On a class of fractional q-Integral inequalities

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Abstract

In the present paper, we use the fractional q-calculus to generate some new integral inequalities for some monotonic functions. Other fractional q-integral results, using convex functions, are also presented.

Keywords: Convex function, fractional q-calculus, q-Integral inequalities.

2010 MSC: 26D15.

1 Introduction

The study of the q-integral inequalities play a fundamental role in the theory of differential equations. We refer the reader to [3, 8, 9, 14] for further information and applications. To motivate our work, we shall introduce some important results. The first one is given in [13], where Ngo et al. proved that for any positive continuous function \( f \) on \([0, 1]\) satisfying \( \int_0^1 f(\tau) d\tau \geq \int_0^1 \tau d\tau \), and for \( \delta > 0 \), the inequalities

\[ \int_0^1 f^{\delta+1}(\tau) d\tau \geq \int_0^1 \tau^\delta f(\tau) d\tau \]  
(1.1)

and

\[ \int_0^1 f^{\delta+1}(\tau) d\tau \geq \int_0^1 \tau f^{\delta}(\tau) d\tau \]  
(1.2)

are valid.

In [11], W.J. Liu, G.S. Cheng and C.C. Li proved that

\[ \int_a^b f^{\alpha+\beta}(\tau) d\tau \geq \int_a^b (\tau - a)\alpha f^{\beta}(\tau) d\tau, \]
(1.3)

for any \( \alpha > 0, \beta > 0 \) and for any positive continuous function \( f \) on \([a, b]\), such that

\[ \int_x^b f^\gamma(\tau) d\tau \geq \int_x^b (\tau - a)^\gamma d\tau; \ \gamma := \min(1, \beta), x \in [a, b]. \]

Recently, Liu et al. [12] proved another interesting form of integral result, and the following inequality

\[ \frac{\int_a^b f^{\beta}(\tau) d\tau}{\int_a^b f^{\gamma}(\tau) d\tau} \geq \frac{\int_a^b (\tau - a)^\delta f^{\beta}(\tau) d\tau}{\int_a^b (\tau - a)^\delta f^{\gamma}(\tau) d\tau}, \beta \geq \gamma > 0, \delta > 0 \]
(1.4)
(where \( f \) is a positive continuous and decreasing function on \([a, b]\)), was proved in this paper. Several interesting inequalities can be found in [12].

Many researchers have given considerable attention to (1), (3) and (4) and a number of extensions and generalizations appeared in the literature (e.g. [4, 5, 6, 7, 10, 11, 15, 16]).

The main purpose of this paper is to establish some new fractional q-integral inequalities on the specific time scales \( T_{t_0} = \{ t : t = t_0 q^n, n \in \mathbb{N} \} \cup \{ 0 \} \), where \( t_0 \in \mathbb{R} \), and \( 0 < q < 1 \). Other fractional q-integral results, involving convex functions, are also presented. Our results have some relationships with those obtained in [12].

2 Notations and Preliminaries

In this section, we provide a summary of the mathematical notations and definitions used in this paper. For more details, one can consult [1, 2].

Let \( t_0 \in \mathbb{R} \). We define

\[
T_{t_0} := \{ t : t = t_0 q^n, n \in \mathbb{N} \} \cup \{ 0 \}, 0 < q < 1.
\] (2.5)

For a function \( f : T_{t_0} \to \mathbb{R} \), the \( \nabla_q \) derivative of \( f \) is:

\[
\nabla_q f(t) = \frac{f(qt) - f(t)}{(q - 1)t}
\] (2.6)

for all \( t \in T \setminus \{ 0 \} \) and its \( \nabla_q \) integral is defined by:

\[
\int_0^t f(\tau) \nabla_q \tau = (1 - q)t \sum_{i=0}^{\infty} q^i f(tq^i)
\] (2.7)

The fundamental theorem of calculus applies to the \( \nabla_q \) derivative and \( \nabla_q \) integral. In particular, we have:

\[
\nabla_q \int_0^t f(\tau) \nabla_q \tau = f(t).
\] (2.8)

If \( f \) is continuous at 0, then

\[
\int_0^t \nabla_q f(\tau) \nabla_q \tau = f(t) - f(0).
\] (2.9)

Let \( T_{t_1}, T_{t_2} \) denote two time scales. Let \( f : T_{t_1} \to \mathbb{R} \) be continuous let \( g : T_{t_1} \to T_{t_2} \) be \( q \)-differentiable, strictly increasing, and \( g(0) = 0 \). Then for \( b \in T_{t_1} \), we have:

\[
\int_0^b f(t) \nabla_q g(t) \nabla t = \int_0^{g(b)} (f \circ g^{-1})(s) \nabla s.
\] (2.10)

The \( q \)-factorial function is defined as follows:

If \( n \) is a positive integer, then

\[
(t - s)^{(n)} = (t - s)(t - qs)(t - q^2s)...(t - q^{n-1}s).
\] (2.11)

If \( n \) is not a positive integer, then

\[
(t - s)^{(n)} = t^n \prod_{k=0}^{\infty} \frac{1 - (\frac{s}{t})q^k}{1 - (\frac{s}{t})q^{n+k}}.
\] (2.12)

The \( \nabla_q \) derivative of the \( q \)-factorial function with respect to \( t \) is

\[
\nabla_q(t - s)^{(n)} = \frac{1 - q^n}{1 - q} (t - s)^{(n-1)},
\] (2.13)
and the $q$-derivative of the $q$-factorial function with respect to $s$ is
\[
\nabla_q (t-s)^{(n)} = -\frac{1-q^n}{1-q}(t-qs)^{(n-1)}.
\]
(2.14)

The $q$-exponential function is defined as
\[
e_q(t) = \prod_{k=0}^{\infty} (1 - q^k t), e_q(0) = 1
\]
(2.15)

The fractional $q$-integral operator of order $\alpha \geq 0$, for a function $f$ is defined as
\[
\nabla_q^{-\alpha} f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - q\tau)^{\alpha-1} f(\tau) \nabla \tau; \quad \alpha > 0, t > 0,
\]
(2.16)

where $\Gamma_q(\alpha) := \frac{1}{1-q} \int_0^1 (\frac{u}{1-q})^{\alpha-1} e_q(qu) \nabla u$.

## 3 Main Results

**Theorem 3.1.** Let $f$ and $g$ be two positive and continuous functions on $T_{t_0}$ such that $f$ is decreasing and $g$ is increasing on $T_{t_0}$. Then for all $\alpha > 0$, $\beta \geq \gamma > 0$, $\delta > 0$, we have
\[
\frac{\nabla_q^{-\alpha} f^\beta(t)}{\nabla_q^{-\alpha} f^\gamma(t)} \leq \frac{\nabla_q^{-\alpha} g^\delta f^\beta(t)}{\nabla_q^{-\alpha} g^\delta f^\gamma(t)}, \quad t > 0.
\]
(3.17)

**Proof.** Let us consider
\[
H(\tau, \rho) := \left( g^\delta(\rho) - g^\delta(\tau) \right) \left( f^\beta(\tau)f^\gamma(\rho) - f^\gamma(\tau)f^\beta(\rho) \right), \tau, \rho \in (0, t), t > 0.
\]
(3.18)

We have
\[
H(\tau, \rho) \geq 0.
\]
(3.19)

Hence, we get
\[
\int_0^t \frac{(t - q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)} H(\tau, \rho) \nabla \tau = g^\delta(\rho)f^\gamma(\rho)\nabla_q^{-\alpha}[f^\beta(t)] + f^\beta(\rho)\nabla_q^{-\alpha}[g^\delta(t)f^\gamma(t)]
\]
\[\quad - f^\gamma(\rho)\nabla_q^{-\alpha}[g^\delta t f^\beta(t)] - g^\delta(\rho)f^\beta(\rho)\nabla_q^{-\alpha}[f^\gamma(t)] \geq 0.
\]
(3.20)

Consequently,
\[
2^{-1} \int_0^t \int_0^t \frac{(t - q\rho)^{(\alpha-1)}(t - q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)^2} H(\tau, \rho) \nabla \tau \nabla \rho = \nabla_q^{-\alpha}[f^\beta(t)]\nabla_q^{-\alpha}[g^\delta(t)f^\gamma(t)]
\]
\[\quad - \nabla_q^{-\alpha}[f^\gamma(t)]\nabla_q^{-\alpha}[g^\delta(t)f^\beta(t)] \geq 0.
\]
(3.21)

Theorem 3.1 is thus proved.

Another result which generalizes Theorem 3.1 is described in the following theorem:

**Theorem 3.2.** Suppose that $f$ and $g$ are two positive and continuous functions on $T_{t_0}$, such that $f$ is decreasing and $g$ is increasing on $T_{t_0}$. Then for all $\alpha > 0$, $\omega > 0$, $\beta \geq \gamma > 0$, $\delta > 0$, we have
\[
\frac{\nabla_q^{-\alpha}[f^\beta(t)]\nabla_q^{-\omega}[g^\delta f^\gamma(t)]}{\nabla_q^{-\alpha}[f^\gamma(t)]\nabla_q^{-\omega}[g^\delta f^\beta(t)]} \geq 1; \quad t > 0.
\]
(3.22)

**Proof.** The relation (3.20) allows us to obtain
\[
\int_0^t \int_0^t \frac{(t - q\rho)^{(\alpha-1)}(t - q\tau)^{(\alpha-1)}}{\Gamma_q(\omega)\Gamma_q(\alpha)} H(\tau, \rho) \nabla \tau \nabla \rho = \nabla_q^{-\alpha}[f^\beta(t)]\nabla_q^{-\omega}[g^\delta f^\gamma(t)]
\]
\[\quad + \nabla_q^{-\omega}[f^\beta(t)]\nabla_q^{-\alpha}[g^\delta f^\gamma(t)] - \nabla_q^{-\alpha}[f^\gamma(t)]\nabla_q^{-\omega}[g^\delta f^\beta(t)] - \nabla_q^{-\omega}[f^\gamma(t)]\nabla_q^{-\alpha}[g^\delta f^\beta(t)] \geq 0,
\]
for any $\omega > 0$.

Hence, we have (3.22).
Remark 3.1. It is clear that Theorem 3.1 would follow as a special case of Theorem 3.2 for $\alpha = \omega$.

The third result is given by the following theorem:

**Theorem 3.3.** Let $f$ and $g$ be two positive continuous functions on $T_{0a}$, such that
\[
\left(f^\delta(\tau)g^\delta(\rho) - f^\delta(\rho)g^\delta(\tau)\right)\left(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\right) \geq 0; \tau, \rho \in (0, t), t > 0.
\] (3.24)

Then we have
\[
\frac{\nabla_q^{-\alpha}[f^{\delta+\beta}(t)]}{\nabla_q^{-\alpha}[f^{\delta+\gamma}(t)]} \geq \frac{\nabla_q^{-\alpha}[g^{\delta}(f^\delta(t))]}{\nabla_q^{-\alpha}[g^{\delta}(f^\gamma(t))],}
\] (3.25)

for any $\alpha > 0, \beta \geq \gamma > 0, \delta > 0$.

**Proof.** We consider the quantity:
\[
K(\tau, \rho) := \left(f^\delta(\tau)g^\delta(\rho) - f^\delta(\rho)g^\delta(\tau)\right)\left(f^\gamma(\rho)f^{\beta}(\tau) - f^\gamma(\tau)f^{\beta}(\rho)\right); \tau, \rho \in (0, t), t > 0
\]
and we use the same arguments as in the proof of Theorem 3.1.

Using two fractional parameters, we obtain the following generalization of Theorem 3.3:

**Theorem 3.4.** Let $f$ and $g$ be two positive continuous functions on $T_{0a}$, such that
\[
\left(f^\delta(\tau)g^\delta(\rho) - f^\delta(\rho)g^\delta(\tau)\right)\left(f^{\beta-\gamma}(\tau) - f^{\beta-\gamma}(\rho)\right) \geq 0; \tau, \rho \in (0, t), t > 0.
\] (3.26)

Then for all $\alpha > 0, \omega > 0, \beta \geq \gamma > 0, \delta > 0$, we have
\[
\frac{\nabla_q^{-\alpha}[f^{\delta+\beta}(t)]}{\nabla_q^{-\alpha}[f^{\delta+\gamma}(t)]} + \frac{\nabla_q^{-\omega}[f^{\delta+\beta}(t)]}{\nabla_q^{-\omega}[f^{\delta+\gamma}(t)]} \geq 1.
\] (3.27)

**Remark 3.2.** Applying Theorem 3.4, for $\alpha = \omega$, we obtain Theorem 3.3.

Involving convex functions, we have the following result:

**Theorem 3.5.** Let $f$ and $h$ be two positive continuous functions on $T_{0a}$ and $f \leq h$ on $T_{0a}$. If $\frac{f}{h}$ is decreasing and $f$ is increasing on $[0, \infty]$, then for any convex function $\phi; \phi(0) = 0$, the inequality
\[
\frac{\nabla_q^{-\alpha}(f(t))}{\nabla_q^{-\alpha}(h(t))} \geq \frac{\nabla_q^{-\alpha}(\phi(f(t)))}{\nabla_q^{-\alpha}(\phi(h(t)))}, t > 0, \alpha > 0
\] (3.28)
is valid.

**Proof.** Using the fact that on $T_{0a}$, $\frac{\phi(f(t))}{f(t)}$ is an increasing function and $\frac{f}{h}$ is a decreasing function, we can write
\[
L(\tau, \rho) \geq 0, \tau, \rho \in (0, t), t > 0,
\] (3.29)
where
\[
L(\tau, \rho) := \frac{\phi(f(\tau))}{f(\tau)}f(\rho)h(\tau) + \frac{\phi(f(\rho))}{f(\rho)}f(\tau)h(\rho)
\] (3.30)
\[
-\frac{\phi(f(\rho))}{f(\rho)}f(\rho)h(\tau) - \frac{\phi(f(\tau))}{f(\tau)}f(\tau)h(\rho), \tau, \rho \in (0, t), t > 0.
\]

Multiplying both sides of (3.29) by $\frac{(\tau^q_{\alpha})^\alpha}{\Gamma_q(\alpha)}$, then integrating the resulting inequality with respect to $\tau$ over $(0, t)$, yields
\[
f(\rho)\nabla_q^{-\alpha}\left[\frac{\phi(f(t))}{f(t)}h(t)\right] + \frac{\phi(f(\rho))}{f(\rho)}h(\rho)\nabla_q^{-\alpha}f(t)
\] (3.31)
\[
-\frac{\phi(f(\rho))}{f(\rho)}f(\rho)\nabla_q^{-\alpha}h(t) - h(\rho)\nabla_q^{-\alpha}\left[\frac{\phi(f(t))}{f(t)}h(t)\right] \geq 0.
\]
With the same arguments as before, we obtain

\[ \nabla_q^{-\alpha} f(t) \left[ \frac{\phi(f(t))}{f(t)} h(t) \right] - \nabla_q^{-\alpha} h(t) \nabla_q^{-\alpha} \left[ \frac{\phi(f(t))}{f(t)} f(t) \right] \geq 0. \quad (3.32) \]

On the other hand, we have

\[ \frac{\phi(f(\tau))}{f(\tau)} \leq \frac{\phi(h(\tau))}{h(\tau)}, \quad \tau \in (0, t), \quad t > 0. \quad (3.33) \]

Therefore,

\[ \frac{(t - q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(\tau) \frac{\phi(f(\tau))}{f(\tau)} \leq \frac{(t - q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(\tau) \frac{\phi(h(\tau))}{h(\tau)}, \quad \tau \in (0, t), \quad t > 0. \quad (3.34) \]

The inequality (3.34) implies that

\[ \nabla_q^{-\alpha} \left[ \frac{\phi(f(t))}{f(t)} h(t) \right] \leq \nabla_q^{-\alpha} \left[ \frac{\phi(h(t))}{h(t)} h(t) \right]. \quad (3.35) \]

Combining (3.32) and (3.35), we obtain (3.28).

To finish, we present to the reader the following result which generalizes the previous theorem:

**Theorem 3.6.** Let \( f \) and \( h \) be two positive continuous functions on on \( T_{1_{0}} \) and \( f \leq h \) on \( T_{1_{0}} \). If \( \frac{f}{h} \) is decreasing and \( f \) is increasing on \( T_{1_{0}} \), then for any convex function \( \phi; \phi(0) = 0 \), we have

\[
\frac{\nabla_q^{-\omega} (f(t)) \nabla_q^{-\omega} (\phi(h(t))) + \nabla_q^{-\omega} (f(t)) \nabla_q^{-\omega} (\phi(h(t)))}{\nabla_q^{-\omega} (h(t)) \nabla_q^{-\omega} (\phi(f(t))) + \nabla_q^{-\omega} (h(t)) \nabla_q^{-\omega} (\phi(f(t)))} \geq 1, \quad \alpha > 0, \quad \omega > 0, \quad t > 0. \quad (3.36)
\]

**Proof.** The relation (3.31) allows us to obtain

\[
\nabla_q^{-\omega} f(t) \omega \left[ \frac{\phi(f(t))}{f(t)} h(t) \right] + \nabla_q^{-\omega} \left[ \frac{\phi(f(t))}{f(t)} h(t) \right] \nabla_q^{-\alpha} f(t) \quad (3.37)
\]

\[- \nabla_q^{-\omega} \left[ \frac{\phi(f(t))}{f(t)} f(t) \right] \nabla_q^{-\omega} h(t) - \nabla_q^{-\omega} h(t) \nabla_q^{-\omega} \left[ \frac{\phi(f(t))}{f(t)} f(t) \right] \geq 0.
\]

On the other hand, we have:

\[
\frac{(t - q\tau)^{(\omega-1)}}{\Gamma_q(\omega)} h(\tau) \frac{\phi(f(\tau))}{f(\tau)} \leq \frac{(t - q\tau)^{(\omega-1)}}{\Gamma_q(\omega)} h(\tau) \frac{\phi(h(\tau))}{h(\tau)}, \quad \tau \in [0, t], \quad t > 0. \quad (3.38)
\]

Integrating both sides of (3.38) with respect to \( \tau \) over \( (0, t) \), yields

\[
\nabla_q^{-\omega} \left[ \frac{\phi(f(t))}{f(t)} h(t) \right] \leq \nabla_q^{-\omega} \left[ \frac{\phi(h(t))}{h(t)} h(t) \right]. \quad (3.39)
\]

By (3.35), (3.37) and (3.39), we get (3.36).

**Remark 3.3.** Applying Theorem (3.6), for \( \alpha = \omega \), we obtain Theorem (3.5).

**References**


Received: March 2, 2013; Accepted: April 17, 2013

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