The $Q_1$-matrix completion problem

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Abstract
A matrix is a $Q_1$-matrix if it is a $Q$-matrix with positive diagonal entries. A digraph $D$ is said to have $Q_1$-completion if every partial $Q_1$-matrix specifying $D$ can be completed to a $Q_1$-matrix. In this paper, necessary and sufficient conditions for a digraph to have $Q_1$-completion are obtained. Later on the relationship among the completion problem of $Q_1$-matrix and some other class of matrices are discussed. Finally, the digraphs of order at most four that include all loops and have $Q_1$-completion are characterized.

Keywords
Partial matrix, Matrix completion, $Q_1$-matrix, $Q_1$-completion, Digraph.

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1. Introduction
A number of researchers studied matrix completion problems for different classes of matrices ([5–10]). The $P_0$-matrix and $P_{0,1}$-completion are studied in [7, 11]. A real $n \times n$ matrix $B_1$ is a $P_0$-matrix ($P$-matrix) if every principal minor of $B_1$ is nonnegative (positive). The matrix $B_1$ is a $P_{0,1}$-matrix if all diagonal entries of $B_1$ are positive and $B_1$ is itself a $P_0$-matrix. In 2009, DeAlba et al. [2] solved the $Q$-matrix completion problem. In this paper it is seen that the property of being a $Q$-matrix is not inherited by principal submatrices, thus the $Q$-matrix completion problem is significantly different from the completion problems studied earlier. One may see [3] for a survey of matrix completion results.

A partial matrix is a rectangular array of numbers in which some entries are specified, while others are free to be chosen. For $\alpha \subseteq \{1, \ldots, n\}$, the principal submatrix $B[\alpha]$ is obtained by deleting from $B$ all rows and columns whose indices are not in $\alpha$. A principal minor is the determinant of a principal submatrix. A pattern for $n \times n$ matrices is a subset of $\{1, \ldots, n\} \times \{1, \ldots, n\}$. A partial matrix specifies a pattern if its specified entries lie exactly in those positions listed in the pattern.

For a given class $\Gamma$ of matrices (e.g., $P_0$, $P$ or $Q$-matrices) a partial $\Gamma$-matrix is a partial matrix for which the specified entries satisfy the properties of a $\Gamma$-matrix. Thus, a partial $Q$-matrix is a partial matrix $M$ in which $S_k(M) > 0$ for every $k \in \{1, 2, \ldots, n\}$ for which all $k \times k$ principal submatrices are fully specified. Similarly a partial $Q_1$-matrix is a partial $Q$-matrix with all specified positive diagonal entries.

A completion of a partial matrix is a specific choice of val-
ues for the unspecified entries. A matrix completion problem asks which partial matrices have completions with a given property. A \( \Gamma \)-completion of a partial \( \Gamma \)-matrix \( M \) is a completion of \( M \) which is a \( \Gamma \)-matrix. The \( \Gamma \)-matrix completion problem studies the properties and classifications of patterns having \( \Gamma \)-completions.

### 1.1 Digraphs

It is observed from the history of matrix completion problems that Graph theory and Matrix completion problems are correlated with each other. Graph theoretic techniques are seen to be very fruitful to solve the matrix completion problems. Any standard reference, for example, [1] and [4] can be used for graph theoretic terminologies. A directed graph or digraph \( D = (V_D,A_D) \) of order \( n > 0 \) is a finite nonempty set \( V_D \), with \( |V_D| = n \) of objects called vertices together with a (possibly empty) set \( A_D \) of ordered pairs of vertices, called arcs or directed edges. We write \( v \in D \) (resp. \( (u,v) \in D \)) to imply \( v \in V_D \) (resp. \( (u,v) \in A_D \)). If \( x = (u,u) \), then \( x \) is called a loop at the vertex \( u \).

A symmetric edge of \( D \) is a pair of arcs \( \{(u,v),(v,u)\} \subseteq A_D \), usually written as \( \{u,v\} \). A (directed) \( u-v \) path \( P \) of length \( k \geq 0 \) in \( D \) is an alternating sequence \( (u = v_0,x_1,v_1,\ldots,x_k,v_k = v) \) of vertices and arcs, where \( v_i, 1 \leq i \leq k \), are distinct vertices and \( x_i = (v_{i-1},v_i) \). Then, the vertices \( v_i \) and the arcs \( x_i \) are said to be on \( P \). Further, if \( k \geq 2 \) and \( u = v \), then a \( u-v \) path is a cycle of length \( k \). We then write \( C_k = (v_1,x_2,\ldots,v_k) \) and call \( C_k \) a \( k \)-cycle in \( D \).

A cycle \( C \) is even (resp. odd) if its length is even (resp. odd). A digraph \( H = (V_H,A_H) \) is a subdigraph of order \( k \) of the digraph \( D \) if \( |V_H| = k \) and \( V_H \subseteq V_D, A_H \subseteq A_D \). A digraph \( D \) is said to be connected (resp. strongly connected) if for every pair \( u,v \) of vertices, \( D \) contains a \( u-v \) path (resp. both a \( u-v \) path and a \( v-u \) path). The maximal connected (resp. strongly connected) subdigraphs of \( D \) are called components (resp. strong components) of \( D \).

A subdigraph \( H \) of \( D \) is an induced subdigraph if \( A_H = (V_H \times V_H) \cap A_D \) (induced by \( V_H \)) and is a spanning subdigraph if \( V_H = V_D \). Again for \( v \in V_D, D - v \) denotes the subdigraph of \( D \) induced by \( V_D \setminus \{v\} \). The complement of a digraph \( D \) is the digraph \( \overline{D} \), where \( V_{\overline{D}} = V_D \) and \( (v,w) \in A_{\overline{D}} \) if and only if \( (v,w) \notin A_D \). A digraph \( D \) is said to be symmetric if \( (u,v) \in D \) implies \( (v,u) \in D \). On the other hand, \( D \) is asymmetric if \( (u,v) \in D \) implies \( (v,u) \notin D \). A complete symmetric digraph on \( n \) vertices, denoted by \( K_n \), is the digraph having all possible arcs (including all loops).

Two digraphs \( D_1 = (V_1,A_1) \) and \( D_2 = (V_2,A_2) \) are isomorphic, if there is a bijection \( \phi : V_1 \rightarrow V_2 \) such that \( A_2 = \{ (\phi(u),\phi(v)) : (u,v) \in A_1 \} \). An unlabelled digraph is an equivalent class of isomorphic digraphs. Choosing a particular member of an unlabelled digraph is referred as a labelling of the unlabelled digraph.

### 1.2 Digraphs with matrices

Let \( \pi \) be a permutation of a nonempty finite set \( V \). The digraph \( D_\pi = (V,A_\pi) \), where \( A_\pi = \{ (v,\pi(v)) : v \in V \} \) is called a permutation digraph. Clearly, each component of a permutation digraph is a loop or a cycle.

A permutation subdigraph \( H \) (of order \( k \)) of a digraph \( D \) is a permutation digraph that is a subdigraph of \( D \) (of order \( k \)). A digraph \( D \) is stratified if \( D \) has a permutation subdigraph of order \( k \) for every \( k = 2,3,\ldots,|D| \). A digraph \( D \) is said to be pseudo-stratified if there exist a vertex \( v \) in \( D \) such that \( D - v \) is stratified.

Let \( B = [b_{ij}] \) be an \( n \times n \) matrix. We have

\[
\det(B) = \sum (\text{sgn}\pi)b_{1\pi(1)}\cdots b_{n\pi(n)}
\]

where the sum is taken over all permutations \( \pi \) of \( \langle n \rangle = \{1,2,\ldots,n\} \).

A signing of a digraph is an assignment of a sign \( + \) or \( - \) to each arc of the digraph. The result of signing of a digraph is called a signed digraph. For an arc \( e \in D \), by \( s(e) \) we mean \( e \) has sign \( s(e) \).

For a \( k \)-cycle in \( C_k \) in \( D \), the sign \( s(C_k) \) is defined to be,

\[
s(C_k) = (-1)^{k+1} \prod_{e \in C_k} s(e)
\]

For a permutation subdigraph \( K \) of \( D \), the sign \( s(K) \) of \( K \) is

\[
s(K) = \prod_{C \in K} s(C)
\]

### 2. Partial \( Q_1 \)-matrix and the \( Q_1 \)-matrix completion problem

A partial \( Q_1 \)-matrix is a partial \( Q \)-matrix in which all specified diagonal entries are positive i.e. a partial \( Q_1 \)-matrix is a partial matrix \( M \) with all specified positive diagonal entries and \( S_k(M) > 0 \) for every \( k \in \{1,2,\ldots,n\} \), whenever all \( k \times k \) principal submatrices are fully specified. Now, a partial \( Q_1 \)-matrix is characterized as follows.

**Proposition 2.1.** Suppose \( M = [a_{ij}] \) is a partial matrix. Then \( M \) is a partial \( Q_1 \)-matrix if and only if exactly one of the following holds:

(i) At least one diagonal entry of \( M \) is unspecified, all specified diagonal entries are positive.

(ii) All diagonal entries are specified and positive; at least one off-diagonal entry is unspecified.

(iii) All entries of \( M \) are specified and \( M \) is a \( Q_1 \)-matrix.

A completion \( B \) of a partial \( Q_1 \)-matrix \( M \) is called a \( Q_1 \)-completion of \( M \), if \( B \) is a \( Q_1 \)-matrix. Since any matrix which is permutation similar to a \( Q_1 \)-matrix is a \( Q_1 \)-matrix, it is evident that if a partial \( Q_1 \)-matrix \( M \) has a \( Q_1 \)-completion, so does any partial matrix which is permutation similar to \( M \).

It can be easily seen that any partial matrix \( M \) with all unspecified diagonal entries has \( Q_1 \)-completion. A completion can be obtained by choosing sufficiently large values for the unspecified diagonal entries. Let \( M \) be a partial \( Q_1 \)-matrix in which the diagonal entries at \( (i,i) \) positions \( (i = k + 1,\ldots,n) \)
are unspecified. In case $M[1, \ldots, k]$ is fully specified, $M$ may not have a $Q_1$-completion. For example, the partial matrix,

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & ? \end{bmatrix},$$

where ? denotes an unspecified entry, does not have $Q$-completion. Indeed, for any completion $B$ of $M$, $S_1(B) = 0$. On the other hand, if $M[1, \ldots, k]$ has an unspecified entry and has a $Q_1$-completion, then $M$ has a $Q_1$-completion. A completion of $M$ can be obtained by choosing sufficiently large values for the unspecified diagonal entries. These above observations are listed in the following results.

**Theorem 2.2.** If a matrix $M$ omits all diagonal entries, then $M$ has $Q_1$-completion.

**Proof.** Suppose $M = [a_{ij}]$ be a partial $Q_1$-matrix. For any $t > 1$, consider a completion $B = [b_{ij}]$ of $M$ by setting all diagonal entries equal to $t$ and rest of the off diagonal entries to be equal to zero. Then, any $k \times k$ principal minor will be of the form $t^k + p(t)$, where $p(t)$ is a polynomial of degree $\leq k - 1$. Now by choosing $t$ large enough, we have $S_k(B) > 0$ for all $k \times k$ principal minors of $B$. Since only finitely many principal minors are to be considered, thus for sufficiently large $t$, $M$ has $Q_1$-completion. \qed

**Theorem 2.3.** Suppose $M$ be a partial $Q_1$-matrix in which the diagonal entry at $(r + 1, r + 1)$ position is unspecified. If the principal submatrix $M[1, \ldots, r]$ of $M$ is not fully specified and has $Q_1$-completion, then $M$ has $Q_1$-completion.

**Proof.** Suppose $M = [a_{ij}]$ be a partial $Q_1$-matrix in which the diagonal entry at $(r + 1, r + 1)$ position is unspecified. Then, $M$ is of the form,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where, $M_{11} = M[1, \ldots, r]$ and $M_{22} = M[r + 1, r + 1]$.

Let $A_1$ be the $Q_1$-matrix completion of $M[1, \ldots, r]$. Then,

$$M' = \begin{bmatrix} A_1 & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

is a partial $Q_1$-matrix, since $M_{22}$ has an unspecified diagonal entry. Now for $t > 0$, consider a completion $B = [b_{ij}]$ of $M'$ obtained by choosing $b_{ii} = t$, $i = r + 1$ and $b_{ij} = 0$ against all other unspecified entries in $M'$. Then $B$ is of the form,

$$B = \begin{bmatrix} A_1 & B_{12} \\ B_{21} & t \end{bmatrix}.$$ 

Since $A_1$ is a $Q_1$-matrix, $S_i(A_1) > 0$ for $1 \leq i \leq r$. For $2 \leq j \leq r + 1$, $S_j(B) = S_j(A_1) + tS_{j-1}(A_1) + s_j$, where $s_j$ is a constant. Now $S_j(B) > 0$ for sufficiently large values of $t$ and clearly $B$ is $Q_1$-matrix. \qed

**Corollary 2.4.** Suppose $M$ be a partial $Q_1$-matrix in which the diagonal entries at $(i, i)$ positions $(i = r + 1, \ldots, n)$ are unspecified. If the principal submatrix $M[1, \ldots, r]$ of $M$ is not fully specified and has $Q_1$-completion, then $M$ has $Q_1$-completion.

The converse of Corollary 2.4 is not true which can be seen from the following example.

**Example 2.5.** Consider the partial matrix,

$$M = \begin{bmatrix} d_1 & a_{12} & a_{13} & ? \\ a_{21} & d_2 & ? & ? \\ a_{31} & a_{32} & d_3 & ? \\ ? & ? & ? & ? \end{bmatrix},$$

where ? denotes the unspecified entries. We show that for any choice of values of the specified entries $M$ has $Q_1$-completions, though there are occasions when $M[1, 2, 3]$ need not have $Q_1$-completion. For $t > 0$, consider the completion $B(t)$ of $M$ defined as follows:

$$B(t) = \begin{bmatrix} d_1 & a_{12} & a_{13} & 0 \\ a_{21} & d_2 & t & 0 \\ a_{31} & a_{32} & d_3 & t \\ t & t & t & t \end{bmatrix}.$$ 

Then,

$$S_1(B(t)) = t + \sum d_i,$$

$$S_2(B(t)) = t^2 + f_1(t),$$

$$S_3(B(t)) = t^3 + f_2(t),$$

$$S_4(B(t)) = d_1t^3 + f_2(t),$$

where $f_i(t)$ is a polynomial in $t$ of degree at most $i$, $i = 1, 2$. Consequently, $B(t)$ is a $Q_1$-matrix for sufficiently large $t$, and therefore $M$ has $Q_1$-completion. On the other hand, the partial $Q_1$-matrix

$$M[1, 2, 3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & ? \\ 1 & 1 & 1 \end{bmatrix},$$

with unspecified entries ?, is the principal submatrix of $M$ induced by its diagonal $\{1, 2, 3\}$. Now one can verify that $M[1, 2, 3]$ does not have $Q_1$-completion, because $\det(M[1, 2, 3]) = 0$ for any completion of $M[1, 2, 3]$.

### 3. Digraphs and $Q_1$-completions

It can be easily seen that an $n \times n$ partial matrix $M$ specifies a digraph $D = \langle (n), A_D \rangle$ if for $1 \leq i, j \leq n$, $(i, j) \in A_D$ if and only if the $(i, j)$-th entry of $M$ is specified. For example, the partial $Q_1$-matrix $M$ in Example 2.5 specifies the digraph $D$ in Figure 1.

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Image: Tree symbol.
Theorem 3.1. Suppose $M$ is a partial $Q_1$-matrix specifying the digraph $D$. If the partial submatrix of $M$ induced by every strongly connected induced subdigraph of $D$ has $Q_1$-completion, then $M$ has $Q_1$-completion.

Proof. We prove the result for the case when $D$ has two strong components $D_1$ and $D_2$. The general result will then follow by induction. By a relabeling of the vertices of $D$, if required, we have

$$M = \begin{pmatrix} M_{11} & M_{12} \\ X & M_{22} \end{pmatrix},$$

where $M_{ii}$ is a partial $Q_1$-matrix specifying $D_i$, $i=1,2$, and all entries in $X$ are unspecified. By the hypothesis, $M_0$ has a $Q_1$-completion $B_0$. Consider the completion

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

by choosing all entries in $X$ as well as all unspecified entries in $M_{12}$ as 0. Then, for $2 \leq k \leq |D|$ we have,

$$S_k(B) = S_k(B_{11}) + S_k(B_{22}) + \sum_{i=1}^{k-1} S_r(B_{11})S_k-s(B_{22}) \geq 0,$$

Here, we mean $S_k(B_{ii}) = 0$ whenever $k$ exceeds the size of $B_{ii}$. Thus $M$ can be completed to a $Q_1$-matrix. \qed

The proof of the following result is similar.

Theorem 3.2. Suppose $M$ is a partial $Q_1$-matrix specifying the digraph $D$. If the partial submatrix of $M$ induced by each component of $D$ has a $Q_1$-completion, then $M$ has a $Q_1$-completion.

The converse of Theorem 3.1 is not true. For example, every partial $Q_1$-matrix specifying the digraph $D$ in Figure 1 has $Q_1$-completion, although the strong component $D'$ induced by vertices $\{1,2,3\}$ does not have $Q_1$-completion (see Example 3.3).

Example 3.3. Consider the digraph $D$ in the Figure 1. We show that $D$ has $Q_1$-completion, but the strong component $D'$ induced by vertices $\{1,2,3\}$ does not have $Q_1$-completion.

The Digraph $D$

Let $M = [a_{ij}]$ be a partial $Q_1$-matrix specifying $D$. Then for $t > 0$, $M$ can be completed to a $Q_1$-matrix $B(t)$ but the principal submatrix induced by the digraph $D'$ i.e. $M[1,2,3]$ does not have $Q_1$-completion (see Example 2.5). To see that $M[1,2,3]$ does not have $Q_1$-completion, consider the partial $Q_1$-matrix

$$M[1,2,3] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & x \\ 1 & 1 & 1 \end{pmatrix},$$

with unspecified entry $x$. Then for any $Q_1$-completion $B$ of $M[1,2,3]$, we have $S_3(B) = 0$ and hence $M[1,2,3]$ does not have $Q_1$-completion.

The property of having $Q_1$-completion is not inherited by induced subdigraphs. This can be also seen from the Example 2.5.

4. The $Q_1$-matrix completion problem

We say that a digraph $D$ has $Q_1$-completion, if every partial $Q_1$-matrix specifying $D$ can be completed to a $Q_1$-matrix. The $Q_1$-matrix completion problem aims at studying and classifying all digraphs $D$ which have $Q_1$-completion.

The property of being a $Q_1$-matrix is preserved under similarity and transposition, but it is not inherited by principal submatrices, as it can easily be verified. Also it is clear that if a digraph $D$ has $Q_1$-completion, then any digraph which is isomorphic to $D$ has $Q_1$-completion.

4.1 Sufficient conditions for $Q_1$-matrix completion

Theorem 4.1. If a digraph $D \neq K_n$ of order $n$ has $Q_1$-completion, then any spanning subdigraph of $D$ has $Q_1$-completion.

Proof. Suppose $H$ be a spanning subdigraph of $D$ and $M_H$ be a partial $Q_1$-matrix specifying the digraph $H$. Consider a partial matrix $M_D$ obtained from $M_H$ by specifying the entries corresponding to $(i,j) \in A_D \setminus A_H$ as 0. Since $D \neq K_n$, by Proposition 2.1, $M_D$ is a partial $Q_1$-matrix specifying $D$. Let $B$ be a $Q_1$-completion of $M_D$. Clearly, $B$ is a $Q_1$-completion of $M_H$. \qed

Theorem 4.2. Suppose $D \neq K_n$ be a digraph such that $\overline{D}$ is stratified. If it is possible to sign the arcs of $\overline{D}$ so that the sign of every cycle is of positive sign, then $D$ has $Q_1$-completion.

Proof. Suppose $M$ be a partial $Q_1$-matrix specifying the digraph $D$. For any $t > 0$, consider a completion $B$ of $M$ by choosing the unspecified entry $x_{ij} = \text{sign}(i,j)t$ (using the sign of the arc in $\overline{D}$). Then for each $k = 2,3,\ldots,n$ we have,

$$S_k(B) = c_k t^k + r_k(t)$$

where $c_k$ is the number of permutation subdigraphs of order $k$ in $D$ and $r_k(t)$ is a polynomial of degree less than $k$. If $D$ contains all loops, then the trace of any partial $Q_1$-matrix specifying $D$ is positive; if $D$ omits a loop, then $S_1(B) = c_1 t + r_0$, where $c_1$ is the number of loops in $D$ and $r_0 \in R$. Now by choosing $t$ sufficiently large, $B$ becomes a $Q_1$-matrix. \qed

Example 4.3. Consider the digraph $D_0$ and its complement $\overline{D_0}$ in Figure 2. It can be easily seen that the digraph $\overline{D_0}$ is stratified. Also it is possible to sign the arcs of $\overline{D_0}$ so that
every cycle in $\overline{D_0}$ is of positive sign. Thus by Theorem 4.2, the digraph $D_0$ has $Q_1$-completion.

However the converse of the Theorem 4.2 is not true which can be seen from the Example 4.6. Although the complement of the digraph $\overline{D}$ is not stratified, but it has $Q_1$-completion [See Example 4.6].

**Corollary 4.4.** Suppose $D \neq K_n$ be a digraph such that $\overline{D}$ has a stratified spanning subdigraph $D_1$. If $D_1$ has a signing in which the sign of every cycle is $+$, then $D$ has $Q_1$-completion.

**Theorem 4.5.** Suppose $D \neq K_n$ be a digraph with all loops such that $\overline{D}$ is pseudo-stratified. If it is possible to sign the arcs of $\overline{D}$ so that the sign of every cycle is of positive sign, then $D$ has $Q_1$-completion.

**Proof.** Let $M = [a_{ij}]$ be a partial $Q_1$-matrix specifying the digraph $D$. Since $\overline{D}$ is pseudo-stratified, there exists a subdigraph $D_1$ of $\overline{D}$ of order $n - 1$ such that $D_1$ is stratified. Suppose $D_1$ obtained from $\overline{D}$ by deleting a vertex say $v_1$ in $\overline{D}$ i.e. $D_1 = \overline{D} - v_1$. For $t > 0$, consider a completion $B(t) = [b_{ij}]$ of $M$ by choosing the unspecified entries as following:

$$b_{ij} = \begin{cases} \text{sgn}(i,j)t, & \text{for } (i,j) \in D_1 \\ 0, & \text{otherwise.} \end{cases}$$

where $\text{sgn}(i,j)$ denotes the sign of the arcs of $\overline{D}$. Since $M$ is a partial $Q_1$-matrix with all specified diagonal entries, thus $d_i > 0, \forall i = 1,2,\ldots,n$. Now we have,

$$S_k(B) = c_k t^k + f_k(t), \quad k \in \{2,3,\ldots,n-1\}$$

$$S_n(B) = d_1 c_n t^{n-1} + f_{n-1}(t)$$

where $c_k$ is the number of permutation subdigraphs of order $k$ in $D$ and $f_k(t)$ is a polynomial of degree less than $k$. Now choosing a sufficiently large value of $t$, we have $S_k(B) > 0, \forall k \in \{1,2,\ldots,n\}$ and hence $B(t)$ is $Q_1$-matrix.

**Example 4.6.** Consider the digraph $\overline{\overline{D}} \neq K_4$ in Figure 3. The complement $\overline{D}$ of the digraph $\overline{D}$ is not stratified although $\overline{D}$ has $Q_1$-completion. Since $\overline{D}$ is pseudo-stratified digraph of order 4, thus it has a stratified subdigraph $D_1$ of order 3 which is obtained by deleting the vertex 2 from $\overline{D}$. Now $\overline{D}$ satisfies the statement of the Theorem 4.5, hence $\overline{D}$ has $Q_1$-completion. To see this consider a partial $Q_1$-matrix

![Figure 2. The Digraphs $D_0$ and $\overline{D_0}$](image)

![Figure 3. The Digraphs $\overline{D}$ and $\overline{\overline{D}}$](image)

$$M = \begin{bmatrix} d_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ -a_{1n} & -a_{2n} & -a_{3n} & \cdots & d_n \end{bmatrix},$$

specifying the digraph $\overline{D}$ with unspecified entries as $?$. For $t > 0$, consider a completion $B$ of $M$ as follows:

$$B = \begin{bmatrix} d_1 & a_{12} & a_{13} & t \\ 0 & d_2 & a_{23} & 0 \\ t & a_{32} & d_3 & -t \\ 0 & 0 & t & d_4 \end{bmatrix}.$$
Let \( D = \text{diag}[d_1, d_2, \ldots, d_n] \). Then \( D > 0 \). We can write \( B = B_0 + D \), where \( B_0 \) is a skew-symmetric real matrix. Since a skew symmetric real matrix is a \( B_0 \)-matrix [7], hence \( S_k(B_0) \geq 0 \) for every \( k \in \{1, 2, \ldots, n\} \). On the other hand, being a positive matrix \( D \), we have \( \forall k \in \{1, 2, \ldots, n\}, S_k(D) > 0 \) and as a result \( S_2(B) > 0 \). Hence the result follows.

**Example 4.8.** Consider the digraph \( D_2 \) in Figure 4. The complement \( \overline{D}_2 \) of \( D_2 \) is neither stratified nor pseudo-stratified, but it has \( Q_1 \)-completion by Theorem 4.7.

![Figure 4. The Digraphs \( D_2 \) and \( \overline{D}_2 \)](image)

**Example 4.9.** The converse of the Theorem 4.7 is not true which can be seen from the Example 3.3. Although the digraph \( D \) is not asymmetric, the digraph \( \overline{D} \) has \( Q_1 \)-completion. From Example 2.5, it is seen that any partial \( Q_1 \)-matrix \( M \) specifying \( D \) can be completed to a \( \overline{Q}_1 \)-matrix.

**Corollary 4.10.** A digraph \( D \) has \( Q_1 \)-completion if it does not contain a cycle of even length.

**Theorem 4.11.** Suppose \( D \neq K_n \) be a digraph with all loops. Suppose \( \overline{D} \) contains a 2-cycle \( \langle v_1, v_2 \rangle \) such that \( \langle v_1, v_2 \rangle \) does not form a permutation subdigraph of order 4 with any 2-cycle in \( D + \langle v_1, v_2 \rangle \). Then \( D \) has \( Q_1 \)-completion.

**Proof.** Suppose \( M = [a_{ij}] \) be a partial \( Q_1 \)-matrix specifying the digraph \( D \). For any \( i > 0 \), consider a completion \( B \) of \( M \) by choosing the unspecified entries as following:

\[
b_{ij} = \begin{cases} 
  t, & \text{if } (i, j) = (v_1, v_2) \in \overline{D} \\
  -t, & \text{if } (i, j) = (v_2, v_1) \in \overline{D} \\
  0, & \text{otherwise.}
\end{cases}
\]

Then we have,

\[
S_1(B) = d_1 + d_2 + d_3 + d_4
\]
\[
S_2(B) = t^2 + f_1(t, a_{ij})
\]
\[
S_3(B) = (d_3 + d_4)t^2 + f_1(t, a_{ij})
\]
\[
S_4(B) = d_3d_4t^2 + f_1(t, a_{ij}),
\]

where \( f_1(t, a_{ij}) \) is a polynomial in \( t \) of degree at most 1. Since \( M \) is a partial \( Q_1 \)-matrix with all specified diagonal entries, thus we have \( S_1(B) > 0 \). Now choosing \( t \) sufficiently large, we have \( S_i(B) > 0 \) for all \( i = 2, 3, 4 \). Hence the result follows.

**Example 4.12.** Consider the digraph \( D_3 \) in Figure 5. The complement \( \overline{D}_3 \) of \( D_3 \) contains a 2-cycle \( \langle 1, 3 \rangle \) which does not form a permutation subdigraph of order 4 with any 2-cycle in \( D_3 + \langle 1, 3 \rangle \). Thus by Theorem 4.11, \( D_3 \) has \( Q_1 \)-completion.

![Figure 5. The Digraphs \( D_3 \) and \( \overline{D}_3 \)](image)

However the converse of the Theorem 4.11 is not true which can be seen from the Example 4.8. Although the digraph \( \overline{D}_2 \) in Figure 4 does not have 2-cycle but \( D_2 \) has \( Q_1 \)-completion.

### 4.2 Necessary conditions for \( Q_1 \)-matrix completion

In this section we will discuss some necessary conditions for a digraph to have \( Q_1 \)-completion.

**Theorem 4.13.** If a digraph \( D \neq K_n \) of order \( n \geq 2 \) contains two vertices \( v_1 \) and \( v_2 \) with indegree or outdegree \( n \), then \( D \) does not have \( Q_1 \)-completion.

**Proof.** Suppose a digraph \( D \) of order \( n \geq 2 \) contains two vertices \( v_1 \) and \( v_2 \) with indegree or outdegree \( n \). Consider a partial \( Q_1 \)-matrix \( M \) specifying \( D \) with all specified entries are exactly 1. Then two columns or rows of \( M \) are equal and for any completion \( B \) of \( M \), we have \( \det B = 0 \). Hence the result follows.

**Theorem 4.14.** Suppose \( D \neq K_n \) be a digraph with all loops such that \( D \) is asymmetric. If \( D \) contains a 2-cycle \( \langle v_1, v_2 \rangle \), then \( D \) does not have \( Q_1 \)-completion.

**Proof.** Suppose that \( D \) has a 2-cycle \( \langle v_1, v_2 \rangle \). Consider a partial \( Q_1 \)-matrix \( M = [a_{ij}] \) specifying \( D \) such that \( d_i = 1 \) \((1 \leq i \leq n)\) and \( a_{v_1v_2}a_{v_2v_1} > \binom{n}{2} \) and rest of all specified entries are zero. Let \( B = [b_{ij}] \) be any completion of \( M \). Then

\[
S_2(B) = \sum_{i \neq j} d_id_j - \sum_{i \neq j} b_{ij}b_{ji} < - \sum_{i,j \notin \{v_1,v_2\}} b_{ij}b_{ji} < 0,
\]

and, therefore, \( B \) is not a \( Q_1 \)-matrix.

**Example 4.15.** Consider the digraph \( D_4 \) in Figure 6. The complement \( \overline{D}_4 \) of \( D_4 \) is asymmetric and \( D_4 \) has 2-cycle \( \langle 1, 3 \rangle \).
Thus by Theorem 4.14, $D_4$ does not have $Q_1$-completion. To see this consider a partial $Q_1$-matrix

$$M = \begin{bmatrix} 1 & ? & 10 & 0 \\ 0 & 1 & ? & 0 \\ 10 & 0 & 1 & 0 \\ ? & ? & ? & 1 \end{bmatrix},$$

specifying the digraph $D_4$. Then for any completion $B$ of $M$, we have $S_2(B) < 0$. Hence, $M$ cannot be completed to a $Q_1$-matrix.

![Figure 6. The Digraphs $D_4$ and $D_4'$](image)

### 5. Relationship theorems

#### 5.1 $Q$-completion and $Q_1$-completion

It is easily seen that a $Q_1$ matrix is a $Q$-matrix but not vice versa. Thus the completion problem of these two classes are partially related.

**Theorem 5.1.** If a digraph $D$ has $Q$-completion, then it must also have $Q_1$-completion.

**Proof.** Suppose $D$ be a digraph that has $Q$-completion and $M$ be a partial $Q_1$-matrix specifying the digraph $D$. Thus $M$ is a partial $Q$-matrix specifying the digraph $D$. Since $D$ has $Q$-completion, thus $M$ can be completed to a $Q$-matrix $B$. Now if $D$ includes all loops, then we are done i.e. $B$ is a $Q_1$-matrix. If $D$ omits at least one loop, then $M$ has at least one unspecified diagonal entry. Now we choose positive numbers to those unspecified diagonal entries of $M$. In that case $M$ is also a partial $Q$-matrix and it can be completed to a $Q$-matrix $B$ which is also a $Q_1$-matrix.

However the converse of the Theorem 5.1 is not true which can be seen from the following example.

**Example 5.2.** The digraph $D_3$ in Figure 5 has $Q_1$-completion (see Example 4.12). But the digraph $D_3$ does not have $Q$-completion (By Theorem 2.8, [2]).

#### 5.2 $P$-completion and $Q_1$-completion

Since a $P$-matrix is a $Q_1$-matrix, thus the completion problem between these two classes are also partially related.

**Theorem 5.3.** Any asymmetric digraph that has $P$-completion also has $Q_1$-completion.

**Proof.** Suppose $D$ be an asymmetric digraph that has $P$-completion and $M$ be a partial $Q_1$-matrix specifying $D$. Since $D$ is asymmetric, hence all principal submatrices of $M$ of order greater than 1 are unspecified. Also being a partial $Q_1$-matrix as well as $P$-matrix, all specified diagonal entries in $M$ are positive. Since $M$ has $P$-completion, thus consider a $P$-matrix completion $B$ of $M$. Now $B$ is also a $Q_1$-completion of $M$ and hence the result follows.

**Theorem 5.4.** Any asymmetric digraph has $Q_1$-completion.

**Proof.** Since any asymmetric digraph has $P$-completion [7], thus by Theorem 5.3 any asymmetric digraph also has $Q_1$-completion.

### 6. Classification of digraphs of small order having $Q_1$-completion

Based on the obtained results in the previous sections, we will classify the digraphs of order at most four that include all loops as to $Q_1$-completion in this section. Again permutation similarity of $Q_1$-matrix implies that if a digraph $D$ has $Q_1$-completion, then any digraph which is isomorphic to $D$ has $Q_1$-completion. Thus any digraph which is obtained by labelling the unlabelled digraph associated to $D$ has $Q_1$-completion.

The nomenclature of the digraphs considered in the sequel are indicated as per their order in the list in [4, Appendix, pp. 233]. Here, $D_p(q,n)$ is the one obtained by attaching a loop at each of the vertices to the $n$-th member in the list of digraphs with $p$ vertices and $q$ (non-loop) arcs in the list.

We will classify the digraphs into a series of following lemmas.

**Lemma 6.1.** For $1 \leq p \leq 4$, the digraphs $D_p(q,n)$ which are listed below do not have $Q_1$-completion.

- $p = 3$; $q = 4$; $n = 3, 4$
- $p = 4$; $q = 5$; $n = 1$
- $p = 4$; $q = 6$; $n = 40, 43$
- $p = 4$; $q = 7$; $n = 16, 22, 29, 36$
- $p = 4$; $q = 8$; $n = 5, 7, 10, 12, 14, 15, 18, 21, 22, 26, 27$
- $p = 4$; $q = 9$; $n = 1, 2, 5, 11, 13$
- $p = 4$; $q = 10$; $n = 1–5$
- $p = 4$; $q = 11$; $n = 1$.

**Proof.** Each of the digraph listed above satisfies the Theorem 4.13 and hence the result follows.

**Lemma 6.2.** For $1 \leq p \leq 4$, the digraphs $D_p(q,n)$ which are listed below do not have $Q_1$-completion.

- $p = 3$; $q = 4$; $n = 2$
- $p = 4$; $q = 7$; $n = 31, 33, 34, 37$
- $p = 4$; $q = 8$; $n = 16, 17–19, 20, 23–25$
- $p = 4$; $q = 9$; $n = 4, 12$.

**Proof.** Each of the digraph satisfies the theorem 4.14 and hence the result follows.
Lemma 6.3. For $1 \leq p \leq 4$, the digraphs $D_p(q,n)$ which are listed below do not have $Q_1$-completion.

- $p = 4$; $q = 6; n = 2$
- $q = 7; n = 4,5$
- $q = 8; n = 1,11$

Proof. Suppose $M = \begin{bmatrix} 1 & 1 & 1 & ? \\ 1 & 1 & 1 & ? \\ ? & ? & ? & 1 \end{bmatrix}$, be a partial $Q_1$ matrix specifying the digraph $D_2(6,2)$ with unspecified entries as $?$. But $M$ cannot be completed to a $Q_1$-matrix since for any completion $B$ of $M$, we have $\det B = 0$. Again any digraph listed in the Lemma 6.3 contains the digraph $D_2(6,2)$ as an induced subgraph, hence the result follows.

Theorem 6.4. For $1 \leq p \leq 4$, the digraphs $D_p(q,n)$ which are listed below have $Q_1$-completion.

- $p = 2$; $q = 0,1,2; n = 1$
- $p = 3$; $q = 0,1; n = 1$
- $q = 2,3; n = 1–4$
- $q = 4,6; n = 1$
- $p = 4$; $q = 0,1; n = 1$
- $q = 2; n = 1–5$
- $q = 3; n = 1–13$
- $q = 4; n = 1–27$
- $q = 5; n = 1–38$
- $q = 6; n = 1–39,41,42,45–48$
- $q = 7; n = 1–5,16–21,24–28,30,32,35,38$
- $q = 8; n = 2–4,8,9$
- $q = 9; n = 3$
- $q = 12; n = 1$. 

Proof. It can be easily seen that $D_p(q,n)$ has $Q_1$-completion if $q = 0$ or it is a complete digraph.

The digraphs $D_2(q,n), q = 1, n = 1; D_3(q,n), q = 1, n = 1; q = 2, n = 2–4; q = 3, n = 3, 4; D_4(q,n), q = 1, n = 1; q = 2, n = 2–5; q = 3, n = 3–13; q = 4, n = 16–27; q = 5, n = 29–38; q = 6, n = 45–48 are asymmetric and hence each of the digraph has $Q_1$-completion by Theorem 5.4.

The digraphs $D_4(q,n), q = 2, n = 1; q = 3, n = 1–3; q = 4, n = 1–9, 13; q = 5, n = 1–3,7–10,12,13,18,20,25,27; q = 6, n = 3–8; q = 7, n = 2; q = 8, n = 2$ have $Q$-completion (see [2]) and by Theorem 5.1, these digraphs have $Q_1$-completion.

The complement of each digraph $D_3(q,n), q = 2, n = 1; q = 3, n = 1,4; q = 4, n = 1; D_4(q,n), q = 4, n = 10–12,14,15; q = 5, n = 4–6,11,14–17,19,21–24,26,28; q = 6, n = 1,9,11,16–22,24,25,27,28,31,33,34,37–39,41,42,44; q = 7, n = 6,11–13,17–20,30,32,35,38; q = 8, n = 13 is pseudo-stratified, hence by Theorem 4.5, they have $Q_1$-completion.

The digraphs $D_4(q,n), q = 6, n = 10,12–15,23,26,29,32,35,36; q = 7, n = 1,3,7–10,14,21,24,25–28; q = 8, n = 3,4,8,9; q = 9, n = 3$ satisfies the statement of the Theorem 4.11, hence they have $Q_1$-completion.

Remark 6.5. In this paper, the $Q_1$-matrix completion is discussed. A few necessary necessary and sufficient conditions for a digraph to have $Q_1$-completion are obtained. But a strong necessary and sufficient condition is still needed. Also most of the digraphs among 218 digraphs of order 4 are classified according to $Q_1$-completion. The following digraphs $D_p(q,n), 1 \leq p \leq 4$ are not classified according to the $Q_1$-completion.

- $p = 4$; $q = 6; n = 30$
- $q = 7; n = 15,23$
- $q = 8; n = 6$. 

References


