On degree of approximation of conjugate series of a Fourier series by product summability

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Abstract

In this paper a theorem on degree of approximation of a function \(f \in \text{Lip}(\alpha, r)\) by product summability \((E, q)(\bar{N}, p_n)\) of conjugate series of Fourier series associated with \(f\) has been proved.

Keywords: Degree of Approximation, \(\text{Lip}(\alpha, r)\) class of function, \((E, q)\) mean, \((\bar{N}, p_n)\) mean, \((E, q)(\bar{N}, p_n)\) product mean, Fourier series, conjugate of the Fourier series, Lebesgue integral.

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1 Introduction

Let \(\sum a_n\) be a given infinite series with the sequence of partial sums \(\{s_n\}\). Let \(\{p_n\}\) be a sequence of positive real numbers such that

\[ P_n = \sum_{\upsilon=0}^{n} p_\upsilon \longrightarrow \infty, \text{ as } n \longrightarrow \infty, (P_{-i} = p_{-i} = 0, i \geq 0). \tag{1.1} \]

The sequence-to-sequence transformation

\[ t_n = \frac{1}{P_n} \sum_{\upsilon=0}^{n} p_\upsilon s_\upsilon, \tag{1.2} \]

defines the sequence \(\{t_n\}\) of the \((\bar{N}, p_n)\)-mean of the sequence \(\{s_n\}\) generated by the sequence of coefficient \(\{p_n\}\). If

\[ t_n \longrightarrow s, \text{ as } n \longrightarrow \infty, \tag{1.3} \]

then the series \(\sum a_n\) is said to be \((\bar{N}, p_n)\) summable to \(s\).

The conditions for regularity of \((\bar{N}, p_n)\)-summability are easily seen to be [1]

\[ \begin{cases} (i) P_n \rightarrow \infty, \text{ as } n \rightarrow \infty, \\ (ii) \sum_{i=0}^{n} p_i \leq C |P_n|, \text{ as } n \rightarrow \infty. \end{cases} \tag{1.4} \]

The sequence-to-sequence transformation, [1]

\[ T_n = \frac{1}{(1 + q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon, \tag{1.5} \]

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defines the sequence \( \{T_n\} \) of the \((E, q)\) mean of the sequence \( \{s_n\} \). If
\[
T_n \to s, \text{ as } n \to \infty,
\]
then the series \( \sum a_n \) is said to be \((E, q)\) summable to \(s\). Clearly \((E, q)\) method is regular. Further, the \((E, q)\) transformation of the \((\bar{N}, p_n)\) transform of \( \{s_n\} \) is defined by
\[
\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{\infty} \binom{n}{k} q^{n-k} T_k
\]
with
\[
\sum_{k=0}^{\infty} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{p_v} s_v \right\}
\]
(1.7)
If
\[
\tau_n \to s, \text{ as } n \to \infty,
\]
then \( \sum a_n \) is said to be \((E, q)(\bar{N}, p_n)\)-summable to \(s\).

Let \( f(t) \) be a periodic function with period \( 2\pi \) and \( L\)-integrable over \((-\pi, \pi)\). The Fourier series associated with \( f \) at any point \( x \) is defined by
\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) = \sum_{n=0}^{\infty} A_n(x),
\]
(1.9)
and the conjugate series of the Fourier Series (1.9) is
\[
\sum_{n=1}^{\infty} \left( b_n \cos nx - a_n \sin nx \right) = \sum_{n=0}^{\infty} B_n(x).
\]
(1.10)
Let \( \bar{s}_n(f : x) \) be the \(n\)-th partial sum of (1.10). The \( L_\infty \)-norm of a function \( f : R \to R \) is defined by
\[
\|f\|_\infty = \sup\{|f(x)| : x \in R\}
\]
(1.11)
and the \( L_v \)-norm is defined by
\[
\|f\|_v = \left( \int_0^{2\pi} |f(x)|^v dx \right)^{\frac{1}{v}}, v \geq 1.
\]
(1.12)
The degree of approximation of a function \( f : R \to R \) by a trigonometric polynomial \( P_n(x) \) of degree \( n \) under norm \( \| \cdot \|_\infty \) is defined by [5]
\[
\| P_n - f \|_\infty = \sup\{|p_n(x) - f(x)| : x \in R\}
\]
(1.13)
and the degree of approximation \( E_n(f) \) a function \( f \in L_v \) is given by
\[
E_n(f) = \min_{P_n} \| P_n - f \|_v.
\]
(1.14)
A function \( f \) is said to satisfy Lipschitz condition (here after we write \( f \in \text{Lip} \alpha \)) if
\[
|f(x+1) - f(x)| = O(|t|^\alpha), 0 < \alpha \leq 1.
\]
(1.15)
and \( f(x) \in \text{Lip}(\alpha, r) \), for \( 0 \leq x \leq 2\pi \), if
\[
\left( \int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{\frac{1}{r}} = O(|t|^\alpha), 0 < \alpha \leq 1, r \geq 1, t > 0.
\]
(1.16)
For a given positive increasing function \( \xi(t) \), the function \( f(x) \in \text{Lip} (\xi(t), r) \), if
\[
\left( \int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{\frac{1}{r}} = O(\xi(t)), r \geq 1, t > 0.
\]
(1.17)
We use the following notation throughout this paper:
\[
\psi(t) = \frac{1}{2} (f(x+t) - f(x-t)),
\]
(1.18)
and
\[
\mathcal{K}_n(t) = \frac{1}{\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{p_v} \cos \frac{1}{2} - \cos \left( v + \frac{1}{2} \right) t \right\}
\]
Further, the method \((E, q)(\bar{N}, P_n)\) is assumed to be regular.
2 Known Theorems

Dealing with the degree of approximation by the product Misra et al. [2] proved the following theorem using \(E, q(N, p_n)\)-mean of Conjugate Series of Fourier series:

**Theorem 2.1.** If \(f\) is a 2\(\pi\)-periodic function of class \(\text{Lip}_\alpha\), then degree of approximation by the product \((E, q)(\bar{N}, p_n)\) summability mean of the conjugate series (1.10) of the Fourier Series (1.9) is given by

\[
\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^{\alpha}}\right),
\]

where \(\tau_n\) is as defined in (1.7).

Very recently Paikray et al. [3] established a theorem on degree of approximation by the product mean \((E, q)(\bar{N}, p_n)\) of the Conjugate Series of Fourier Series of a function of class \(\text{Lip}_\alpha, r\). They proved:

**Theorem 2.2.** If \(f\) is a 2\(\pi\)-Periodic function of class \(\text{Lip}_\alpha, r\), then degree of approximation by the product \((E, q)(\bar{N}, p_n)\) summability mean on the Conjugate Series (1.10) of the Fourier series (1.9) is given by

\[
\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^{\alpha}}\right),
\]

where \(\tau_n\) is as defined in (1.7).

3 Main Theorem

In this paper, we have proved a theorem on degree of approximation by the product mean \((E, q)(\bar{N}, p_n)\) of the conjugate series of the Fourier series of a function of class \(\text{Lip}(\xi(t), r)\). We prove:

**Theorem 3.3.** Let \(\xi(t)\) be a positive increasing function and \(f\) a 2\(\pi\)-periodic function of the class \(\text{Lip}(\xi(t), r), r \geq 1, t > 0\). Then degree of approximation by the product \((E, q)(\bar{N}, p_n)\) summability means on the Conjugate Series (1.10) of the Fourier series (1.9) is given by

\[
\|\tau_n - f\|_\infty = O\left(\left(\frac{1}{(n+1)^{\frac{1}{2}}}\right)\right),
\]

where \(\tau_n\) is as defined in (1.7).

4 Required Lemmas

We require the following Lemmas to prove the theorem.

**Lemma 4.1.**

\[
|\hat{K}_n(t)| = O(n), \quad 0 \leq t \leq \frac{1}{n+1}.
\]

**Proof.** For \(0 \leq t \leq \frac{1}{n+1}\), we have \(\sin nt \leq n \sin t\) then

\[
|\hat{K}_n(t)| = \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \cos \frac{t}{2} - \cos \left( v + \frac{1}{2} \right) \frac{t}{2} \sin \frac{t}{2} \right\} \right|
\]

\[
\leq \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \cos \frac{t}{2} - \cos \left( \frac{2\sin^2 \frac{v}{2} + \sin vt \cdot \sin \frac{t}{2} \right) \sin \frac{t}{2} \right\} \right|
\]

\[
\leq \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left( \cos \frac{t}{2} \left( 2\sin^2 \frac{vt}{2} + \sin vt \right) + \sin vt \right) \right\} \right|
\]

\[
\leq \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \left( O(\sin vt) + O(\sin vt) \right) \right\} \right|
\]

\[
= O(n).
\]
This proves the lemma. □

**Lemma 4.2.**

\[ |\tilde{K}_n(t)| = O\left(\frac{1}{t}\right), \text{for } \frac{1}{n+1} \leq t \leq \pi. \]

**Proof.** For \( \frac{1}{n+1} \leq t \leq \pi \), by Jordan’s lemma, we have \( \sin\left(\frac{t}{n}\right) \geq \frac{t}{n} \). Then

\[
|\tilde{K}_n(t)| = \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{p=0}^{k} p \cos \frac{k}{2} - \cos \left(\frac{v + \frac{1}{2}}{2}\right) \right\} \right|
\leq \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{p=0}^{k} p \cos \frac{k}{2} - \cos \frac{v}{2} \cdot \cos \frac{k}{2} \right\} \right|
\leq \frac{\pi}{2\pi(1+q)^n t} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{p=0}^{k} p \right\} \right|
= \frac{1}{2(1+q)^n t} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \right|
= O\left(\frac{1}{t}\right).
\]

This proves the lemma. □

**5 Proof of Theorem 3.1**

Using Riemann-Lebesgue theorem, we have for the \( n \)-th partial sum \( \tilde{s}_n(f : x) \) of the conjugate Fourier series (1.10) of \( f(x) \), following Titchmarsh [4]

\[
\tilde{s}_n(f : x) - f(x) = \frac{2}{\pi} \int_{0}^{\pi} \psi(t) K_n dt,
\]

the \((N, p_n)\) transform of \( \tilde{s}_n(f : x) \) using (1.2) is given by

\[
t_n - f(x) = \frac{2}{\pi P_n} \int_{0}^{\pi} \psi(t) \sum_{k=0}^{n} p_k \cos \frac{n}{2} - \sin \left(\frac{n + \frac{1}{2}}{2}\right) t \frac{dt}{2 \sin \left(\frac{t}{2}\right)},
\]

denoting the \((E, q)(N, p_n)\) transform of \( \tilde{s}_n(f : x) \) by \( \tau_n \), we have

\[
\|\tau_n - f\| = \frac{1}{\pi(1+q)^n} \int_{0}^{\pi} \psi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{p=0}^{k} p \cos \frac{k}{2} - \sin \left(\frac{v + \frac{1}{2}}{2}\right) t \right\} dt
\leq \int_{0}^{\pi} \psi(t) K_n(t) dt
= \left\{ \int_{0}^{\pi \frac{1}{n+1}} + \int_{\pi \frac{1}{n+1}}^{\pi} \right\} \psi(t) K_n(t) dt
= I_1 + I_2, \text{ say.}
\]

Now
\[ |I_1| = \frac{2}{\pi (1+q)^n} \left| \int_0^{1/n} \psi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} p_v \cos \left( \frac{v}{2} - \cos \left( \frac{v+1}{2} \right) t \right) \right\} dt \right| \]

\[ = \left| \int_0^{1/n} \psi(t) \tilde{K}_n(t) dt \right| \]

\[ = \left( \int_0^{1/n} \left( \frac{\psi(t)}{\xi(t)} \right)^{\frac{r}{2}} dt \right)^{\frac{1}{r}} \left( \int_0^{1/n} (\xi(t)\tilde{K}_n(t))^s dt \right)^{\frac{1}{s}}, \text{ using Holder's inequality} \]

\[ = O(1) \left( \int_0^{1/n} \xi(t)^{n\ast} dt \right)^{\frac{1}{2}} \]

\[ = O\left( \xi \left( \frac{1}{n+1} \right) \left( \frac{n\ast}{n+1} \right)^{\frac{1}{2}} \right) \]

\[ = O\left( \xi \left( \frac{1}{n+1} \right) \frac{1}{(n+1)^{\frac{1}{2} - 1}} \right) \]

\[ = O\left( \xi \left( \frac{1}{n+1} \right) \frac{1}{(n+1)^{\frac{1}{2}}} \right) \]

\[ = O\left( (n+1)^{\frac{1}{2}} \xi \left( \frac{1}{n+1} \right) \right). \quad (5.2) \]

Next

\[ |I_2| \leq \left( \int_0^{\pi} \left( \frac{\phi(t)}{\xi(t)} \right)^{\frac{r}{2}} dt \right)^{\frac{1}{r}} \left( \int_0^{\pi} (\xi(t)\tilde{K}_n(t))^s dt \right)^{\frac{1}{s}}, \text{ using Holder's inequality} \]

\[ = O(1) \left( \int_0^{\pi} \left( \frac{\xi(t)}{t} \right)^{s} dt \right)^{\frac{1}{s}}, \text{ using Lemma 4.1} \]

\[ = O(1) \left( \int_0^{n+1} \left( \frac{\xi \left( \frac{1}{y} \right)}{\frac{1}{y}} \right)^s dy \right)^{\frac{1}{s}}. \quad (5.3) \]

Since \( \xi(t) \) is a positive increasing function, so is \( \xi(1/y)/(1/y) \). Using second mean value theorem we get

\[ = O\left( (n+1) \xi \left( \frac{1}{n+1} \right) \left( \int_0^{n+1} dy \right)^{\frac{1}{2}} \right), \text{ for some } \frac{1}{\pi} \leq \delta \leq n+1 \]

\[ = O\left( (n+1)^{\frac{1}{2}} \xi \left( \frac{1}{n+1} \right) \right) \]

Then from (5.2) and (5.3), we have

\[ |\tau_n - f(x)| = O\left( (n+1)^{\frac{1}{2}} \xi \left( \frac{1}{n+1} \right) \right), \text{ for } r \geq 1. \]

\[ \| \tau_n - f(x) \|_{\infty} = \sup_{-\pi < x < \pi} |\tau_n - f(x)| = O\left( (n+1)^{\frac{1}{2}} \xi \left( \frac{1}{n+1} \right) \right), r \geq 1. \]

This completes the proof of the theorem.

**References**


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