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The Q_1-matrix completion problem

Kalyan Sinha

Abstract
A matrix is a Q_1-matrix if it is a Q-matrix with positive diagonal entries. A digraph D is said to have Q_1-completion if every partial Q_1-matrix specifying D can be completed to a Q_1-matrix. In this paper, necessary and sufficient conditions for a digraph to have Q_1-completion are obtained. Later on the relationship among the completion problem of Q_1-matrix and some other class of matrices are discussed. Finally, the digraphs of order at most four that include all loops and have Q_1-completion are characterized.

Keywords
Partial matrix, Matrix completion, Q_1-matrix, Q_1-completion, Digraph.

AMS Subject Classification
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Department of Mathematics, PSTDS Vidyapith, Chinsurah, Hooghly, West Bengal-712305, India.
***Corresponding author**: kalyansinha90@gmail.com
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1. Introduction

A real $n \times n$ matrix $B = [b_{ij}]$ is a *Q-matrix* if for every $k \in \{1,2,\ldots,n\}$, $S_k(B) > 0$, where $S_k(B)$ is the sum of all $k \times k$ principal minors of B. The matrix B is Q_1-matrix if all diagonal entries are positive and for every $k \in \{1,2,\ldots,n\}$, $S_k(B) > 0$. Clearly a Q_1-matrix is a Q-matrix but not conversely.

A number of researchers studied matrix completion problems for different classes of matrices ([5–10]). The P_0-matrix and $P_{0,1}$-completion are studied in [7, 11]. A real $n \times n$ matrix B_1 is a *P_0-matrix* (P-matrix) if every principal minor of B_1 is nonnegative (positive). The matrix B_1 is a $P_{0,1}$-matrix if all diagonal entries of B_1 are positive and B_1 is itself a P_0-matrix. In 2009, DeAlba *et al.* [2] solved the Q-matrix completion problem. In this paper it is seen that the property of being a Q-matrix is not inherited by principal submatrices, thus the Q-matrix completion problem is significantly different from the completion problems studied earlier. One may see [3] for a survey of matrix completion results.

A *partial matrix* is a rectangular array of numbers in which some entries are specified, while others are free to be chosen. For $\alpha \subseteq \{1,\ldots,n\}$, the principal submatrix $B[\alpha]$ is obtained by deleting from B all rows and columns whose indices are not in α. A principal minor is the determinant of a principal submatrix. A *pattern* for $n \times n$ matrices is a subset of $\{1,\ldots,n\} \times \{1,\ldots,n\}$. A partial matrix *specifies a pattern* if its specified entries lie exactly in those positions listed in the pattern.

For a given class Γ of matrices (e.g., P_0, P or Q-matrices) a *partial Γ-matrix* is a partial matrix for which the specified entries satisfy the properties of a Γ-matrix. Thus, a *partial Q-matrix* is a partial matrix M in which $S_k(M) > 0$ for every $k \in \{1,2,\ldots,n\}$ for which all $k \times k$ principal submatrices are fully specified. Similarly a *partial Q_1-matrix* is a partial Q-matrix with all specified positive diagonal entries.

A *completion* of a partial matrix is a specific choice of values for the unspecified entries. A *matrix completion problem* asks which partial matrices have completions with a given property. A Γ-*completion* of a partial Γ-matrix M is a completion of M which is a Γ-matrix. The Γ-*matrix completion problem* studies the properties and classifications of patterns having Γ-completions.

1.1 Digraphs

It is observed from the history of matrix completion problems that Graph theory and Matrix completion problems are correlated with each other. Graph theoretic techniques are seen to be very fruitful to solve the matrix completion problems. Any standard reference, for example, [1] and [4] can be use for graph theoretic terminologies. A *directed graph* or *digraph* $D=(V_D,A_D)$ of order $n>0$ is a finite nonempty set V_D, with $|V_D|=n$ of objects called *vertices* together with a (possibly empty) set A_D of ordered pairs of vertices, called *arcs* or *directed edges*. We write $v\in D$ (resp. $(u,v)\in D$) to imply $v\in V_D$ (resp. $(u,v)\in A_D$). If $x=(u,u)$, then x is called a *loop* at the vertex u.

A *symmetric edge* of D is a pair of arcs $\{(u,v),(v,u)\}\subset A_D$, usually written as $\{u,v\}$. A (*directed*) u-v *path* P of length $k\geq 0$ in D is an alternating sequence $(u=v_0,x_1,v_1,\ldots,x_k,v_k=v)$ of vertices and arcs, where v_i, $1\leq i\leq k$, are distinct vertices and $x_i=(v_{i-1},v_i)$. Then, the vertices v_i and the arcs x_i are said to be on P. Further, if $k\geq 2$ and $u=v$, then a u-v path is a *cycle* of length k. We then write $C_k=\langle v_1,v_2,\ldots,v_k\rangle$ and call C_k a k-cycle in D.

A cycle C is *even* (resp. *odd*) if its length is even (resp. odd). A digraph $H=(V_H,A_H)$ is a *subdigraph of order* k of the digraph D if $|V_H|=k$ and $V_H\subseteq V_D$, $A_H\subseteq A_D$. A digraph D is said to be connected (resp. strongly connected) if for every pair u,v of vertices, D contains a u-v path (resp. both a u-v path and a v-u path). The maximal connected (resp. strongly connected) subdigraphs of D are called *components* (resp. *strong components*) of D.

A subdigraph H of D is an *induced subdigraph* if $A_H=(V_H\times V_H)\cap A_D$ (*induced by* V_H) and is a *spanning subdigraph* if $V_H=V_D$. Again for $v\in V_D$, $D-v$ denotes the subdigraph of D induced by $V_D\setminus\{v\}$. The *complement of a digraph* D is the digraph $\overline{D}$, where $V_{\overline{D}}=V_D$ and $(v,w)\in A_{\overline{D}}$ if and only if $(v,w)\notin A_D$. A digraph D is said to be *symmetric* if $(u,v)\in D$ implies $(v,u)\in D$. On the other hand, D is *asymmetric* if $(u,v)\in D$ implies $(v,u)\notin D$. A *complete symmetric digraph* on n vertices, denoted by K_n, is the digraph having all possible arcs (including all loops).

Two digraphs $D_1=(V_1,A_1)$ and $D_2=(V_2,A_2)$ are *isomorphic*, if there is a bijection $\phi:V_1\to V_2$ such that $A_2=\{(\phi(u),\phi(v)):(u,v)\in A_1\}$. An *unlabelled* digraph is an equivalent class of isomorphic digraphs. Choosing a particular member of an unlabelled digraph is referred as a *labelling* of the unlabelled digraph.

1.2 Digraphs with matrices

Let π be a permutation of a nonempty finite set V. The digraph $D_\pi=(V,A_\pi)$, where $A_\pi=\{\big(v,\pi(v)\big):v\in V\}$ is called a *permutation digraph*. Clearly, each component of a permutation digraph is a loop or a cycle.

A *permutation subdigraph* H (of order k) of a digraph D is a permutation digraph that is a subdigraph of D (of order k). A digraph D is *stratified* if D has a permutation subdigraph of order k for every $k=2,3,\ldots,|D|$. A digraph D is said to be pseudo-stratified if there exist a vertex v in D such that $D-v$ is stratified.

Let $B=[b_{ij}]$ be an $n\times n$ matrix. We have

$$\det(B)=\sum(\operatorname{sgn}\pi)b_{1\pi(1)}\cdots b_{n\pi(n)} \tag{1.1}$$

where the sum is taken over all permutations π of $\langle n\rangle=\{1,2,\ldots,n\}$.

A signing of a digraph is an assignment of a sign $+$ or $-$ to each arc of the digraph. The result of signing of a digraph is called a *signed digraph*. For an arc $e\in D$, by $s(e)$ we mean e has sign $s(e)$.

For a k-cycle in C_k in D, the sign $s(C_k)$ is defined to be,

$$s(C_k)=(-1)^{k+1}\prod_{e\in C_k}s(e)$$

For a permutation subdigraph K of D, the sign $s(K)$ of K is

$$s(K)=\prod_{C\in K}s(C)$$

2. Partial Q_1-matrix and the Q_1-matrix completion problem

A *partial* Q_1*-matrix* is a partial Q-matrix in which all specified diagonal entries are positive i.e. a *partial* Q_1*-matrix* is a partial matrix M with all specified positive diagonal entries and $S_k(M)>0$ for every $k\in\{1,2,\ldots,n\}$, whenever all $k\times k$ principal submatrices are fully specified. Now, a partial Q_1-matrix is characterized as follows.

Proposition 2.1. *Suppose $M=[a_{ij}]$ is a partial matrix. Then M is a partial Q_1-matrix if and only if exactly one of the following holds:*
(i) At least one diagonal entry of M is unspecified, all specified diagonal entries are positive.
(ii) All diagonal entries are specified and positive; at least one off-diagonal entry is unspecified.
(iii) All entries of M are specified and M is a Q_1-matrix.

A completion B of a partial Q_1-matrix M is called a Q_1-*completion* of M, if B is a Q_1-matrix. Since any matrix which is permutation similar to a Q_1-matrix is a Q_1-matrix, it is evident that if a partial Q_1-matrix M has a Q_1-completion, so does any partial matrix which is permutation similar to M.

It can be easily seen that any partial matrix M with all unspecified diagonal entries has Q_1-completion. A completion can be obtained by choosing sufficiently large values for the unspecified diagonal entries. Let M be a partial Q_1-matrix in which the diagonal entries at (i,i) positions $(i=k+1,\ldots,n)$

are unspecified. In case $M[1,\dots,k]$ is fully specified, M may not have a Q_1-completion. For example, the partial matrix,

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & ? \end{bmatrix},$$

where ? denotes an unspecified entry, does not have Q_1-completion. Indeed, for any completion B of M, $S_3(B) = 0$. On the other hand, if $M[1,\dots,k]$ has an unspecified entry and has a Q_1-completion, then M has a Q_1-completion. A completion of M can be obtained by choosing sufficiently large values for the unspecified diagonal entries. These above observations are listed in the following results.

Theorem 2.2. *If a matrix M omits all diagonal entries, then M has Q_1-completion.*

Proof. Suppose $M = [a_{ij}]$ be a partial Q_1-matrix. For any $t > 1$, consider a completion $B = [b_{ij}]$ of M by setting all diagonal entries equal to t and rest of the off diagonal entries to be equal to zero. Then, any $k \times k$ principal minor will be of the form $t^k + p(t)$, where $p(t)$ is a polynomial of degree $\leq k-1$. Now by choosing t large enough, we have $S_k(B) > 0$ for all $k \times k$ principal minors of B. Since only finitely many principal minors are to be considered, thus for sufficiently large t, M has Q_1-completion. □

Theorem 2.3. *Suppose M be a partial Q_1-matrix in which the diagonal entry at $(r+1,r+1)$ position is unspecified. If the principal submatrix $M[1,\dots,r]$ of M is not fully specified and has Q_1-completion, then M has Q_1-completion.*

Proof. Suppose $M = [a_{ij}]$ be a partial Q_1-matrix in which the diagonal entry at $(r+1,r+1)$ position is unspecified. Then, M is of the form,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where, $M_{11} = M[1,\dots,r]$ and $M_{22} = M[r+1,r+1]$.

Let A_1 be the Q_1-matrix completion of $M[1,\dots,r]$. Then,

$$M' = \begin{bmatrix} A_1 & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

is a partial Q_1-matrix, since M_{22} has an unspecified diagonal entry. Now for $t > 0$, consider a completion $B = [b_{ij}]$ of M' obtained by choosing $b_{ii} = t,\ i = r+1$ and $b_{ij} = 0$ against all other unspecified entries in M'. Then B is of the form,

$$B = \begin{bmatrix} A_1 & B_{12} \\ B_{21} & t \end{bmatrix}.$$

Since A_1 is a Q_1-matrix, $S_i(A_1) > 0$ for $1 \leq i \leq r$. For $2 \leq j \leq r+1$,

$$S_j(B) = S_j(A_1) + tS_{j-1}(A_1) + s_j,$$

where s_j is a constant. Now $S_j(B) > 0$ for sufficiently large values of t and clearly B is Q_1-matrix. □

Corollary 2.4. *Suppose M be a partial Q_1-matrix in which the diagonal entries at (i,i) positions $(i = r+1,\dots,n)$ are unspecified. If the principal submatrix $M[1,\dots,r]$ of M is not fully specified and has Q_1-completion, then M has Q_1-completion.*

The converse of Corollary 2.4 is not true which can be seen from the following example.

Example 2.5. *Consider the partial matrix,*

$$M = \begin{bmatrix} d_1 & a_{12} & a_{13} & ? \\ a_{21} & d_2 & ? & ? \\ a_{31} & a_{32} & d_3 & ? \\ a_{41} & ? & ? & ? \end{bmatrix},$$

where ? denotes the unspecified entries. We show that for any choice of values of the specified entries M has Q_1-completions, though there are occasions when $M[1,2,3]$ need not have Q_1-completion. For $t > 0$, consider the completion $B(t)$ of M defined as follows:

$$B(t) = \begin{bmatrix} d_1 & a_{12} & a_{13} & 0 \\ a_{21} & d_2 & t & 0 \\ a_{31} & a_{32} & d_3 & t \\ a_{41} & t & -t & t \end{bmatrix}.$$

Then,

$$\begin{aligned} S_1(B(t)) &= t + \sum d_i, \\ S_2(B(t)) &= t^2 + f_1(t), \\ S_3(B(t)) &= t^3 + f_2(t), \\ S_4(B(t)) &= d_1 t^3 + f_2(t), \end{aligned}$$

where $f_i(t)$ is a polynomial in t of degree at most i, $i = 1,2$. Consequently, $B(t)$ is a Q_1-matrix for sufficiently large t, and therefore M has Q_1-completion. On the other hand, the partial Q_1-matrix

$$M[1,2,3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & ? \\ 1 & 1 & 1 \end{bmatrix},$$

with unspecified entries ?, is the principal submatrix of M induced by its diagonal $\{1,2,3\}$. Now one can verify that $M[1,2,3]$ does not have Q_1-completion, because $\det(M[1,2,3]) = 0$ *for any completion of* $M[1,2,3]$.

3. Digraphs and Q_1-completions

It can be easily seen that an $n \times n$ partial matrix M *specifies* a digraph $D = (\langle n \rangle, A_D)$ if for $1 \leq i,j \leq n$, $(i,j) \in A_D$ if and only if the (i,j)-th entry of M is specified. For example, the partial Q_1-matrix M in Example 2.5 specifies the digraph D in Figure 1.

Theorem 3.1. *Suppose M is a partial Q_1-matrix specifying the digraph D. If the partial submatrix of M induced by every strongly connected induced subdigraph of D has Q_1-completion, then M has Q_1-completion.*

Proof. We prove the result for the case when D has two strong components D_1 and D_2. The general result will then follow by induction. By a relabeling of the vertices of D, if required, we have

$$M = \begin{bmatrix} M_{11} & M_{12} \\ X & M_{22} \end{bmatrix},$$

where M_{ii} is a partial Q_1-matrix specifying D_i, $i = 1,2$, and all entries in X are unspecified. By the hypothesis, M_{ii} has a Q_1-completion B_{ii}. Consider the completion

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

by choosing all entries in X as well as all unspecified entries in M_{12} as 0. Then, for $2 \le k \le |D|$ we have,

$$S_k(B) = S_k(B_{11}) + S_k(B_{22}) + \sum_{r=1}^{k-1} S_r(B_{11})S_{k-r}(B_{22}) \ge 0,$$

Here, we mean $S_k(B_{ii}) = 0$ whenever k exceeds the size of B_{ii}. Thus M can be completed to a Q_1-matrix. □

The proof of the following result is similar.

Theorem 3.2. *Suppose M is a partial Q_1-matrix specifying the digraph D. If the partial submatrix of M induced by each component of D has a Q_1-completion, then M has a Q_1-completion.*

The converse of Theorem 3.1 is not true. For example, every partial Q_1-matrix specifying the digraph D in Figure 1 has Q_1-completion, although the strong component D' induced by vertices $\{1,2,3\}$ does not have Q_1-completion (see Example 3.3).

Example 3.3. *Consider the digraph D in the Figure 1. We show that D has Q_1-completion, but the strong component D' induced by vertices $\{1,2,3\}$ does not have Q_1-completion.*

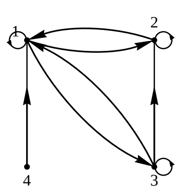

Figure 1. The Digraph D

Let $M = [a_{ij}]$ be a partial Q_1-matrix specifying D. Then for $t > 0$, M can be completed to a Q_1-matrix $B(t)$ but the principal submatrix induced by the digraph D' i.e. $M[1,2,3]$ does not have Q_1-completion (see Example 2.5). To see that $M[1,2,3]$ does not have Q_1-completion, consider the partial Q_1-matrix

$$M[1,2,3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & x \\ 1 & 1 & 1 \end{bmatrix},$$

with unspecified entry x. Then for any Q_1-completion B of $M[1,2,3]$, we have $S_3(B) = 0$ and hence $M[1,2,3]$ does not have Q_1-completion.

The property of having Q_1-completion is not inherited by induced subdigraphs. This can be also seen from the Example 2.5.

4. The Q_1-matrix completion problem

We say that a digraph D has Q_1-*completion*, if every partial Q_1-matrix specifying D can be completed to a Q_1-matrix. The Q_1-*matrix completion problem* aims at studying and classifying all digraphs D which have Q_1-completion.

The property of being a Q_1-matrix is preserved under similarity and transposition, but it is not inherited by principal submatrices, as it can easily be verified. Also it is clear that if a digraph D has Q_1-completion, then any digraph which is isomorphic to D has Q_1-completion.

4.1 Sufficient conditions for Q_1-matrix completion

Theorem 4.1. *If a digraph $D \neq K_n$ of order n has Q_1-completion, then any spanning subdigraph of D has Q_1-completion.*

Proof. Suppose H be a spanning subdigraph of D and M_H be a partial Q_1-matrix specifying the digraph H. Consider a partial matrix M_D obtained from M_H by specifying the entries corresponding to $(i, j) \in A_D \setminus A_H$ as 0. Since $D \neq K_n$, by Proposition 2.1, M_D is a partial Q_1-matrix specifying D. Let B be a Q_1-completion of M_D. Clearly, B is a Q_1-completion of M_H. □

Theorem 4.2. *Suppose $D \neq K_n$ be a digraph such that $\overline{D}$ is stratified. If it is possible to sign the arcs of $\overline{D}$ so that the sign of every cycle is of positive sign, then D has Q_1-completion.*

Proof. Suppose M be a partial Q_1-matrix specifying the digraph D. For any $t > 0$, consider a completion B of M by choosing the unspecified entry $x_{ij} = sgn(i, j)t$ (using the sign of the arc in $\overline{D}$). Then for each $k = 2,3,\ldots,n$, we have,

$$S_k(B) = c_k t^k + r_k(t)$$

where c_k is the number of permutation subdigraphs of order k in D and $r_k(t)$ is a polynomial of degree less than k. If D contains all loops, then the trace of any partial Q_1-matrix specifying D is positive; if D omits a loop, then $S_1(B) = c_1 t + r_0$, where c_1 is the number of loops in D and $r_0 \in R$. Now by choosing t sufficiently large, B becomes a Q_1-matrix. □

Example 4.3. *Consider the digraph D_0 and its complement $\overline{D_0}$ in Figure 2. It can be easily seen that the digraph $\overline{D_0}$ is stratified. Also it is possible to sign the arcs of $\overline{D_0}$ so that*

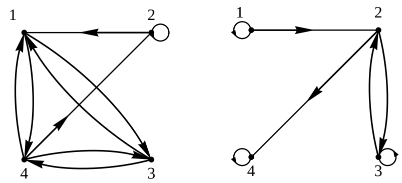

Figure 2. The Digraphs D_0 and $\overline{D_0}$

every cycle in $\overline{D_0}$ is of positive sign. Thus by Theorem 4.2, the digraph D_0 has Q_1-completion.

However the converse of the Theorem 4.2 is not true which can be seen from the Example 4.6. Although the complement of the digraph $\widehat{D}$ is not stratified, but it has Q_1-completion [See Example 4.6].

Corollary 4.4. *Suppose $D \neq K_n$ be a digraph such that $\overline{D}$ has a stratified spanning subdigraph D_1. If D_1 has a signing in which the sign of every cycle is $+$, then D has Q_1-completion.*

Theorem 4.5. *Suppose $D \neq K_n$ be a digraph with all loops such that $\overline{D}$ is pseudo-stratified. If it is possible to sign the arcs of $\overline{D}$ so that the sign of every cycle is of positive sign, then D has Q_1-completion.*

Proof. Let $M = [a_{ij}]$ be a partial Q_1-matrix specifying the digraph D. Since $\overline{D}$ is pseudo-stratified, there exists a subdigraph D_1 of $\overline{D}$ of order $n-1$ such that D_1 is stratified. Suppose D_1 is obtained from $\overline{D}$ by deleting a vertex say v_1 in $\overline{D}$ i.e. $D_1 = \overline{D} - v_1$. For $t > 0$, consider a completion $B(t) = [b_{ij}]$ of M by choosing the unspecified entries as following:

$$b_{ij} = \begin{cases} sgn(i,j)t, & \text{for all } (i,j) \in D_1 \\ 0, & \text{otherwise.} \end{cases}$$

where $sgn(i,j)$ denotes the sign of the arcs of $\overline{D}$. Since M is a partial Q_1-matrix with all specified diagonal entries, thus $d_i > 0, \forall\, i = 1,2,\dots,n$. Now we have,

$$\begin{aligned} S_k(B) &= c_k t^k + f_k(t),\ k \in \{2,3,\dots,n-1\} \\ S_n(B) &= d_1 c_{n-1} t^{n-1} + f_{n-1}(t) \end{aligned}$$

where c_k is the number of permutation subdigraphs of order k in D and $f_k(t)$ is a polynomial of degree less than k. Now choosing a sufficiently large value of t, we have $S_k(B) > 0$, $\forall\, k \in \{1,2,\dots,n\}$ and hence $B(t)$ is Q_1-matrix. $\square$

Example 4.6. *Consider the digraph $\widehat{D} \neq K_4$ in Figure 3. The complement $\overline{\widehat{D}}$ of the digraph $\widehat{D}$ is not stratified although $\widehat{D}$ has Q_1-completion. Since $\overline{\widehat{D}}$ is pseudo-stratified digraph of order 4, thus it has a stratified subdigraph D_1 of order 3 which is obtained by deleting the vertex 2 from $\overline{\widehat{D}}$. Now $\widehat{D}$ satisfies the statement of the Theorem 4.5, hence $\widehat{D}$ has Q_1-completion. To see this consider a partial Q_1-matrix*

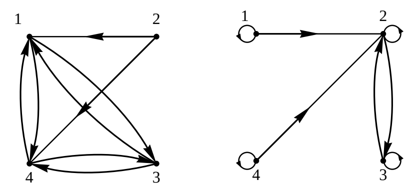

Figure 3. The Digraphs $\widehat{D}$ and $\overline{\widehat{D}}$

$$M = \begin{bmatrix} d_1 & a_{12} & a_{13} & ? \\ ? & d_2 & a_{23} & ? \\ ? & a_{32} & d_3 & ? \\ ? & ? & ? & d_4 \end{bmatrix},$$

specifying the digraph $\widehat{D}$ with unspecified entries as ?. For $t > 0$, consider a completion B of M as follows:

$$B = \begin{bmatrix} d_1 & a_{12} & a_{13} & t \\ 0 & d_2 & a_{23} & 0 \\ t & a_{32} & d_3 & -t \\ 0 & 0 & t & d_4 \end{bmatrix}.$$

Since M is a partial Q_1-matrix, hence all $d_i > 0\ \ \forall\, i = 1,2,3,4$. Now, we have,

$$\begin{aligned} S_1(B) &= d_1 + d_2 + d_3 + d_4 \\ S_2(B) &= t^2 + f_1(t) \\ S_3(B) &= t^3 + f_2(B) \\ S_4(B) &= d_2 t^3 + f_2(B) \end{aligned}$$

where $f_i(t)$ denotes a polynomial of t of total degree i. Now sufficiently large values of $t > 0$, B becomes a Q_1-matrix.

However the converse of the Theorem 4.5 is not true which can be seen from the Example 4.8.

Theorem 4.7. *Every asymmetric digraph D has Q_1-completion.*

Proof. Suppose $M = [a_{ij}]$ be a partial Q_1-matrix specifying the digraph D. Since M is a partial Q_1-matrix, thus the specified diagonal entries of M are positive. Consider a completion B of M obtained by setting all unspecified diagonal entries d_i as 1 and all unspecified pairs x_{ij}, x_{ji} to 0. We set all other x_{ij} to $-a_{ji}$ and we have,

$$B = \begin{bmatrix} d_1 & a_{12} & a_{13} & \dots & a_{1n} \\ -a_{12} & d_2 & a_{23} & \dots & a_{2n} \\ -a_{13} & -a_{23} & d_3 & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ -a_{1n} & -a_{2n} & -a_{3n} & \dots & d_n \end{bmatrix}.$$

Let $D = diag[d_1, d_2, \ldots, d_n]$. Then $D > 0$. We can write $B = B_0 + D$, where B_0 is a skew-symmetric real matrix. Since a skew symmetric real matrix is a P_0-matrix [7], hence $S_k(B_0) \geq 0$ for every $k \in \{1, 2, \ldots, n\}$. On the other hand, being a positive matrix D, we have $\forall k \in \{1, 2, \ldots, n\}$, $S_k(D) > 0$ and as a result $S_k(B) > 0$. Hence the result follows. ☐

Example 4.8. *Consider the digraph D_2 in Figure 4. The complement $\overline{D_2}$ of D_2 is neither stratified nor pseudo-stratified, but it has Q_1-completion by Theorem 4.7.*

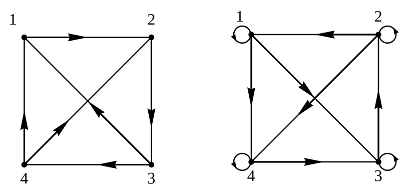

Figure 4. The Digraphs D_2 and $\overline{D_2}$

Example 4.9. *The converse of the Theorem 4.7 is not true which can be seen from the Example 3.3. Although the digraph D is not asymmetric, but the digraph D has Q_1-completion. From Example 2.5, it is seen that any partial Q_1-matrix M specifying D can be completed to a Q_1-matrix.*

Corollary 4.10. *A digraph D has Q_1-completion if it does not contain a cycle of even length.*

Theorem 4.11. *Suppose $D \neq K_4$ be a digraph with all loops. Suppose $\overline{D}$ contains a 2-cycle $\langle v_1, v_2 \rangle$ such that $\langle v_1, v_2 \rangle$ does not form a permutation subdigraph of order 4 with any 2-cycle in $D + \langle v_1, v_2 \rangle$. Then D has Q_1-completion.*

Proof. Suppose $M = [a_{ij}]$ be a partial Q_1-matrix specifying the digraph D. For any $t > 0$, consider a completion B of M by choosing the unspecified entries as following:

$$b_{ij} = \begin{cases} t, & \text{if } (i,j) = (v_1, v_2) \in \overline{D} \\ -t, & \text{if } (i,j) = (v_2, v_1) \in \overline{D} \\ 0, & \text{otherwise.} \end{cases}$$

Then we have,

$$\begin{aligned} S_1(B) &= d_1 + d_2 + d_3 + d_4 \\ S_2(B) &= t^2 + f_1(t, a_{ij}) \\ S_3(B) &= (d_3 + d_4)t^2 + f_1(t, a_{ij}) \\ S_4(B) &= d_3 d_4 t^2 + f_1(t, a_{ij}), \end{aligned}$$

where $f_1(t, a_{ij})$ is a polynomial in t of degree at most 1. Since M is a partial Q_1-matrix with all specified diagonal entries, thus we have $S_1(B) > 0$. Now choosing t sufficiently large, we have $S_i(B) > 0$ for all $i = 2, 3, 4$. Hence the result follows. ☐

Example 4.12. *Consider the digraph D_3 in Figure 5. The complement $\overline{D_3}$ of D_3 contains a 2-cycle $\langle 1, 3 \rangle$ which does not form a permutation subdigraph of order 4 with any 2-cycle in $D_3 + \langle 1, 3 \rangle$. Thus by Theorem 4.11, D_3 has Q_1-completion.*

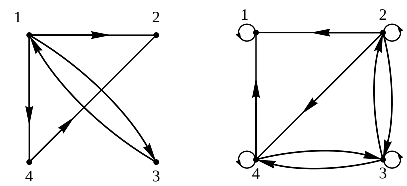

Figure 5. The Digraphs D_3 and $\overline{D_3}$

However the converse of the Theorem 4.11 is not true which can be seen from the Example 4.8. Although the digraph $\overline{D_2}$ in Figure 4 does not have 2-cycle but D_2 has Q_1-completion.

4.2 Necessary conditions for Q_1-matrix completion

In this section we will discuss some necessary conditions for a digraph to have Q_1-completion.

Theorem 4.13. *If a digraph $D \neq K_n$ of order $n \geq 2$ contains two vertices v_1 and v_2 with indegree or outdegree n, then D does not have Q_1-completion.*

Proof. Suppose a digraph D of order $n \geq 2$ contains two vertices v_1 and v_2 with indegree or outdegree n. Consider a partial Q_1-matrix M specifying D with all specified entries are exactly 1. Then two columns or rows of M are equal and for any completion B of M, we have $\det B = 0$. Hence the result follows. ☐

Theorem 4.14. *Suppose $D \neq K_n$ be a digraph with all loops such that $\overline{D}$ is asymmetric. If D contains a 2-cycle $\langle v_1, v_2 \rangle$, then D does not have Q_1-completion.*

Proof. Suppose that D has a 2-cycle $\langle v_1, v_2 \rangle$. Consider a partial Q_1-matrix $M = [a_{ij}]$ specifying D such that $d_i = 1$ $(1 \leq i \leq n)$ and $a_{v_1 v_2} a_{v_2 v_1} > \begin{pmatrix} n \\ 2 \end{pmatrix}$ and rest of all specified entries are zero. Let $B = [b_{ij}]$ be any completion of M. Then

$$S_2(B) = \sum_{i \neq j} d_i d_j - \sum_{i \neq j} b_{ij} b_{ji} \; < \; - \sum_{i,j \notin \{v_1, v_2\}} b_{ij} b_{ji} \; < \; 0,$$

and, therefore, B is not a Q_1-matrix. ☐

Example 4.15. *Consider the digraph D_4 in Figure 6. The complement $\overline{D_4}$ of D_4 is asymmetric and D_4 has 2-cycle $\langle 1, 3 \rangle$.*

Thus by Theorem 4.14, D_4 does not have Q_1-completion. To see this consider a partial Q_1-matrix

$$M = \begin{bmatrix} 1 & ? & 10 & 0 \\ 0 & 1 & ? & 0 \\ 10 & 0 & 1 & 0 \\ ? & ? & ? & 1 \end{bmatrix},$$

specifying the digraph D_4. Then for any completion B of M, we have $S_2(B) < 0$. Hence, M cannot be completed to a Q_1-matrix.

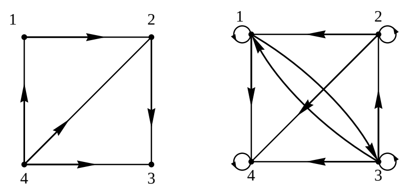

Figure 6. The Digraphs D_4 and $\overline{D_4}$

5. Relationship theorems

5.1 Q-completion and Q_1-completion

It is easily seen that a Q_1 matrix is a Q-matrix but not vice versa. Thus the completion problem of these two classes are partially related.

Theorem 5.1. *If a digraph D has Q-completion, then it must also have Q_1-completion.*

Proof. Suppose D be a digraph that has Q-completion and M be a partial Q_1-matrix specifying the digraph D. Thus M is a partial Q-matrix specifying the digraph D. Since D has Q-completion, thus M can be completed to a Q-matrix B. Now if D includes all loops, then we are done i.e. B is a Q_1-matrix. If D omits at least one loop, then M has atleast one unspecified diagonal entry. Now we choose positive numbers to those unspecified diagonal entries of M. In that case M is also a partial Q-matrix and it can be completed to a Q-matrix B which is also a Q_1-matrix. □

However the converse of the Theorem 5.1 is not true which can be seen from the following example.

Example 5.2. *The digraph D_3 in Figure 5 has Q_1-completion (See Example 4.12). But the digraph D_3 does not have Q-completion (By Theorem 2.8, [2]).*

5.2 P-completion and Q_1-completion

Since a P-matrix is a Q_1-matrix, thus the completion problem between these two classes are also partially related.

Theorem 5.3. *Any asymmetric digraph that has P-completion also has Q_1-completion.*

Proof. Suppose D be a asymmetric digraph that has P-completion and M be a partial Q_1-matrix specifying D. Since D is asymmetric, hence all principal submatrices of M of order greater than 1 are unspecified. Also being a partial Q_1-matrix as well as P-matrix, all specified diagonal entries in M are positive. Since M has P-completion, thus consider a P-matrix completion B of M. Now B is also a Q_1-completion of M and hence the result follows. □

Theorem 5.4. *Any asymmetric digraph has Q_1-completion.*

Proof. Since any asymmetric digraph has P-completion [7], thus by Theorem 5.3 any asymmetric digraph also has Q_1-completion. □

6. Classification of digraphs of small order having Q_1-completion

Based on the obtained results in the previous sections, we will classify the digraphs of order at most four that include all loops as to Q_1-completion in this section. Again permutation similarity of Q_1-matrix implies that if a digraph D has Q_1-completion, then any digraph which is isomorphic to D has Q_1-completion. Thus any digraph which is obtained by labelling the unlabelled digraph associated to D has Q_1-completion.

The nomenclature of the digraphs considered in the sequel are indicated as per their order in the list in [4, Appendix, pp. 233]. Here, $D_p(q,n)$ is the one obtained by attaching a loop at each of the vertices to the n-th member in the list of digraphs with p vertices and q (non-loop) arcs in the list.

We will classify the digraphs into a series of following lemmas.

Lemma 6.1. *For $1 \le p \le 4$, the digraphs $D_p(q,n)$ which are listed below do not have Q_1-completion.*

$$\begin{array}{lll} p=3; & q=4; & n=3,4 \\ & q=5; & n=1 \\ p=4; & q=6; & n=40,43 \\ & q=7; & n=16,22,29,36 \\ & q=8; & n=5,7,10,12,14,15,18,21,22,26,27 \\ & q=9; & n=1,2,5,11,13 \\ & q=10; & n=1\text{–}5 \\ & q=11; & n=1. \end{array}$$

Proof. Each of the digraph listed above satisfies the Theorem 4.13 and hence the result follows. □

Lemma 6.2. *For $1 \le p \le 4$, the digraphs $D_p(q,n)$ which are listed below do not have Q_1-completion.*

$$\begin{array}{lll} p=3; & q=4; & n=2 \\ p=4; & q=7; & n=31,33,34,37 \\ & q=8; & n=16,17\text{–}19,20,23\text{–}25 \\ & q=9; & n=4,12. \end{array}$$

Proof. Each of the digraph satisfies the theorem 4.14 and hence the result follows. □

Lemma 6.3. *For $1 \le p \le 4$, the digraphs $D_p(q,n)$ which are listed below do not have Q_1-completion.*

$$\begin{array}{lll} p=4; & q=6; & n=2 \\ & q=7; & n=4,5. \\ & q=8; & n=1,11. \end{array}$$

Proof. Suppose

$$M = \begin{bmatrix} 1 & 1 & 1 & ? \\ 1 & 1 & 1 & ? \\ 1 & 1 & 1 & ? \\ ? & ? & ? & 1 \end{bmatrix},$$

be a partial Q_1 matrix specifying the digraph $D_4(6,2)$ with unspecified entries as ?. But M cannot be completed to a Q_1-matrix since for any completion B of M, we have $\det B = 0$. Again any digraph listed in the Lemma 6.3 contains the digraph $D_4(6,2)$ as an induced subdigraph, hence the result follows. □

Theorem 6.4. *For $1 \le p \le 4$, the digraphs $D_p(q,n)$ which are listed below have Q_1-completion.*

$$\begin{array}{lll} p=2; & q=0,1,2; & n=1 \\ p=3; & q=0,1; & n=1 \\ & q=2,3; & n=1\text{–}4 \\ & q=4,6; & n=1 \\ p=4; & q=0,1; & n=1 \\ & q=2; & n=1\text{–}5 \\ & q=3; & n=1\text{–}13 \\ & q=4; & n=1\text{–}27 \\ & q=5; & n=1\text{–}38 \\ & q=6; & n=1,3\text{–}29,31\text{–}39,41,42,45\text{–}48 \\ & q=7; & n=1\text{–}3,6\text{–}14,17\text{–}21,24\text{–}28,30,32,35,38 \\ & q=8; & n=2\text{–}4,8,9 \\ & q=9; & n=3 \\ & q=12; & n=1. \end{array}$$

Proof. It can be easily seen that $D_p(q,n)$ has Q_1-completion if $q=0$ or it is a complete digraph.

The digraphs $D_2(q,n), q=1, n=1$; $D_3(q,n), q=1, n=1$; $q=2, n=2$–4; $q=3, n=2,3$, $D_4(q,n), q=1, n=1$; $q=2, n=2$–5; $q=3, n=3$–13; $q=4, n=16$–27; $q=5, n=29$–38; $q=6, n=45$–48 are asymmetric and hence each of the digraph has Q_1-completion by Theorem 5.4.

The digraphs $D_4(q,n), q=2, n=1$; $q=3, n=1$–3; $q=4, n=1$–$9,13$; $q=5, n=1$–$3,7$–$10,12,13,18,20,25,27$; $q=6, n=3$–8; $q=7, n=2$; $q=8, n=2$ have Q-completion (see [2]) and by Theorem 5.1, these digraphs have Q_1-completion.

The complement of each digraph $D_3(q,n), q=2, n=1$; $q=3, n=1,4$; $q=4, n=1$, $D_4(q,n), q=4, n=10$–$12,14,15$; $q=5, n=4$–$6,11,14$–$17,19,21$–$24,26,28$; $q=6, n=1,9,11,16$–$22,24,25,27,28,31,33,34,37$–$39,41,42,44$; $q=7, n=6,11$–$13,17$–$20,30,32,35,38$; $q=8, n=13$ is pseudo-stratified, hence by Theorem 4.5, they have Q_1-completion.

The digraphs $D_4(q,n), q=6, n=10,12$–$15,23,26,29,32,35,36$; $q=7, n=1,3,7$–$10,14,21,24,25$–28; $q=8, n=3,4,8,9$; $q=9, n=3$ satisfies the statement of the Theorem 4.11, hence they have Q_1-completion. □

Remark 6.5. *In this paper, the Q_1-matrix completion is discussed. A few necessary necessary and sufficient conditions for a digraph to have Q_1-completion are obtained. But a strong necessary and sufficient condition is still needed. Also most of the digraphs among 218 digraphs of order 4 are classified according to Q_1-completion. The following digraphs $D_p(q,n)$, $1 \le p \le 4$ are not classified according to the Q_1-completion.*

$$\begin{array}{lll} p=4; & q=6; & n=30, \\ & q=7; & n=15,23 \\ & q=8; & n=6. \end{array}$$

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