Operators in 2-fuzzy $n - n$ inner product space

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Abstract
In this paper various 2-fuzzy operators are introduced in 2-fuzzy $n - n$ inner product space and the properties of 2-fuzzy self-adjoint, 2-fuzzy normal, 2-fuzzy unitary and 2-fuzzy projection operators are studied.

Keywords
2-fuzzy $n - n$ inner product space, 2-fuzzy self-adjoint operator, 2-fuzzy normal operator, 2-fuzzy unitary operator, 2-fuzzy projection operator.

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1. Introduction

Gahler [4] introduced the theory of 2-norm on a linear space in 1964. In 1984 Katsaras [7] gave the notion of fuzzy norm on a linear space. Further, fuzzy normed spaces were defined in various ways by Cheng and Mordeson [2] and by Bag and Samanta [1]. R.M. Somasundaram and Thangaraj Beaula [9] introduced the notion of fuzzy 2-normed linear space, $\{F(X),N\}$. The concept of 2-inner product space was introduced by C.R. Diminnie, S. Gahler and A. White [5]. Parijat Sinha, Ghanshayam Lal and Divya Mishra introduced the concept of fuzzy 2-inner product space and the notion of $\alpha - 2$-norm in [8]. The notions of fuzzy inner product space and of fuzzy normed linear space were established in [6]. Also, Vijayabalaji and Thillaigovindan [10] introduced the fuzzy $n$-inner product space as a generalization of the concept of $n$-inner product space given by Y.J. Cho, M. Matic and J. Pecaric in [3]. Thangaraj Beaula and Daniel Evans introduced the concept of 2-fuzzy $n - n$ inner product space in [11] as an extension of [10]. In this paper operators are introduced in 2-fuzzy $n - n$ inner product space and their properties are studied.

2. Preliminaries

Definition 2.1. Let $n \in N$ and $X$ be a real linear space of dimension greater or equal to $n$. Then a real valued function $\|,\ldots,\|$ on $X^n$ is called a n-norm on $X$, if it satisfies the following four properties

\begin{itemize}
  \item[i)] $\|x_1,\ldots,x_n\| = 0$ if and only if $x_1,\ldots,x_n$ linearly dependent.
  \item[ii)] $\|x_1,\ldots,x_n\|$ is invariant under any permutation
  \item[iii)] $\|ax_1,\ldots,ax_n\| = \|x_1,\ldots,x_n\|$, for any $a$ is a real number
  \item[iv)] $\|x_1,\ldots,x_{n-1},y+z\| \\
             \leq \|x_1,\ldots,x_{n-1},y\| + \|x_1,\ldots,x_{n-1},z\|
\end{itemize}

The pair $(X,\|,\ldots,\|)$ is called a n-normed linear space.

Definition 2.2. Let $X$ be a nonempty set, let $F(X)$ be the set of all fuzzy sets in $X$ and let $K$ be the field of real numbers. Then $F(X)$ becomes a linear space over the field $K$, where the addition and scalar multiplication are defined by $f + g = \{(x,y)\} = \{x+y,\mu \wedge \eta: (x,y) \in f \text{ and } (y,\eta) \in g\}$ and $kf = \{(kf,y)\}: (x,y) \in f, k \in K$. 

The linear space $F(X)$ is said to be a normed space, if, every $f \in F(X)$, is associated with a non-negative real number $\|f\|$ called the norm of $f$ in such a way that

(i) $\|f\| = 0$, if and only if $f = 0$. For $\|f\| = 0$ $\Leftrightarrow \{ \|(x, \mu)/\mu \in \mathbb{R} \}$ then $\mathbb{R}$ for all $t$ $\Leftrightarrow x = 0, \mu \in (0, 1) \Rightarrow f = 0, \|(x, \mu)\|

(ii) $\|kf\| = \|k\||f||, k \in K$. For $\|kf\| = \{ \|k(x, \mu)/\mu \in \mathbb{R} \}$ then $\mathbb{R}$ for all $t$ $\Rightarrow k \in K = \{ \|k\||(x, \mu)/\mu \in \mathbb{R} \} = \|kf\|

(iii) $\|f + g\| \leq \|f\| + \|g\|$ for every $f, g \in F(X)$. For $\|f + g\|$
\[
\{ \|(x, \mu) + (y, \eta)\| : x, y \in \mathbb{R}, \mu, \eta \in (0, 1) \}
\]
\[
\{ \|(x + y), \mu \wedge \eta\|/\mu, x, y \in \mathbb{R}, \mu, \eta \in (0, 1) \}
\]
\[
\leq \{ \|(x, \mu \wedge \eta)\| + \|(y, \mu \wedge \eta)\|/\mu \in \text{and f and (y, \eta) \in g} \} = \|f\| + \|g\|.
\]

**Definition 2.3.** Let $F(X^n)$ be a linear space over a real field. A fuzzy subset $N$ of $F(X^n)$ is said to be 2-fuzzy $n$-$n$ norm if and only if

(N1) For all $t \in R, t \leq 0, N(f_1, \ldots, f_n, t) = 0$

(N2) For all $t \in R, t > 0, N(f_1, \ldots, f_n, t) = 1$ if and only if $f_1, \ldots, f_n$ are linearly dependent

(N3) $N(f_1, \ldots, f_n, t) = 0$ is invariant under any permutation of $f_1, \ldots, f_n$.

(N4) For all $t \in R, t > 0, N(f_1, \ldots, f_n, t) = N(f_1, \ldots, f_n, t)$

(N5) For all $s, t \in R, N(f_1, \ldots, f_n + f_n, s + t) \geq \min\{N(f_1, \ldots, f_n, t), N(f_1, \ldots, f_n, t)\}$

(N6) $N(f_1, \ldots, f_n, t)$ is a non-decreasing function of $t \in R$

The space $(F(X^n), N)$ is called a 2-fuzzy $n$-$n$ normed linear space.

**Definition 2.4.** Let $F(X^n)$ be a linear space over $\mathbb{C}$. Define a fuzzy subset $\eta$ defined as a mapping from $|F(X^n)|^{n+1} \times \mathbb{C}$ to $[0, 1]$ such that $(f_1, \ldots, f_n, f_{n+1}) \in [F(X^n)]^{n+1}$, $\alpha \in \mathbb{C}$ satisfying the following conditions

(i) $f, h, \in F(X), s, t \in s$
\[
\eta(f_1 + g, h, f_2, \ldots, f_n, t) + |s|) \geq \min\{\eta(f_1 + g, f_2, \ldots, f_n, t), \eta(g, h, f_2, \ldots, f_n, t)\}
\]

(ii) $f, h, \in F(X), s, t \in C$
\[
\eta(f_1 + g, f_2, \ldots, f_n, t + |s|) \geq \min\{\eta, f_1 + g, f_2, \ldots, f_n, t), \eta(g, h, f_2, \ldots, f_n, t)\}
\]

(iii) $f, h, \in F(X), s, t \in C$
\[
\eta(f_1 + g, f_2, \ldots, f_n, t) + |s|) \geq \min\{\eta(f_1 + g, f_2, \ldots, f_n, t), \eta(g, h, f_2, \ldots, f_n, t)\}
\]

(3) 2-Fuzzy operators

Let $T$ be a 2-fuzzy operator on 2-fuzzy $n$-$n$ inner product space $F(X^n)$. Then $T$ gives rise to a 2-fuzzy operator $T^*$ on $[F(X^n)]^*$ where $T^*$ is defined by $(T^*H)f = H(Tf)$. Let $f \in F(X^n)$ and $Hf$ its corresponding 2-fuzzy functional in $[F(X^n)]^*$ operate with $T^*$ on $Hf$ to obtain a 2-fuzzy functional $Hg = T^*Hf$ and return to its corresponding 2-fuzzy set $g$ in $F(X^n)$. There are three mappings here as,
\[
f \rightarrow Hf \rightarrow T^*Hf = Hg \rightarrow g
\]
write $g = T^*f$ and call this new mapping $T^*$ to map $F(X^n)$ into itself the adjoint of $T$. The same symbol is used for the adjoint of $T$ as for its conjugate since these two mappings are the same if $F(X^n)$ and $[F(X^n)]^*$ are identified by means of the natural correspondence.

It can be observed that
\[
(T^*H)f = Hf(Tf) = < Th, f > \alpha
\]
and
\[
(T^*H)f = Hf(h) < h, g > \alpha = < h, T^*f > \alpha
\]
so that
\[
< Th, f > \alpha = < h, T^*f > \alpha
\]
Theorem 3.1.

\[ \langle f + h, g \rangle = \langle f, g \rangle + \langle h, g \rangle \]
where \( \langle f, g \rangle = \inf \{ t : \eta(f, g, f_2, \ldots, f_n, t) \geq \alpha \} \)

Proof.

\[ \langle f + h, g \rangle = \inf \{ t : \eta(f, g, f_2, \ldots, f_n, t) \geq \alpha \} \]
+ \[ \eta(h, g, f_2, \ldots, f_n, t) \geq \alpha \]
= \[ \inf \{ t + s : \eta(f, g, f_2, \ldots, f_n, t + s) \geq \alpha \} \]

Conversely for any \( \epsilon > 0 \)

\[ A = \min \{ (1 - \eta(f, g, f_2, \ldots, f_n, < f, g > - \alpha - \frac{\epsilon}{2}), \]
\( (1 - \eta(h, g, f_2, \ldots, f_n, < h, g > - \alpha - \frac{\epsilon}{2})) \} \]

\[ = \min \{ (1 - \eta(-f, g, f_2, \ldots, f_n, < -f, g > + \alpha + \frac{\epsilon}{2}), \]
\( \eta(-h, g, f_2, \ldots, f_n, < -h, g > + \alpha + \frac{\epsilon}{2}) \} \]

By the definition of infimum
\( \eta(f, g, f_2, \ldots, f_n, < f, g > - \frac{\epsilon}{2}) < \alpha \)

Hence \( 1 - \eta(f, g, f_2, \ldots, f_n, < f, g > - \frac{\epsilon}{2}) < 1 - \alpha \).

Similarly \( 1 - \eta(h, g, f_2, \ldots, f_n, < h, g > - \frac{\epsilon}{2}) < 1 - \alpha \).

Then
\[ \min \{ (1 - \eta(f, g, f_2, \ldots, f_n, < f, g > - \alpha - \frac{\epsilon}{2}), \]
\( (1 - \eta(h, g, f_2, \ldots, f_n, < h, g > - \alpha - \frac{\epsilon}{2})) \} > 1 - \alpha \]

Hence \( A < \alpha \)

which implies
\( \eta(f + h, g, f_2, \ldots, f_n, < f + h, g > - \alpha - \epsilon) \leq A < \epsilon \)
\( (i.e.) < f + h, g > = < f, g > + \langle h, g \rangle - \epsilon \)

Since \( \epsilon \) is arbitrary,
\[ < f + h, g > = < f, g > + < h, g > \] (3.2)

From (3.1) and (3.2)
\[ < f + h, g > = < f, g > + < h, g > \] .

3.1 Linearity

\( T^* \) is linear.

Consider for any \( f, g \in F(X^n) \) and for all \( h \in F(X^n) \)

\[ < h, T^*(f + h) > = < h, T^*f + T^*h > \]
\[ = \inf \{ t_1 + t_2 : \eta(T^*f, f_2, \ldots, f_n, t_1 + t_2) \geq \alpha \} \]
\[ = \inf \{ t_1 + t_2 : \min \{ \eta(T^*f, f_2, \ldots, f_n, t_1), \]
\( \eta(T^*h, f_2, \ldots, f_n, t_2) \} \geq \alpha \} \]
\[ = \inf \{ t_1 + t_2 : \eta(h, T^*f, f_2, \ldots, f_n, t_1) \]
\( \geq \alpha \} \]
\[ = \inf \{ t_1 + t_2 : \eta(h, T^*f, f_2, \ldots, f_n, t_1) \]
\( \geq \alpha \} \]

Consider

\[ < h, T^n f + T^n g > = \inf \{ t_1 + t_2 : \eta(h, T^n f + T^n g, f_2, \ldots, f_n, t_1 + t_2) \geq \alpha \} \]
\[ = \inf \{ t_1 + t_2 : \min \{ \eta(h, T^n f, f_2, \ldots, f_n, t_1), \]
\( \eta(h, T^n g, f_2, \ldots, f_n, t_2) \} \geq \alpha \} \]
\[ = \inf \{ t_1 + t_2 : \eta(h, T^n f, f_2, \ldots, f_n, t_1) \]
\( \geq \alpha \} \]
\[ = \inf \{ t_1 + t_2 : \eta(h, T^n g, f_2, \ldots, f_n, t_2) \geq \alpha \} \]

From (3.3) and (3.4)
\[ < h, T^n(f + g) > = < h, T^n f + T^n g > \]

hence
\[ T^n(f + g) = T^n f + T^n g \]

Consider
\[ \| T^n f, f_1, \ldots, f_n \|_2 \]
\[ = \| T^n f, T^n f \|_2 \]
\[ = \| T^n f, f \|_2 \]
\[ \leq \| T^n f, f_1, \ldots, f_n \|_2 \]
\[ \leq \| T^n(f + f_1, \ldots, f_n) \|_2 \]

hence
\[ \| T^n f, f_1, \ldots, f_n \|_2 \leq \| T^n f, f_1, \ldots, f_n \|_2 \]

Theorem 3.2. The 2-fuzzy adjoint operator \( T \rightarrow T^* \) satisfies the following properties

\( (i) \) \( (T_1 + T_2)^* = T_1^* + T_2^* \)
The reverse inequality follows from Theorem 3.1
Therefore, \((T_1 + T_2)^* = T_1^* + T_2^*\)

(v) Consider
\[
\|T^* f, f_2, \ldots, f_n\|_\alpha \\
\leq \|T f, f_2, \ldots, f_n\|_\alpha
\]  
(3.5)

Applying (3.5) for \(T^*\)
\[
\|T^* f, f_2, \ldots, f_n\|_\alpha \\
\leq \|T^* f, f_2, \ldots, f_n\|_\alpha
\]
\[
\|T f, f_2, \ldots, f_n\|_\alpha \\
\leq \|T f, f_2, \ldots, f_n\|_\alpha
\]  
(3.6)

From (3.5) and (3.6)
\[
\|T^* f, f_2, \ldots, f_n\|_\alpha \leq \|T f, f_2, \ldots, f_n\|_\alpha
\]

\[\Box\]

### 3.2 2-Fuzzy self adjoint operator

\(T \in \beta(\mathcal{F}(X^n))\), \(T\) is said to be 2-fuzzy self adjoint when
\(T = T^*, 0^* = 0, I^* = I\)

\[
< f, 0^* g >_\alpha = < 0 f, g >_\alpha \\
= \inf \{t : \eta(0, f, g, f_2, \ldots, f_n, t) \geq \alpha\} \\
= \inf \{t : \eta(0, g, f_2, \ldots, f_n, t) \geq \alpha\} \\
= 0.
\]

Now to prove if \(A_1, A_2\) are 2-fuzzy self adjoint then \((\beta_1 A_1 + \beta_2 A_2)^*\) is also 2-fuzzy self adjoint.
<h, (β₁A₁ + β₂A₂) * g > α
\leq h(β₁A₁ + β₂A₂)g > α
= \inf\{t : \eta((β₁A₁ + β₂A₂)h, g, f₂, \ldots, fₙ, t) \geq \alpha\}
= \inf\{t₁ + t₂ : \eta((β₁A₁ + β₂A₂)h, g, f₂, \ldots, fₙ, t) \geq \alpha\}
\geq \inf\{t₁ + t₂ : \min[\eta(β₁A₁h, g, f₂, \ldots, fₙ, t) \geq \alpha, \eta(β₂A₂h, g, f₂, \ldots, fₙ, t) \geq \alpha]\}
= \inf\{t₁ + t₂ : \min[\eta(β₁A₁h, g, f₂, \ldots, fₙ, t) \geq \alpha, \eta(β₂A₂h, g, f₂, \ldots, fₙ, t) \geq \alpha]\}

the reverse inequality follows from Theorem 3.1
Hence
(β₁A₁ + β₂A₂)* = β₁A₁* + β₂A₂*

**Theorem 3.3.** If A₁, A₂ are 2-fuzzy self adjoint then their product A₁A₂ is also 2-fuzzy self adjoint if and only if A₁A₂ = A₂A₁.

**Proof.** Since we have (A₁A₂)* = A₂*A₁
Let A₁A₂ = A₂A₁
(A₁A₂)* = A₂*A₁ = A₁A₂.
Hence the product is 2-fuzzy self adjoint
Conversely assume that the product is 2-fuzzy self adjoint
Consider (A₁A₂)* = A₂*A₁ = A₁A₂.
Since (A₁A₂)* = A₂*A₁,
we have A₁A₂ = A₂A₁.

**Theorem 3.4.** If T is a 2-fuzzy operator for which
\[ \eta(T^{*}f, f, f₂, \ldots, fₙ, t) = 0 \]
for all f then T = 0.

**Proof.** Consider
\[ \eta(T(β₁f + β₂g), β₁f + β₂g, f₂, \ldots, fₙ, t) \]
\[ \geq \min[\eta(Tf, f, f₂, \ldots, fₙ, t) \frac{t}{|β₁|}, \eta(Tf, f₂, \ldots, fₙ, t) \frac{t}{|β₂|}, (\times)\eta(Tg, f, f₂, \ldots, fₙ, t) \frac{t}{|β₂|}] \]
\[ \Rightarrow \eta(Tg, g, f₂, \ldots, fₙ, t) = 0 \]
If T = 0, then \( \eta(0, f, g, f₂, \ldots, fₙ, t) = 0 \)
when \( T \neq 0 \), put \( g = Tf \)
then
\[ \eta(Tf, Tf, f₂, \ldots, fₙ, t) = 0 \]
\[ \Rightarrow \|Tf, f₂, \ldots, fₙ\|α = 0 \]
\[ \Rightarrow T = 0 \]
\[ \Rightarrow T = 0. \]

### 3.3 2-Fuzzy normal operator

An operator N is said to be 2-fuzzy normal if it commutes with its adjoint i.e. \( NN^* = N^*N \).

**Theorem 3.5.** An operator T is 2-fuzzy normal if and only if
\[ \|T^*f, f₂, \ldots, fₙ\|α = \|Tf, f₂, \ldots, fₙ\|α \]
for all f.

**Proof.** By replacing f by Nf (3.7) becomes
\[ \|NNf, f₂, \ldots, fₙ\|α = \|NN^*f, f₂, \ldots, fₙ\|α \]
\[ \Rightarrow \|N^2f, f₂, \ldots, fₙ\|α = \|NN^*f, f₂, \ldots, fₙ\|α \]
\[ \|N^2f, f₂, \ldots, fₙ\|α \]
\[ = \inf\{t : \eta(N^2f, N^2f, f₂, \ldots, fₙ, t) ≥ α\} \]
\[ = \inf\{t : \eta(NNf, NNf, f₂, \ldots, fₙ) ≥ α\} \]
\[ = \inf\{t : \eta(N^*Nf, N^*Nf, f₂, \ldots, fₙ) ≥ α\} \]
\[ = \|N^*Nf, f₂, \ldots, fₙ\|α \]

By Theorem 3.2,
\[ \|N^*Nf, f₂, \ldots, fₙ\|α = \|Nf, f₂, \ldots, fₙ\|α \]
(3.8)

From (3.7) and (3.8)
\[ \|N^2f, f₂, \ldots, fₙ\|α = \|Nf, f₂, \ldots, fₙ\|α \]

\[ \Box \]
3.4 2-Fuzzy unitary operator

An operator $T$ is said to be 2-fuzzy unitary if $T^*T = T^*-T = I$.

**Theorem 3.7.** If $T$ is a 2-fuzzy operator on a 2-fuzzy $n$-n Hilbert space $H(X^n)$, then the following conditions are equivalent to one another.

(i) $T^*T = I$

(ii) $<Tf,Tg> = <f,g>$ for all $f,g \in H(X^n)$

(iii) $\|Tf,f_2,\ldots,f_n\|_a = \|f,f_2,\ldots,f_n\|_a$

**Proof.** (i) $\Rightarrow$ (ii)

Given $T^*T = I$, $<Tf,Tg> = <f,g>$

$$= <f,T^*Tg> = <f,g>$$

$$= <f,f>$$

(hence)

$$\|Tf,f_2,\ldots,f_n\|_a = \|f,f_2,\ldots,f_n\|_a$$

(ii) $\Rightarrow$ (iii)

Given $<Tf,Tg> = <f,g>$

Taking $f = g$

$$<Tf,Tf> = <f,f>$$

$$\|Tf,f_2,\ldots,f_n\|_a = \|f,f_2,\ldots,f_n\|_a$$

(iii) $\Rightarrow$ (i) Given $\|Tf,f_2,\ldots,f_n\|_a = \|f,f_2,\ldots,f_n\|_a$

Therefore $\|Tf,f_2,\ldots,f_n\|_a = \|f,f_2,\ldots,f_n\|_a$

$$\Rightarrow <Tf,Tf> = <f,f>$$

$$\Rightarrow <Tf,T^*f> = <f,f>$$

$$\Rightarrow <T^*Tf,f> = <f,f>$$

$$\Rightarrow <(T^*T - I)f,f> = 0$$

$$\Rightarrow T^*T - I = 0$$

$$\Rightarrow T^*T = I.$$

$$\square$$

**Theorem 3.8.** An operator $T$ on a 2-fuzzy $n$-n Hilbert space $H(X^n)$ is unitary if and only if it is an isomorphism of $H(X^n)$ onto itself.

**Proof.** Let $T$ be a 2-fuzzy unitary operator on $H(X^n)$. Then from the definition of the unitary operator, it is invertible. Hence it is onto. Also $T^*T = I$.

But by Theorem 3.7

$$\|Tf,f_2,\ldots,f_n\|_a = \|f,f_2,\ldots,f_n\|_a$$

hence $T$ is an isometric isomorphism of $H(X^n)$ onto itself.

Conversely, let $T$ be an isometric isomorphism of $H(X^n)$ onto itself, then $T$ is 1-1 and onto and $T^{-1}$ exists.

But $\|Tf,f_2,\ldots,f_n\|_a = \|f,f_2,\ldots,f_n\|_a$, by Theorem 3.7

$T^*T = I$

Hence

$$(T^*T)^{-1} = T^{-1}$$

$$\Rightarrow T^*(TT^{-1}) = T^{-1}$$

$$\Rightarrow T^*T = T^{-1}$$

Pre multiply (3.7) by $T$

$$TT^* = T^{-1}T = I$$

Post multiply (3.7) by $T$

$$T^*T = T^{-1}T = I$$

$\Rightarrow T$ is 2-fuzzy unitary.

$\square$

4. 2-Fuzzy projection

A projection $P$ on a 2-fuzzy $n$-n Hilbert space $H(X^n)$ is an operator $P$ on $H(X^n)$ such that $P^2 = P$ and $P^* = P$.

**Theorem 4.1.** If $P$ is a projection on a 2-fuzzy $n$-n Hilbert space with range $M$ and null space $N$, then $M \perp N$ if and only if $P$ is self adjoint and $N = M^\perp$.

**Proof.** Let $P$ be a 2-fuzzy projection on $H(X^n)$ with the range $M$ and null space $N$.

Then $H(X^n) = M \oplus N$.

Let $M \cap N$

Now to prove $P$ is 2-fuzzy self adjoint. Each $h \in H(X^n)$ can be written uniquely in the form $h = f + g$, where $f \in M$ and $g \in N$.

Here $Ph = f$ and since

$M \cap N, <f,g> = 0$

(4.1)

From (4.1)

$$<Ph,h> = <f,h>$$

$$= <f,f + g>$$

$$= <f,f>_a + <f,g>_a$$

$$= <f,f>_a$$

Also

$$<P^*h,h> = <h,Ph>$$

$$= <h,f>_a$$

$$= <f,g,f>_a$$

$$= <f,f>_a$$

$$\Rightarrow <P^*h,h> = <f,f>_a$$

$$\Rightarrow <(P^* - P)h,h> = 0$$

$$\Rightarrow P^* = P.$$

Therefore $P$ is 2-fuzzy self adjoint.

Conversely assume $P$ is 2-fuzzy self adjoint. Now to prove $M \perp N$.

Let $f \subset M, g \subset N$
Then $P f = f$, $P g = 0$

\[
< f, g >_\alpha = < P f, g >_\alpha = < f, P^* g >_\alpha = < f, P g >_\alpha < f, 0 >_\alpha = 0
\]

$\Rightarrow M \perp N$. 

To prove in $P$ is a 2-fuzzy projection on $H$ with range $M$ and null space $N$, then $M \perp N$.

$N = M^\perp$

Let $f \in N$, then $f \in M^\perp \Rightarrow N \subset M^\perp$ if $N \neq M^\perp$, assume $N$ is a proper closed subspace of $M^\perp$

then exists a non zero $h_0 \in M^\perp$ such that $h_0 \perp N$. But $h_0 \in M^\perp$ implies $h_0 \perp M$.

Therefore $h_0 \perp M$ and $h_0 \perp N$.

Since $\mathcal{F}(X^n) = M \oplus N$, $h_0$ but $\mathcal{F}(X^n)$

$\Rightarrow h_0 = 0$ leads to a contradiction

$\Rightarrow N = M^\perp$.

References