Existence of solutions of a coupled system of functional integro-differential equations of arbitrary (fractional) orders

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Abstract
In this paper, we study the existence of solutions for a coupled system of functional integro-differential equations.

Keywords

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1. Introduction
During the past few years, some investigators have established a lot of useful and interesting coupled system of functional equations, in order to achieve various goals; see [1–6], and the references cited therein.

In this paper, we are concerned with the coupled system of functional integro-differential equations

\[
\begin{align*}
\frac{dx}{dt} &= f_1(t, D^\alpha y(t), D^\beta x(t)), \quad t \in (0, T], \\
\frac{dy}{dt} &= f_2(t, D^\alpha y(t), D^\beta x(t)), \quad t \in (0, T],
\end{align*}
\]

(1.1)

with the nonlocal condition

\[
\begin{align*}
x(0) + \sum_{k=1}^{n} a_k x(\tau_k) &= x_0, \quad a_k > 0, \quad \tau_k (0, T), \\
y(0) + \sum_{j=1}^{n} b_j y(\eta_j) &= y_0, \quad b_j > 0, \quad \eta_j (0, T).
\end{align*}
\]

(1.2)

The existence of at least one solution \((x, y), x, y \in C^1[0, T]\) is proved, under certain conditions. Also, under other assumptions, the existence of at least one solution \((x, y), x, y \in AC[0, T]\) of the problem (1.1)-(1.2) we be proved.

2. Preliminaries
In this section, we giving the following definitions and theorems which used in our results:

Definition 2.1. [7] The fractional order integral of order \(\alpha\) of \(f \in L^1\) is defined by

\[I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds.\]

Definition 2.2. [7] The Caputo fractional-order derivative \(D^\alpha_a\) of order \(\alpha \in (0, 1]\) of the absolutely continuous function
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3. Main Results

Consider the initial value problem of the coupled system of functional integro-differential equations (1.1) with the condition (1.2).

Let $D^\alpha x(t) = u(t), D^\alpha y(t) = v(t)$ in (1.1), we obtain

$$u(t) = \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s, v(s), u(s)) \, ds, \quad t \in (0,T],$$

$$v(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, v(s), u(s)) \, ds, \quad t \in (0,T],$$

where

$$x(t) = x(0) + I^{1-\beta} u(t), \quad t \in (0,T],$$

$$y(t) = y(0) + I^{1-\alpha} v(t), \quad t \in (0,T],$$

(3.1)

3.1 Continuous solution

We discuss the existence of at least one continuous solution $(u, v), u, v \in C[0,T]$ of the coupled system of functional integral equations (3.1), under the following assumptions:

1. $f_i : [0,T] \times \mathbb{R}^2 \to \mathbb{R}, i = 1, 2$ satisfy Caratheodory condition i. e. $f_i$ are measurable in $t$ for any $u, v \in \mathbb{R}$ and continuous in $u, v$ for almost all $t \in [0,T]$. There exist two functions $m_i(t) \in L^1[0,T]$, such that

$$|f_i(t, v, u)| \leq m_i(t).$$

2. $\sup_{t \in [0,T]} I^{1-\beta} m_1(t) \leq M_1, \quad \sup_{t \in [0,T]} I^{1-\alpha} m_2(t) \leq M_2.$

Let $X$ be the Banach space of all order pairs $(u, v)$ with the norm

$$||(u, v)||_X = ||u||_C + ||v||_C = \sup_{t \in [0,T]} |u(t)| + \sup_{t \in [0,T]} |v(t)|.$$

Definition 3.1. By a solution of the coupled system of functional integral equations (3.1), we mean a function $(u, v), u, v \in C[0,T]$ that satisfies (3.1).

Theorem 3.2. Let the assumptions (1)-(2) be satisfied, then the coupled system of functional integral equations (3.1) has at least one continuous solution $(u, v), u, v \in C[0,T].$

Proof. Define the operator $A$ associated with the coupled system of functional integral equations (3.1) by

$$A(u, v) = \left( \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s, v(s), u(s)) \, ds, \right)$$

$$\frac{d}{dt} \left( \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, v(s), u(s)) \, ds \right).$$

Let $Q_r = \{(u, v) \in \mathbb{R}^2 : ||u|| \leq r_1, ||v|| \leq r_2, ||(u, v)|| \leq r_1 + r_2 = r\}$, where $r = M_1 + M_2$.

Then we have, for $(u, v) \in Q_r$

$$A(u, v) = \left( \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} f_1(s, v(s), u(s)) \, ds, \right)$$

$$\frac{d}{dt} \left( \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f_2(s, v(s), u(s)) \, ds \right).$$
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\[
\int_{0}^{t} (t-s)^{-\beta} f_1(s,v(s),u(s)) ds \leq \int_{0}^{t} (t-s)^{-\beta} |f_1(s,v(s),u(s))| ds \\
\leq \int_{0}^{t} (t-s)^{-\beta} m_1(s) ds \\
\leq M_1. 
\]

Hence
\[
\| \int_{0}^{t} (t-s)^{-\beta} f_1(s,v(s),u(s)) ds \|_C \leq M_1. \tag{3.5}
\]

And
\[
\int_{0}^{t} (t-s)^{-\alpha} f_2(s,v(s),u(s)) ds \leq \int_{0}^{t} (t-s)^{-\alpha} |f_2(s,v(s),u(s))| ds \\
\leq \int_{0}^{t} (t-s)^{-\alpha} m_2(s) ds \\
\leq M_2. 
\]

Hence
\[
\| \int_{0}^{t} (t-s)^{-\alpha} f_2(s,v(s),u(s)) ds \|_C \leq M_2. \tag{3.6}
\]

From (3.5) and (3.6), we get
\[
\| A(u,v) \|_\infty \leq M_1 + M_2. 
\]

This prove that \( A : Q \rightarrow Q \), and the class of function \( \{ A_n(u,v) \} \) is uniformly bounded in \( Q \).

Let \( t_1, t_2 \in (0,T) \) such that \( |t_2 - t_1| < \delta \), then
\[
A(u(t_2),v(t_2)) - A(u(t_1),v(t_1)) = \\
\left( \int_{0}^{t_2} (t-s)^{-\beta} f_1(s,v(s),u(s)) ds, \right. \\
\left. \int_{0}^{t_2} (t-s)^{-\alpha} f_2(s,v(s),u(s)) ds \right) \\
- \left( \int_{0}^{t_1} (t-s)^{-\beta} f_1(s,v(s),u(s)) ds, \right. \\
\left. \int_{0}^{t_1} (t-s)^{-\alpha} f_2(s,v(s),u(s)) ds \right) \\
= \left( \int_{0}^{t_2} (t-s)^{-\beta} f_1(s,v(s),u(s)) ds, \right. \\
\left. \int_{0}^{t_2} (t-s)^{-\alpha} f_2(s,v(s),u(s)) ds \right) \\
- \left( \int_{0}^{t_1} (t-s)^{-\beta} f_1(s,v(s),u(s)) ds, \right. \\
\left. \int_{0}^{t_1} (t-s)^{-\alpha} f_2(s,v(s),u(s)) ds \right) \\
\leq \left( \int_{0}^{t_2} (t-s)^{-\beta} m_1(s) ds, \right. \\
\left. \int_{0}^{t_2} (t-s)^{-\alpha} m_2(s) ds \right) \\
- \left( \int_{0}^{t_1} (t-s)^{-\beta} m_1(s) ds, \right. \\
\left. \int_{0}^{t_1} (t-s)^{-\alpha} m_2(s) ds \right) \\
\leq \left( \int_{0}^{t_2} (t-s)^{-\beta} m_1(s) ds, \right. \\
\left. \int_{0}^{t_2} (t-s)^{-\alpha} m_2(s) ds \right) \\
- \left( \int_{0}^{t_1} (t-s)^{-\beta} m_1(s) ds, \right. \\
\left. \int_{0}^{t_1} (t-s)^{-\alpha} m_2(s) ds \right). 
\]

Hence
\[
\int_{0}^{t_2} (t-s)^{-\beta} f_1(s,v(s),u(s)) ds \\
- \int_{0}^{t_1} (t-s)^{-\beta} f_1(s,v(s),u(s)) ds \\
\leq \int_{0}^{t_2} (t-s)^{-\beta} m_1(s) ds \\
- \int_{0}^{t_1} (t-s)^{-\beta} m_1(s) ds \\
\leq \int_{0}^{t_2} (t-s)^{-\beta} m_1(s) ds. 
\tag{3.7}
\]

And
\[
\int_{0}^{t_2} (t-s)^{-\alpha} f_2(s,v(s),u(s)) ds \\
- \int_{0}^{t_1} (t-s)^{-\alpha} f_2(s,v(s),u(s)) ds \\
\leq \int_{0}^{t_2} (t-s)^{-\alpha} m_2(s) ds \\
- \int_{0}^{t_1} (t-s)^{-\alpha} m_2(s) ds \\
\leq \int_{0}^{t_2} (t-s)^{-\alpha} m_2(s) ds. 
\tag{3.8}
\]

Hence
\[
\int_{0}^{t_2} (t-s)^{-\alpha} f_2(s,v(s),u(s)) ds \\
- \int_{0}^{t_1} (t-s)^{-\alpha} f_2(s,v(s),u(s)) ds \\
\leq \int_{0}^{t_2} (t-s)^{-\alpha} m_2(s) ds. 
\tag{3.9}
\]
From (3.7) and (3.8), we get
\[ \|A(u(t), v(t)) - A(u(t), v(t))\| \leq \int_t^T (t - s)^{-\beta} m_1(s)ds + \int_t^T (t^2 - s)^{-\alpha} m_2(s)ds. \]

This means that the class of functions \( A(u,v) \) is equi-continuous in \( Q_t^a \).

Let \( v_n \to v \) and \( u_n \to u \), then from continuity of the two functions \( f_i \), we obtain \( f_1(t, v_n(t), u_n(t)) \to f_1(t, v(t), u(t)) \) and \( f_2(t, v_n(t), u_n(t)) \to f_2(t, v(t), u(t)) \) as \( n \to \infty \).

Using assumptions (1)-(2) and Lebesgue dominated convergence Theorem [8] we obtain
\[ \lim_{n \to \infty} A(u_n, v_n) = A(u, v). \]

Then \( v_n \to v, u_n \to u \) \( \Rightarrow \) \( A(u_n, v_n) \to A(u, v) \) as \( n \to \infty \). This means that the operator \( A \) is continuous.

Then by Schauder fixed point Theorem 2.6 there exist at least

**3.2 Integrable solution**

We discuss the existence of at least one integrable solution \( (\eta, \omega), \eta, \omega \in L^1[0, T] \) of the coupled system of functional integral equations (3.1). The proof is completed.

**Definition 3.3.** By a solution of the coupled system of functional integral equations (3.3), we mean a function \( (\eta, \omega) \), \( \eta, \omega \in L^1[0, T] \) that satisfies (3.3).

**Theorem 3.4.** Let the assumptions (I)-(II) be satisfied, then the coupled system of functional integral equations (3.3) has at least one integrable solution \( (\eta, \omega) \), \( \eta, \omega \in L^1[0, T] \).

**Proof.** Define the operator \( A \) associated with the coupled system of functional integral equations (3.3) by
\[ A(\eta, \omega) = \left( f_1(t, t^{1-\alpha} \omega(t), t^{1-\beta} \eta(t)), f_2(t, t^{1-\alpha} \omega(t), t^{1-\beta} \eta(t)) \right). \]

Let \( Q_p = \{ (\eta, \omega) \in \mathbb{R}^2 : ||\eta|| \leq p_1, ||\omega|| \leq p_2, ||(\eta, \omega)|| \leq p_1 + p_2 = p \}, \) where \( p = \frac{[\|a_1\|_1 + \|a_2\|_1]}{1 - (b_2k + b_1s)}. \)

Let \( (u, v) \in Q_p \), then
\[ A(\eta, \omega) = \left( f_1(t, t^{1-\alpha} \omega(t), t^{1-\beta} \eta(t)), f_2(t, t^{1-\alpha} \omega(t), t^{1-\beta} \eta(t)) \right). \]

But
\[ |f_1(t, t^{1-\alpha} \omega(t), t^{1-\beta} \eta(t))| \leq |a_1(t)| + b_1 \int_0^t (t - s)^{\alpha - 1} \omega(s)ds, \]
\[ + b_1 \int_0^t (t - s)^{\beta - 1} \eta(s)ds. \]

Integrating the above inequality from 0 to \( T \) and making the change of variable we have
\[ \int_0^T |f_1(s, s^{1-\alpha} \omega(t), s^{1-\beta} \eta(t))|ds \leq \int_0^T |a_1(s)|ds \]
\[ + b_1 \int_0^T \int_0^t (t - s)^{\alpha - 1} \omega(s)dsdt, \]
\[ + b_1 \int_0^T \int_0^t (t - s)^{\beta - 1} \eta(s)dsdt \]
\[ \leq ||a_1||_1 + b_1 \int_0^T \int_0^T (t - s)^{\alpha - 1} \omega(s)dsdt \]
\[ + b_1 \int_0^T \int_0^T (t - s)^{\beta - 1} \eta(s)dsdt \]
\[ \leq ||a_1||_1 + b_1 \frac{(T)^{1-\alpha}}{\Gamma(2-\alpha)} ||\omega||_{L^1} + b_1 \frac{(T)^{1-\beta}}{\Gamma(2-\beta)} ||\eta||_{L^1}. \]

Hence
\[ ||f_1(t, t^{1-\alpha} \omega(t), t^{1-\beta} \eta(t))||_{L^1} \]
\[ \leq ||a_1||_1 + b_1 \frac{(T)^{1-\alpha}}{\Gamma(2-\alpha)} ||\omega||_{L^1} + b_1 \frac{(T)^{1-\beta}}{\Gamma(2-\beta)} ||\eta||_{L^1}. \]
And

$$|f_2(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))|$$

$$\leq |a_2(t)| + b_2 \int_0^t \frac{(t-s)^{-\alpha}}{(1-\alpha)} \omega(s)ds$$

$$+ b_2 \int_0^t \frac{(t-s)^{-\beta}}{(1-\beta)} \eta(s)ds,$$

Integrating the above inequality from \(0\) to \(T\) and making the change of variable we have

$$\int_0^T |f_1(s, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))ds| \leq \int_0^T |a_2(s)|ds$$

$$+ b_2 \int_0^T \int_0^t \frac{(t-s)^{-\alpha}}{(1-\alpha)} \omega(s)dtds$$

$$+ b_2 \int_0^T \int_0^t \frac{(t-s)^{-\beta}}{(1-\beta)} \eta(s)dtds$$

$$\leq ||a_2||_{L^1} + b_2 \int_0^T \frac{(t-s)^{-\alpha}}{(1-\alpha)} \omega(s)dts$$

$$+ b_2 \int_0^T \frac{(t-s)^{-\beta}}{(1-\beta)} \eta(s)dts$$

$$\leq ||a_2||_{L^1} + b_2 \frac{(T)^{1-\alpha}}{(2-\alpha)} ||\omega||_{L^1} + b_2 \frac{(T)^{1-\beta}}{(2-\beta)} ||\eta||_{L^1}.$$  \hspace{1cm} (3.11)

Hence

$$||f_2(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))||_{L^1}$$

$$\leq ||a_2||_{L^1} + b_2 \frac{(T)^{1-\alpha}}{(2-\alpha)} ||\omega||_{L^1} + b_2 \frac{(T)^{1-\beta}}{(2-\beta)} ||\eta||_{L^1},$$

From (3.10) and (3.11), we get

$$||A(\eta, \omega)||$$

$$\leq ||a_1||_{L^1} + b_1 \frac{(T)^{1-\alpha}}{(2-\alpha)} ||\omega||_{L^1} + b_1 \frac{(T)^{1-\beta}}{(2-\beta)} ||\eta||_{L^1}$$

$$+ ||a_2||_{L^1} + b_2 \frac{(T)^{1-\alpha}}{(2-\alpha)} ||\omega||_{L^1} + b_2 \frac{(T)^{1-\beta}}{(2-\beta)} ||\eta||_{L^1}$$

$$\leq ||a_1||_{L^1} + ||a_2||_{L^1} + (b_1 + b_2)(p_1 + p_2) = p,$$

where, \(k = \max \{\frac{(T)^{1-\alpha}}{(2-\alpha)}, \frac{(T)^{1-\beta}}{(2-\beta)}\}.

This prove that \(A : Q_p \to Q_p\), and the class of function \(\{A_n(\eta, \omega)\}\) is uniformly bounded on \(Q_p\).

Let \(\Omega\) be bounded subset of \(Q_p\) and \(A : Q_p \to Q_p\). Then \(A(\Omega)\) is also bounded on \(Q_p\).

Let \((\eta, \omega) \in \Omega\), then

\[
\begin{align*}
(A(\eta, \omega))_h - A(\eta, \omega) &= \\
&= \left(1 - \frac{h}{T}\right) \int_0^T f_1(\theta, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))d\theta, \\
&+ \frac{1}{h} \int_0^{T-h} f_2(\theta, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))d\theta - \left(f_1(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t)), f_2(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))\right) \\
&= \frac{1}{h} \int_0^{T-h} f_1(\theta, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))d\theta - f_1(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t)) \\
&- f_1(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))d\theta \\
&- f_2(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))d\theta.
\end{align*}
\]

But

\[
\begin{align*}
&\int_0^T \frac{1}{h} \int_0^{T-h} \left(f_1(\theta, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t)) \\
&- f_1(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))\right)d\theta dt \\
&\leq \frac{1}{h} \int_0^T \int_0^{T-h} |f_1(\theta, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t)) - f_1(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))|d\theta dt.
\end{align*}
\]

Since \(f_1 \in L^1[0, T]\), It follows that

\[
\begin{align*}
&\frac{1}{h} \int_0^{T-h} |f_1(\theta, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t)) - f_1(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))|d\theta \to 0 \quad as \quad h \to 0,
\end{align*}
\]

And

\[
\begin{align*}
&\frac{1}{h} \int_0^{T-h} |f_2(\theta, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t)) - f_2(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))|d\theta dt \\
&\leq \int_0^T \frac{1}{h} \int_0^{T-h} |f_2(\theta, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t)) - f_2(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))|d\theta dt.
\end{align*}
\]

Since \(f_2 \in L^1[0, T]\), It follows that

\[
\begin{align*}
&\frac{1}{h} \int_0^{T-h} |f_2(\theta, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t)) - f_2(t, l^{1-\alpha} \omega(t), l^{1-\beta} \eta(t))|d\theta \to 0 \quad as \quad h \to 0,
\end{align*}
\]

then \((A(\mu, v))_h \to (A(\eta, \omega))\) uniformly. Hence \(A(\Omega)\) is relatively compact.

Hence \(A\) is compact operator.
Let \( \{(\eta_n, \omega_n)\} \subset Q_p \) and \( (\eta_n, \omega_n) \to (\eta, \omega) \)
\[
\lim_{n \to \infty} A(\eta_n, \omega_n) = \lim_{n \to \infty} \left( f_1(t, I^{1-\alpha} \omega_n(t), I^{1-\beta} \eta_n(t)),
\quad f_2(t, I^{1-\alpha} \omega_n(t), I^{1-\beta} \eta_n(t)) \right) = A(\eta, \omega).
\]

Then \( \omega_n \to \omega, \eta_n \to \eta \Rightarrow A(\eta_n, \omega_n) \to A(\eta, \omega) \) as \( n \to \infty \).
This mean that the operator \( A \) is continuous operator. Then by Schauder fixed point Theorem 2.6 there exist at least one solution \( \eta, \omega \in L^1[0, T] \) of the coupled system of functional integral equations (3.3). The proof is completed. 

\[\square\]

### 3.3 The coupled system of functional integro-differential equations

**Theorem 3.5.** Let the assumptions of Theorem 3.2 be satisfied, then there exist at least one solution \( x, y \in C^1[0, T] \) of the initial value problem of the coupled system of functional integro-differential equation (1.1) with the nonlocal condition (1.2).

**Proof.** The solution of the coupled system of functional integro-differential equation (1.1) is given by
\[
\begin{align*}
x(t) &= x(0) + \int_0^t \frac{t-s}{\Gamma(1-\alpha)} u(s)ds, \quad t \in (0, T],
\quad y(t) = y(0) + \int_0^t \frac{t-s}{\Gamma(1-\alpha)} v(s)ds, \quad t \in (0, T],
\end{align*}
\]
Using the nonlocal condition (1.2), we get
\[
\begin{align*}
x(\tau_k) &= x(0) + \sum_{j=1}^{\tau_k} \frac{\tau_k-s}{\Gamma(1-\alpha)} u(s)ds,
\quad y(\eta_j) = y(0) + \int_0^{\eta_j} \frac{\eta_j-s}{\Gamma(1-\alpha)} v(s)ds,
\end{align*}
\]
and
\[
\begin{align*}
\sum_{k=1}^{n} a_k x(\tau_k) &= x(0) \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k \int_0^{(\tau_k-s)/\Gamma(1-\alpha)} u(s)ds,
\quad \sum_{j=1}^{n} b_j y(\eta_j) = y(0) \sum_{j=1}^{n} b_j + \sum_{j=1}^{n} b_j \int_0^{(\eta_j-s)/\Gamma(1-\alpha)} v(s)ds,
\end{align*}
\]
Since,
\[
\begin{align*}
\sum_{k=1}^{n} a_k x(\tau_k) &= x(0) - x(0),
\quad \sum_{j=1}^{n} b_j y(\eta_j) = y(0) - y(0),
\end{align*}
\]
then
\[
\begin{align*}
x(0) - x(0) &= x(0) \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k \int_0^{(\tau_k-s)/\Gamma(1-\alpha)} u(s)ds,
\quad y(0) - y(0) = y(0) \sum_{j=1}^{n} b_j + \sum_{j=1}^{n} b_j \int_0^{(\eta_j-s)/\Gamma(1-\alpha)} v(s)ds,
\end{align*}
\]
and
\[
\begin{align*}
(1 + \sum_{k=1}^{n} a_k) x(0) &= x(0) - \sum_{k=1}^{n} a_k \int_0^{(\tau_k-s)/\Gamma(1-\alpha)} u(s)ds,
\quad (1 + \sum_{j=1}^{n} b_j) y(0) = y(0) - \sum_{j=1}^{n} b_j \int_0^{(\eta_j-s)/\Gamma(1-\alpha)} v(s)ds.
\end{align*}
\]

Hence
\[
\begin{align*}
x(t) &= \frac{1}{1 + \sum_{k=1}^{n} a_k} x_0 + \int_0^t \frac{t-s}{\Gamma(1-\alpha)} u(s)ds,
\quad y(t) = \frac{1}{1 + \sum_{j=1}^{n} b_j} y_0 + \int_0^t \frac{t-s}{\Gamma(1-\alpha)} v(s)ds.
\end{align*}
\]

Where \( u(t), v(t) \) is defined by the coupled system of functional integral equation (3.1). Then from Theorem 3.2, we can deduce the existence of at least one solution of \( x, y, x, y \in C^1[0, T] \) the problem (1.1) and (1.2). The proof is completed. 

\[\square\]

**Theorem 3.6.** Let the assumptions of Theorem 3.4 be satisfied, then there exist at least one solution \( x, y \in AC[0, T] \) of the initial value problem of the coupled system of functional integro-differential equation (1.1) with the nonlocal condition (1.2).

**Proof.** The solution of the coupled system of functional integro-differential equation (1.1) is given by
\[
\begin{align*}
x(t) &= x(0) + \int_0^t \eta(s)ds, \quad t \in (0, T],
\quad y(t) = y(0) + \int_0^t \omega(s)ds, \quad t \in (0, T],
\end{align*}
\]
Using the nonlocal condition (1.2), we get
\[
\begin{align*}
x(\tau_k) &= x(0) + \int_0^{\tau_k} \eta(s)ds,
\quad y(\eta_j) = y(0) + \int_0^{\eta_j} \omega(s)ds,
\end{align*}
\]
and
\[
\begin{align*}
\sum_{k=1}^{n} a_k x(\tau_k) &= x(0) \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k \int_0^{\tau_k} \eta(s)ds,
\quad \sum_{j=1}^{n} b_j y(\eta_j) = y(0) \sum_{j=1}^{n} b_j + \sum_{j=1}^{n} b_j \int_0^{\eta_j} \omega(s)ds,
\end{align*}
\]
Since,
\[
\begin{align*}
\sum_{k=1}^{n} a_k x(\tau_k) &= x(0) - x(0),
\quad \sum_{j=1}^{n} b_j y(\eta_j) = y(0) - y(0),
\end{align*}
\]
then
\[
\begin{align*}
x(0) - x(0) &= x(0) \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k \int_0^{\tau_k} \eta(s)ds,
\quad y(0) - y(0) = y(0) \sum_{j=1}^{n} b_j + \sum_{j=1}^{n} b_j \int_0^{\eta_j} \omega(s)ds,
\end{align*}
\]
and
\[
\begin{align*}
(1 + \sum_{k=1}^{n} a_k) x(0) &= x(0) - \sum_{k=1}^{n} a_k \int_0^{\tau_k} \eta(s)ds,
\quad (1 + \sum_{j=1}^{n} b_j) y(0) = y(0) - \sum_{j=1}^{n} b_j \int_0^{\eta_j} \omega(s)ds.
\end{align*}
\]
Hence
\[ x(t) = \frac{1}{1 + \sum_{k=1}^{n} a_k} \sum_{k=1}^{n} a_k \int_{0}^{t} \eta(s) ds \]
\[ - \frac{1}{1 + \sum_{k=1}^{n} a_k} \sum_{k=1}^{n} a_k \int_{0}^{t} \eta(s) ds , \]
\[ y(t) = \frac{1}{1 + \sum_{j=1}^{n} b_j} \sum_{j=1}^{n} b_j \int_{0}^{t} \omega(s) ds \]
\[ - \frac{1}{1 + \sum_{j=1}^{n} b_j} \sum_{j=1}^{n} b_j \int_{0}^{t} \omega(s) ds , \]

where \( \eta(t), \omega(t) \) is defined by the coupled system of functional integral equation (3.3). Then from Theorem 3.4, we can deduce the existence of at least one solution of \((x, y)\), \(x, y \in AC[0, T]\) the problem (1.1) and (1.2). The proof is completed.

4. Conclusion

In this paper, we proved the existence of at least one continuous solution of the coupled system of functional integro-differential equations with the nonlocal condition, under certain conditions.

Also, under other assumptions, we proved the existence of at least one absolutely continuous solution the coupled system of functional integro-differential equations with the nonlocal condition.

References
