To formulate the problem mathematically we define the minimum advantage for a given path as follows.

**Definition 1.4. Minimum Advantage** For any path $x \in P$, the minimum advantage for $x$ corresponding to the scenario $S$ is $A(x) = \min_{y \in E} \{A(x, w^y) - \min_{y \in P} A(y, w^y)\}$, where $A(x, w^y) = \sum_{(i,j) \in E} w_{ij}^x x_{ij}$.

Hence the optimization problem whose solution is to be found becomes

$$A^* = \max_{x \in P} A(x) = \max_{x \in P} \min_{y \in S} \{A(x, w^y) - \min_{y \in P} A(y, w^y)\} \quad (1)$$

**Theorem 1.1.** Problem (1) is NP-hard.

**Proof.** Proof is obvious, since the complexity for the problem remains NP-hard even for the planar directed acyclic graphs with nodes of degrees less than or equal to three and integer values for the extremes $w_{ij}^\pm$.

**Definition 1.5. Scenario** For an arbitrary path $x \in P$, we define the scenario $s(x) \in S$ as

$$w_{ij}^s(x) = \begin{cases} w_{ij}^1 & \text{if } (i,j) \in x \\ w_{ij}^+ & \text{otherwise} \end{cases}$$

The scenario $s(x)$ will play an important role in the mixed integer formulation of maxmin advantage problem. In fact under this scenario, the minimum advantage for $x$ is reached. To analyze the mixed integer programming problem for (1) in the next section we use the following notations.

i. $A^s = \min_{x \in P} A(x, w^s)$  
ii. $A^s(x) = A(x, w^s) - A^s$  
iii. $A(x) = \min_{y \in S} A^s(x)$

**Proposition 1.1.** For every $x \in P$, we have $A(x) = A^s(x) = A(x, w^s(x)) - A^s(x)$

## 2 The MIP Formulation

In this section, the MIP formulation is proposed in terms of the notation defined in the previous section. First, the value $A^s(x)$ can be obtained as the optimal objective value of the flow problem of a product from first node to $n^{th}$ node in $G$ with $n$ nodes.

$$A^s(x) = \min \sum_{i,j \in E} [w_{ij}^1 + (w_{ij}^- + w_{ij}^+) x_{ij}] y_{ij} \text{ such that}$$

$$\sum_{j: (i,j) \in E} y_{ij} - \sum_{k: (k,j) \in E} y_{kj} = \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } i = n \\ 0 & \text{if } i \neq 1, n \end{cases} \quad (2)$$

It is not necessary that the variables $y_{ij} \in \{0,1\}$ since (2) has unimodular constraints, that is, every extreme point for the feasible set has integer(or binary) components. Hence the linear programming problem has at least one optimal solution with $y_{ij} \in \{0,1\}$.

**Theorem 2.2.** $A^s(x)$ is a convex function.

**Proof.** Consider $\sum_{i,j \in E} [w_{ij}^1 + (w_{ij}^- + w_{ij}^+) (\lambda x_{ij} + (1-\lambda) x_{ij}')] y_{ij}$

$$= \lambda \sum_{i,j \in E} [w_{ij}^- + (w_{ij}^- + w_{ij}^+) x_{ij}] y_{ij} + (1-\lambda) \sum_{i,j \in E} [w_{ij}^+ + (w_{ij}^- + w_{ij}^+) x_{ij}'] y_{ij}$$

For all $\lambda \in [0,1]$ and $A^{s(\lambda x + (1-\lambda) x')}$ is given by the minimum of the above expression. Also from (2), $A^s(x)$ is the optimal objective value for a linear optimization problem, whose dual optimization problem is given by $A^d(x) = \max a_n - a_1 - - - - - (3)$
\[ st \ a_j - a_i \leq w_{ij}^- + (w_{ij}^- + w_{ij}^+)x_{ij}(i, j) \in E \]

If \((a_1, a_2, \ldots, a_n)\) is a feasible solution of (3), then \(a_1 + \lambda, a_2 + \lambda, \ldots, a_n + \lambda\) is also a feasible solution of (3) for any value of \(\lambda\). Hence a new constraint \(a_1 = 0\) is added to the set of constraints.

\[ A^{\alpha(x)} = \max a_n \]

such that \(a_j - a_i \leq w_{ij}^- + (w_{ij}^- + w_{ij}^+)x_{ij}(i, j) \in E \quad —– (4)\)

\(a_1 = 0\)

where all the \(a_i's\) are nonnegative. Hence on introducing \(a_1 = 0\) and \(x_{ij} \in [0, 1]\), we have \(a_i \geq a_j \geq 0\) for all \((i, j) \in E\)

Using the proposition 2.1 and formulation (2), the problem can be equivalently written as \(A^* = \max[\sum w_{ij}^+ x_{ij} - A^{\alpha(x)}]\) such that

\[ \sum_{j: (i, j) \in E} x_{ij} - \sum_{k: (k, i) \in E} x_{ki} = \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } i = n \\ 0 & \text{if } i \neq 1, n \end{cases} \]

\(x_{ij} \in \{0, 1\}, (i, j) \in E\) where \(A(x, 2^{\alpha(x)}) = \sum_{(i, j) \in E} w_{ij}^+ x_{ij}\)

Since the objective function of the above problem is a piecewise convex function, we cannot delete the binary constraints on the variables and hence it can reach its maximum out of the extreme point set of the feasible polyhedron.

Hence from (4), the mixed integer linear formulation of the problem is

\[ A^* = \max \sum_{(i, j) \in E} w_{ij}^+ x_{ij} - a_n \]

such that

\[ \sum_{j: (i, j) \in E} x_{ij} - \sum_{k: (k, i) \in E} x_{ki} = \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } i = n \\ 0 & \text{if } i \neq 1, n \end{cases} \]

\(a_j - a_i + (w_{ij}^- + w_{ij}^+)x_{ij} \leq w_{ij}^- (i, j) \in E\)

where \(x_{ij} \in \{0, 1\}, (i, j) \in E\)

\[ \square \]

### 3 Proposed method to find critical path

Step 1: Consider a project network where all the activities are trapezoidal fuzzy numbers.

Step 2: Find the \(\alpha\)-cut of each trapezoidal fuzzy number and it is represented as the duration of the corresponding activity.

Step 3: The left \(\alpha\)-cut is taken as \(w_{ij}^-\) and the right \(\alpha\)-cut is taken as \(w_{ij}^+\).

Step 4: On solving the mixed integer problem we get the critical path.

### 4 Numerical example to find critical path using the MIP formulation

Consider the following network whose critical path is to be found

Fig 5.1 Project Network
The duration of each activity and their $\alpha$-cut is represented in the following table.

**Table 5.2:** Duration and $\alpha$-cut of the Network

<table>
<thead>
<tr>
<th>Activity</th>
<th>Duration</th>
<th>$\alpha$-cut</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>(25,28,32,35)</td>
<td>[25+3(\alpha),35-3(\alpha)]</td>
</tr>
<tr>
<td>1-3</td>
<td>(40,55,65,70)</td>
<td>[40+15(\alpha),70-5(\alpha)]</td>
</tr>
<tr>
<td>2-4</td>
<td>(32,37,43,48)</td>
<td>[32+5(\alpha),48-5(\alpha)]</td>
</tr>
<tr>
<td>3-4</td>
<td>(20,25,35,40)</td>
<td>[20+5(\alpha),40-5(\alpha)]</td>
</tr>
<tr>
<td>2-5</td>
<td>(35,38,42,45)</td>
<td>[35+3(\alpha),45-3(\alpha)]</td>
</tr>
<tr>
<td>4-7</td>
<td>(60,65,75,85)</td>
<td>[60+5(\alpha),85-10(\alpha)]</td>
</tr>
<tr>
<td>3-6</td>
<td>(42,45,55,60)</td>
<td>[42+3(\alpha),60-5(\alpha)]</td>
</tr>
<tr>
<td>5-7</td>
<td>(65,75,85,90)</td>
<td>[65+10(\alpha),90-5(\alpha)]</td>
</tr>
<tr>
<td>6-7</td>
<td>(15,18,22,26)</td>
<td>[15+3(\alpha),26-4(\alpha)]</td>
</tr>
</tbody>
</table>

**Case i:** When $\alpha=0$, the completion time of each activity is given as:

<table>
<thead>
<tr>
<th>Activity</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>(25,35)</td>
</tr>
<tr>
<td>1-3</td>
<td>(40,70)</td>
</tr>
<tr>
<td>2-4</td>
<td>(32,48)</td>
</tr>
<tr>
<td>3-4</td>
<td>(20,40)</td>
</tr>
<tr>
<td>2-5</td>
<td>(35,45)</td>
</tr>
<tr>
<td>4-7</td>
<td>(60,85)</td>
</tr>
<tr>
<td>3-6</td>
<td>(42,60)</td>
</tr>
<tr>
<td>5-7</td>
<td>(65,90)</td>
</tr>
<tr>
<td>6-7</td>
<td>(15,26)</td>
</tr>
</tbody>
</table>

The problem can be formulated as:

$$\max [35x_{12} + 70x_{13} + 48x_{24} + 40x_{34} + 45x_{25} + 85x_{47} + 60x_{36} + 90x_{57} + 26x_{67} - a_7]$$

such that

$$x_{12} + x_{13} = 1$$
$$-x_{12} + x_{24} + x_{25} = 0$$
$$-x_{13} + x_{34} + x_{36} = 0$$
$$-x_{24} - x_{34} + x_{47} = 0$$
\[-x_{25} + x_{57} = 0\]
\[-x_{36} + x_{67} = 0\]
\[-x_{47} - x_{57} = -1\]
\[a_2 - a_1 - 10x_{12} \leq 25\]
\[a_3 - a_1 - 30x_{12} \leq 40\]
\[a_4 - a_3 - 20x_{34} \leq 20\]
\[a_4 - a_2 - 16x_{24} \leq 32\]
\[a_5 - a_2 - 10x_{25} \leq 35\]
\[a_7 - a_4 - 25x_{47} \leq 60\]
\[a_6 - a_3 - 18x_{36} \leq 42\]
\[a_7 - a_6 - 11x_{67} \leq 15\]
\[a_7 - a_5 - 25x_{57} \leq 65\]
\[a_1 = 0\]

Case ii: When \(\alpha = 1\), the completion time of each activity is given as

<table>
<thead>
<tr>
<th>Activity</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>(28,32)</td>
</tr>
<tr>
<td>1-3</td>
<td>(55,65)</td>
</tr>
<tr>
<td>2-4</td>
<td>(37,43)</td>
</tr>
<tr>
<td>3-4</td>
<td>(25,35)</td>
</tr>
<tr>
<td>2-5</td>
<td>(38,42)</td>
</tr>
<tr>
<td>4-7</td>
<td>(65,75)</td>
</tr>
<tr>
<td>3-6</td>
<td>(45,55)</td>
</tr>
<tr>
<td>5-7</td>
<td>(75,85)</td>
</tr>
<tr>
<td>6-7</td>
<td>(18,22)</td>
</tr>
</tbody>
</table>

The problem can be formulated as \(\max [32x_{12} + 65x_{13} + 43x_{24} + 35x_{34} + 42x_{25} + 75x_{47} + 55x_{36} + 85x_{57} + 22x_{67} - a_7]\)

such that
\[x_{12} + x_{13} = 1\]
\[-x_{12} + x_{24} + x_{25} = 0\]
\[-x_{13} + x_{34} + x_{36} = 0\]
\[-x_{24} - x_{34} + x_{47} = 0\]
\[-x_{25} + x_{57} = 0\]
\[-x_{36} + x_{67} = 0\]
\[-x_{47} - x_{67} - x_{57} = -1\]
\[a_2 - a_1 - 4x_{12} \leq 28\]
\[a_3 - a_1 - 10x_{12} \leq 55\]
\[a_4 - a_2 - 6x_{34} \leq 37\]
\[a_4 - a_3 - 10x_{24} \leq 25\]
\[a_5 - a_2 - 4x_{25} \leq 38\]
\[a_7 - a_4 - 10x_{47} \leq 65\]
\[a_6 - a_3 - 10x_{36} \leq 45\]
\[a_7 - a_5 - 10x_{67} \leq 75\]
\[a_7 - a_6 - 4x_{57} \leq 18\]
\[a_1 = 0\]

In both the cases, we can find the solution using TORA software, where we get the optimal solution as \(x_{13} = x_{34} = x_{47} = 1\) and the remaining variables to be zero. Hence the critical path is 1-3-4-7.

5 Bi-objective formulation

Let \(x^a \in P\) be a path that maximizes \(a_i(x)\) and \(x^b \in P\) minimizing \(\sum_{(i,j) \in E} w_{ij}^+ x_{ij}\). Both the paths are not efficient solutions of the biobjective problem. Particularly, \(x^b\) gives the minimal execution of the project when all the intervals are uncertain.
Let \( \gamma_0 = \max\{a_n^+ - a_n(x^b), \sum_{(i,j) \in E} w^+_{ij} x^a_{ij} - \sum_{(i,j) \in E} w^+_{ij} x^b_{ij}\} \)
and \( x^0 = \arg \max\{A(x^a), A(x^b)\} \)

**Theorem 5.3.** The path \( x^0 \in P \) is a \( \gamma_0 \)-approximation of the problem (1).

**Proof.** For every path \( x \in P \),
\[
\sum_{(i,j) \in E} w^+_{ij} x_{ij} - a_n(x) \leq \sum_{(i,j) \in E} w^+_{ij} x^0_{ij} - a_n(x)
\]
\[
= -a_n(x^b) + \sum_{(i,j) \in E} [w^+_{ij} x^b_{ij} - a_n(x) + a_n(x^b)]
\]
\[
\leq -a_n(x^b) + \sum_{(i,j) \in E} [w^+_{ij} x^a_{ij} + a_n(x) - a_n(x^b)]
\]
\[
\leq \sum_{(i,j) \in E} [w^+_{ij} x^a_{ij} + \gamma_0 - a_n(x^0)] - a_n(x^0)
\]

Thus \( \gamma_0 \leq A(x) \leq \gamma_0 \).

It can be observed that if \( \sum_{(i,j) \in E} w^+_{ij} x^a_{ij} = \sum_{(i,j) \in E} w^+_{ij} x^b_{ij} \), then \( \gamma_0 = 0 \) implying that is an optimal solution of both objective functions and it is called as an ideal point. If \( \gamma_0 \) is a negative value, the \( \gamma_0 \) approximation can be improved by means of the solution of the sub problem given as
\[
\max a_n(x) \text{ ~---(5) ~---(5)}
\]

such that \( x \in P, \sum_{(i,j) \in E} w^+_{ij} x_{ij} < \sum_{(i,j) \in E} w^+_{ij} x^a_{ij} \).

On solving this problem, a better solution can be obtained for the problem (1). Let \( x' \) be an optimal solution of (5), where \( c = \sum_{(i,j) \in E} w^+_{ij} x^a_{ij} \) and if \( x^1 \) reaches the maximal objective value among \( \{x^a, x^b, x^c\} \) that is \( x^1 = \arg \max\{A(x^a), A(x^b), A(x^c)\} \).

Let \( \gamma_1 = \max\{a_n^+ - a_n(x^b), \sum_{(i,j) \in E} w^+_{ij} x^a_{ij} - \sum_{(i,j) \in E} w^+_{ij} x^b_{ij}\} \).

It can be found that \( x' \) is a feasible solution of (5), since \( c = \sum_{(i,j) \in E} w^+_{ij} x^a_{ij} < \sum_{(i,j) \in E} w^+_{ij} x^c_{ij} \), which implies that \( \gamma_1 \geq \gamma_0 \).

**Theorem 5.4.** The path \( x^1 \in P \) is a \( \gamma_1 \)-approximation of the problem (1).

**Proof.** If \( x \in P \) implies \( \sum_{(i,j) \in E} w^+_{ij} x^a_{ij} \geq \sum_{(i,j) \in E} w^+_{ij} x^b_{ij} \), then
\[
A(x) = \sum_{(i,j) \in E} w^+_{ij} x_{ij} - a_n(x) \geq \sum_{(i,j) \in E} w^+_{ij} x^a_{ij} - a_n(x^a)
\]
\[
= A(x^1) \leq A(x^a).
\]

On the other hand, \( c < \sum_{(i,j) \in E} w^+_{ij} x_{ij} \), hence \( a_n(x) \leq a_n(x^c) \).
\[
\sum_{(i,j) \in E} w^+_{ij} x_{ij} - a_n(x) \leq \sum_{(i,j) \in E} w^+_{ij} x^c_{ij} - \sum_{(i,j) \in E} w^+_{ij} x^b_{ij}
\]
\[
\leq A(x^c) + \sum_{(i,j) \in E} w^+_{ij} x^a_{ij} - \sum_{(i,j) \in E} w^+_{ij} x^b_{ij}
\]
\[
\leq A(x^1) + \sum_{(i,j) \in E} w^+_{ij} x^a_{ij} - \sum_{(i,j) \in E} w^+_{ij} x^b_{ij}
\]

Hence \( x^1 \in P \) is a \( \gamma_1 \)-approximation.

This is an iterative process to solve the problem P.

6 Conclusion

In this paper, we have proposed a new approach called maxmin advantage approach to find the critical path of a project network. Since, we consider the \( \gamma \)-cut of a fuzzy number, this method is suitable not only for trapezoidal but also for triangular fuzzy numbers. In addition, we have formulated the critical path problem as a linear programming problem, which paves the way to further research in finding critical path in project networks.

References


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