A Variant of Jensen’s Inequalities

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Abstract

In this paper, we give an estimate from below and from above of a variant of Jensen’s Inequalities for convex functions in the discrete and continuous cases.

Keywords: Convex functions, Jensen inequalities, Integral inequalities.

2000 MSC: 52A40, 52A41.

1 Introduction and main results

Throughout this note, we write \(I\) and \(\bar{I}\) for the intervals \([a, b]\) and \((a, b)\) respectively. A function \(f\) is said to be convex on \(I\) if \(\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y)\) for all \(x, y \in I\) and \(0 \leq \lambda \leq 1\). Conversely, if the inequality always holds in the opposite direction, the function is said to be concave on the interval. A function \(f\) that is continuous function on \(I\) and twice differentiable on \(\bar{I}\) is convex on \(I\) if \(f''(x) \geq 0\) for all \(x \in \bar{I}\) (concave if the inequality is flipped).

Let \(x_1 \leq x_2 \leq \ldots \leq x_n\) be real numbers and \(\lambda_k \ (1 \leq k \leq n)\) be positive weights associated with \(x_k\) and whose sum is unity. Then the famous Jensen’s discrete and continuous inequalities \cite{2} read:

\begin{align}
\textbf{Theorem A.} \hspace{1cm} & \text{If } \varphi \text{ is a convex function on an interval containing the } x_k, \text{ then } \\
& \varphi \left( \sum_{k=1}^{n} \lambda_k x_k \right) \leq \sum_{k=1}^{n} \lambda_k \varphi(x_k). \hspace{1cm} (1.1)
\end{align}

\begin{align}
\textbf{Theorem B.} \hspace{1cm} & \text{Let } \varphi \text{ be a convex function on } I \subset \mathbb{R}, \text{ let } f : [c, d] \rightarrow I \text{ and } p : [c, d] \rightarrow (0, +\infty) \text{ be continuous functions on } [c, d]. \text{ Then } \\
& \varphi \left( \frac{\int_{c}^{d} p(x) f(x) \, dx}{\int_{c}^{d} p(x) \, dx} \right) \leq \frac{\int_{c}^{d} p(x) \varphi(f(x)) \, dx}{\int_{c}^{d} p(x) \, dx}. \hspace{1cm} (1.2)
\end{align}

If \(\varphi\) is strictly convex, then inequality in (1.2) is strict.

In \cite{3}, S. M. Malamud gave some complements to the Jensen and Chebyshev inequalities and in \cite{1}, I. Budimir, S. S. Dragomir, J. E. Pečarić obtained some results which counterpart Jensen’s discret inequality. Recently, A. McD. Mercer \cite{4} studied a variant of the inequality (1.1) and have obtained:

\begin{align}
\textbf{Theorem C.} \hspace{1cm} & \text{If } \varphi \text{ is a convex function on an interval of positive real numbers containing the } x_k, \text{ then } \\
& \varphi \left( x_1 + x_n - \sum_{k=1}^{n} \lambda_k x_k \right) \leq \varphi(x_1) + \varphi(x_n) - \sum_{k=1}^{n} \lambda_k \varphi(x_k). \hspace{1cm} (1.3)
\end{align}

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Our purpose in this paper is to give an estimate, from below and from above, of a variant of Jensen’s discrete and continuous cases inequalities for convex functions. We obtain the following results:

**Theorem 1.1.** Assume that \( \varphi \) is a convex function on \( I \) containing the \( x_k \) and \( \lambda_k \) (\( 1 \leq k \leq n \)) are positive weights associated with \( x_k \) and whose sum is unity. Then

\[
2\varphi \left( \frac{a + b}{2} \right) - \sum_{k=1}^{n} \lambda_k \varphi \left( x_k \right) \leq \varphi \left( a + b - \sum_{k=1}^{n} \lambda_k x_k \right) \\
\leq \varphi \left( a \right) + \varphi \left( b \right) - \sum_{k=1}^{n} \lambda_k \varphi \left( x_k \right). 
\]  

(1.4)

If \( \varphi \) is strictly convex, then inequalities in (1.4) are strict.

**Remark 1.1.** If \([a, b] = [x_1, x_n]\), then the result of Theorem C is given by the right-hand of inequalities (1.4).

**Theorem 1.2.** Let \( \varphi \) be a convex function on \( I \subset \mathbb{R} \), let \( f : [c, d] \rightarrow I \) and \( p : [c, d] \rightarrow (0, +\infty) \) be continuous functions on \([c, d]\). Then

\[
2\varphi \left( \frac{a + b}{2} \right) - \frac{\int_{c}^{d} p(x) \varphi \left( f \left( x \right) \right) dx}{\int_{c}^{d} p(x) dx} \leq \varphi \left( a + b - \int_{c}^{d} p(x) f \left( x \right) dx \right) \\
\leq \varphi \left( a \right) + \varphi \left( b \right) - \frac{\int_{c}^{d} p(x) \varphi \left( f \left( x \right) \right) dx}{\int_{c}^{d} p(x) dx}. 
\]  

(1.5)

If \( \varphi \) is strictly convex, then inequalities in (1.5) are strict.

**Corollary 1.1.** Under the hypotheses of Theorem 1.1, we have

\[
\left| \varphi \left( a + b - \sum_{k=1}^{n} \lambda_k x_k \right) + \sum_{k=1}^{n} \lambda_k \varphi \left( x_k \right) \right| \\
\leq \max \left\{ 2 \left| \varphi \left( \frac{a + b}{2} \right) \right|, \left| \varphi \left( a \right) + \varphi \left( b \right) \right| \right\}. 
\]  

(1.6)

**Corollary 1.2.** Under the hypotheses of Theorem 1.2, we have

\[
\left| \varphi \left( a + b - \int_{c}^{d} p(x) f \left( x \right) dx \right) + \int_{c}^{d} p(x) \varphi \left( f \left( x \right) \right) dx \right| \\
\leq \max \left\{ 2 \left| \varphi \left( \frac{a + b}{2} \right) \right|, \left| \varphi \left( a \right) + \varphi \left( b \right) \right| \right\}. 
\]  

(1.7)

In [5], S. Simić have obtained an upper global bound without a differentiability restriction on \( f \). Namely, he proved the following:

**Theorem D.** If \( \varphi \) is a convex function on \( I \) containing the \( x_k \) and \( \lambda_k \) (\( 1 \leq k \leq n \)) are positive weights associated with \( x_k \) and whose sum is unity, then

\[
\sum_{k=1}^{n} \lambda_k \varphi \left( x_k \right) - \varphi \left( \sum_{k=1}^{n} \lambda_k x_k \right) \leq \varphi \left( a \right) + \varphi \left( b \right) - 2\varphi \left( \frac{a + b}{2} \right). 
\]  

(1.8)

In the following, we improve this result by proving:
Theorem 1.3. If $\varphi$ is a convex function on $I$ containing the $x_k$ and $\lambda_k$ ($1 \leq k \leq n$) are positive weights associated with $x_k$ and whose sum is unity, then

$$0 \leq \sum_{k=1}^{n} \lambda_{\sigma(k)} \varphi(a + b - x_k) - \varphi\left( a + b - \sum_{k=1}^{n} \lambda_k x_k \right)$$

$$+ \sum_{k=1}^{n} \lambda_{\sigma(k)} \varphi(x_k) - \varphi\left( \sum_{k=1}^{n} \lambda_k x_k \right) \leq \varphi(a) + \varphi(b) - 2\varphi\left( \frac{a + b}{2} \right)$$

holds for all permutation $\sigma(k)$ of $\{1, 2, ..., n\}$.

Theorem 1.4. Let $\varphi$ be a convex function on $I : [c, d] \to I$ and $p : [c, d] \to (0, +\infty)$ be continuous functions on $[c, d]$. Then

$$0 \leq \int_{c}^{d} p(x) \varphi(f(x)) \, dx - \varphi\left( \int_{c}^{d} p(x) f(x) \, dx \right)$$

$$\leq \int_{c}^{d} p(x) \varphi(a + b - f(x)) \, dx - \varphi\left( a + b - \int_{c}^{d} p(x) f(x) \, dx \right)$$

$$+ \int_{c}^{d} p(x) \varphi(f(x)) \, dx - \varphi\left( \int_{c}^{d} p(x) f(x) \, dx \right) \leq \varphi(a) + \varphi(b) - 2\varphi\left( \frac{a + b}{2} \right).$$

Corollary 1.3. If $\varphi$ is a convex function on $I : [0, 1] \to I$ is a continuous function on $[0, 1]$, then

$$0 \leq \int_{0}^{1} \varphi(f(x)) \, dx - \varphi\left( \int_{0}^{1} f(x) \, dx \right)$$

$$\leq \varphi\left( a + b - \int_{0}^{1} f(x) \, dx \right) - \int_{0}^{1} \varphi(a + b - f(x)) \, dx + \int_{0}^{1} \varphi(f(x)) \, dx$$

$$- \varphi\left( \int_{0}^{1} f(x) \, dx \right) \leq \varphi(a) + \varphi(b) - 2\varphi\left( \frac{a + b}{2} \right).$$

Corollary 1.4. If $\varphi$ is a convex function on $I$ containing the $x_k$ and $\lambda_k$ ($1 \leq k \leq n$) are positive weights associated with $x_k$ and whose sum is unity, then

$$0 \leq \sum_{k=1}^{n} \lambda_k \varphi(x_k) - \varphi\left( \sum_{k=1}^{n} \lambda_k x_k \right)$$

$$\leq \sum_{k=1}^{n} \lambda_k \varphi(a + b - x_k) - \varphi\left( a + b - \sum_{k=1}^{n} \lambda_k x_k \right) + \sum_{k=1}^{n} \lambda_k \varphi(x_k) - \varphi\left( \sum_{k=1}^{n} \lambda_k x_k \right)$$

$$\leq \varphi(a) + \varphi(b) - 2\varphi\left( \frac{a + b}{2} \right).$$

Remark 1.2. If $\varphi$ is a concave function, then the above inequalities are opposite.

2 Lemma

Towards proving these theorems we shall need the following lemma.

Lemma 2.1. Let $\varphi$ be convex function on $I = [a, b]$. Then, we have

$$2\varphi\left( \frac{a + b}{2} \right) \leq \varphi(a + b - x) + \varphi(x) \leq \varphi(a) + \varphi(b).$$
Proof. Let \( \varphi \) be a convex function on \( I \). Then, we have

\[
\varphi \left( \frac{a+b}{2} \right) = \varphi \left( \frac{a+b-x+x}{2} \right) \leq \frac{1}{2} \left( \varphi(a+b-x) + \varphi(x) \right). \tag{2.2}
\]

If we choose \( x = \lambda a + (1 - \lambda) b \) (\( 0 \leq \lambda \leq 1 \)) in (2.2), then we obtain

\[
\frac{1}{2} \left( \varphi(a+b-x) + \varphi(x) \right) = \frac{1}{2} \left( \varphi(a+b - (\lambda a + (1 - \lambda) b)) + \varphi(\lambda a + (1 - \lambda) b) \right)
= \frac{1}{2} \left( \varphi(\lambda b + (1 - \lambda) a) + \varphi(\lambda a + (1 - \lambda) b) \right). \tag{2.3}
\]

By using the convexity of \( \varphi \), we get

\[
\frac{1}{2} \left( \varphi(\lambda b + (1 - \lambda) a) + \varphi(\lambda a + (1 - \lambda) b) \right) \leq \frac{1}{2} (\varphi(a) + \varphi(b)). \tag{2.4}
\]

Thus, by (2.2), (2.3) and (2.4), we obtain

\[
\varphi \left( \frac{b+a}{2} \right) \leq \frac{1}{2} \left( \varphi(a+b-x) + \varphi(x) \right) \leq \frac{1}{2} (\varphi(a) + \varphi(b)). \tag{2.4}
\]

3 Proof of Theorems

Proof of Theorem 1.1. Let \( \varphi \) be a convex function and let \( \lambda_k \) (\( 0 \leq k \leq n \)) be positive weights associated with \( x_k \) and whose sum is unity. Then, by using inequality (1.1), we have

\[
\varphi \left( a + b - \sum_{k=1}^{n} \lambda_k x_k \right) = \varphi \left( \sum_{k=1}^{n} \lambda_k (a + b) - \sum_{k=1}^{n} \lambda_k x_k \right)
= \varphi \left( \sum_{k=1}^{n} \lambda_k (a + b - x_k) \right) \leq \sum_{k=1}^{n} \lambda_k \varphi(a + b - x_k). \tag{3.1}
\]

By Lemma 2.1, we get

\[
\varphi \left( \sum_{k=1}^{n} \lambda_k (a + b - x_k) \right) \leq \sum_{k=1}^{n} \lambda_k (\varphi(a) + \varphi(b) - \varphi(x_k))
= \varphi(a) + \varphi(b) - \sum_{k=1}^{n} \lambda_k \varphi(x_k). \tag{3.2}
\]

From (3.1) and (3.2), we obtain

\[
\varphi \left( a + b - \sum_{k=1}^{n} \lambda_k x_k \right) \leq \varphi(a) + \varphi(b) - \sum_{k=1}^{n} \lambda_k \varphi(x_k),
\]

which is the right-hand of inequalities in (1.4). Now, using Lemma 2.1, we obtain

\[
2\varphi \left( \frac{a+b}{2} \right) - \varphi \left( \sum_{k=1}^{n} \lambda_k x_k \right) \leq \varphi \left( a + b - \sum_{k=1}^{n} \lambda_k x_k \right). \tag{3.3}
\]

Since \( \varphi \) is a convex function, then from (3.3) and inequality (1.1), we deduce that

\[
2\varphi \left( \frac{a+b}{2} \right) - \sum_{k=1}^{n} \lambda_k \varphi(x_k) \leq 2\varphi \left( \frac{a+b}{2} \right) - \varphi \left( \sum_{k=1}^{n} \lambda_k x_k \right)
\leq \varphi \left( a + b - \sum_{k=1}^{n} \lambda_k x_k \right),
\]
which is the left-hand of inequalities in (1.4). This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2.** Let \( \varphi \) be a convex function. Then, by using inequality (1.2), we have

\[
\varphi \left( a + b - \frac{\int_I^d p(x) f(x) dx}{\int_I^d p(x) dx} \right) = \varphi \left( \frac{\int_I^d p(x) (a + b - f(x)) dx}{\int_I^d p(x) dx} \right) 
\leq \frac{\int_I^d p(x) (a + b - f(x)) dx}{\int_I^d p(x) dx}.
\]

By Lemma 2.1, we get

\[
\frac{\int_I^d p(x) \varphi(a + b - f(x)) dx}{\int_I^d p(x) dx} \leq \frac{\int_I^d p(x) (\varphi(a) + \varphi(b) - \varphi(f(x))) dx}{\int_I^d p(x) dx} = \varphi(a) + \varphi(b) - \frac{\int_I^d p(x) \varphi(f(x)) dx}{\int_I^d p(x) dx}.
\]

From (3.4) and (3.5), we obtain

\[
\varphi \left( a + b - \frac{\int_I^d p(x) f(x) dx}{\int_I^d p(x) dx} \right) \leq \varphi(a) + \varphi(b) - \frac{\int_I^d p(x) \varphi(f(x)) dx}{\int_I^d p(x) dx},
\]

which is the right-hand inequalities in (1.5). Using now Lemma 2.1, we obtain

\[
2\varphi \left( \frac{a + b}{2} \right) \leq \varphi \left( \frac{\int_I^d p(x) f(x) dx}{\int_I^d p(x) dx} \right) + \varphi \left( a + b - \frac{\int_I^d p(x) f(x) dx}{\int_I^d p(x) dx} \right).
\]

This implies

\[
2\varphi \left( \frac{a + b}{2} \right) - \varphi \left( \frac{\int_I^d p(x) f(x) dx}{\int_I^d p(x) dx} \right) \leq \varphi \left( a + b - \frac{\int_I^d p(x) f(x) dx}{\int_I^d p(x) dx} \right).
\]

Since \( \varphi \) is a convex function, then from (3.7) and inequality (1.2), we deduce that

\[
2\varphi \left( \frac{a + b}{2} \right) - \frac{\int_I^d p(x) \varphi(f(x)) dx}{\int_I^d p(x) dx} \leq 2\varphi \left( \frac{a + b}{2} \right) - \varphi \left( \frac{\int_I^d p(x) f(x) dx}{\int_I^d p(x) dx} \right) \leq \varphi \left( a + b - \frac{\int_I^d p(x) f(x) dx}{\int_I^d p(x) dx} \right).
\]

The left-hand of inequalities in (1.5) is proved. This completes the proof of Theorem 1.2.

**Proof of Theorem 1.3.** By using Lemma 2.1, we obtain for all \( x_k \in I \)

\[
2\varphi \left( \frac{a + b}{2} \right) \leq \varphi(a + b - x_k) + \varphi(x_k) \leq \varphi(a) + \varphi(b).
\]

Multiplying (3.8) by \( \lambda_{\sigma(k)} \) and adding, we get

\[
2\varphi \left( \frac{a + b}{2} \right) \leq \sum_{k=1}^n \lambda_{\sigma(k)} \varphi(a + b - x_k) + \sum_{k=1}^n \lambda_{\sigma(k)} \varphi(x_k) \leq \varphi(a) + \varphi(b).
\]

On other hand by Lemma 2.1, we have

\[
2\varphi \left( \frac{a + b}{2} \right) \leq \varphi \left( a + b - \sum_{k=1}^n \lambda_k x_k \right) + \varphi \left( \sum_{k=1}^n \lambda_k x_k \right) \leq \varphi(a) + \varphi(b).
\]
This implies
\[- (\varphi(a) + \varphi(b)) \leq -\varphi \left( a + b - \sum_{k=1}^{n} \lambda_k x_k \right) - \varphi \left( \sum_{k=1}^{n} \lambda_k x_k \right) \]
\[\leq -2\varphi \left( \frac{a + b}{2} \right). \quad (3.10)\]

By addition from (3.9) and (3.10), we get our result.

**Proof of Theorem 1.4.** By using Lemma 2.1, we obtain for all \( f(x) \in I \)
\[2\varphi \left( \frac{a + b}{2} \right) \leq \varphi (a + b - f(x)) + \varphi (f(x)) \leq \varphi(a) + \varphi(b). \quad (3.11)\]

Multiplying (3.11) by \( p(x) \) and integrating over \([c, d]\), we get
\[2\varphi \left( \frac{a + b}{2} \right) \leq \int_{c}^{d} p(x) \varphi (a + b - f(x)) \frac{dx}{\int_{c}^{d} p(x) dx} + \int_{c}^{d} p(x) \varphi (f(x)) \frac{dx}{\int_{c}^{d} p(x) dx} \]
\[\leq \varphi(a) + \varphi(b). \quad (3.12)\]

On other hand by Lemma 2.1, we have
\[2\varphi \left( \frac{a + b}{2} \right) \leq \varphi \left( a + b - \int_{c}^{d} p(x) f(x) \frac{dx}{\int_{c}^{d} p(x) dx} \right) + \varphi \left( \int_{c}^{d} p(x) f(x) \frac{dx}{\int_{c}^{d} p(x) dx} \right) \]
\[\leq \varphi(a) + \varphi(b). \quad (3.13)\]

This implies
\[- (\varphi(a) + \varphi(b)) \leq -\varphi \left( a + b - \int_{c}^{d} p(x) f(x) \frac{dx}{\int_{c}^{d} p(x) dx} \right) - \varphi \left( \int_{c}^{d} p(x) f(x) \frac{dx}{\int_{c}^{d} p(x) dx} \right) \]
\[\leq -2\varphi \left( \frac{a + b}{2} \right). \quad (3.14)\]

By addition from (3.13) and (3.14), we get our result.

**4 Applications**

Let \( x_k \in [a, b] \) \((b > a > 0)\), \( \lambda_k \in [0,1] \) such that \( \sum_{k=1}^{n} \lambda_k = 1 \). Then, by Theorem 1.1 and Theorem 1.3 for \( \varphi(x) = -\ln x \), we obtain respectively
\[\sqrt{ab} \leq \sqrt{A'G + AG'} \leq \frac{a + b}{2}\]
and
\[1 \leq \frac{A'A'}{GG'} \leq \frac{a + b}{2\sqrt{ab}}, \]
where \( A = \sum_{k=1}^{n} \lambda_k x_k, \quad G = \prod_{k=1}^{n} x_k^{\lambda_k}, \quad A' = a + b - \sum_{k=1}^{n} \lambda_k x_k \) and \( G' = \prod_{k=1}^{n} (a + b - x_k)^{\lambda_k} \).
References


Received: April 17, 2013; Accepted: June 28, 2013

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