Curvature tensor of almost $C(\lambda)$ manifolds

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Abstract

The present paper deals with certain characterization of curvature conditions on Pseudo-projective and Quasi-conformal curvature tensor on almost $C(\lambda)$ manifolds. The main object of the paper is to study the flatness of the Pseudo-projective, Quasi-conformal curvature tensor, $\xi$-Pseudo-projective, $\xi$-Quasi-conformal curvature tensor on almost $C(\lambda)$ manifolds.

Keywords: Almost $C(\lambda)$ manifolds, Pseudo-projective curvature tensor, Quasi-conformal curvature tensor, $\xi$-Pseudo-projectively flat, $\xi$-Quasi-conformally flat, $\eta$-Einstein.


1 Introduction

In 1981, D. Janssen and L. Vanhecke [6] have introduced the notion of almost $C(\lambda)$ manifolds. Further Z. Olszak and R. Rosca [11] investigated such manifolds. Again S. V. Kharitonava [8] studied conformally flat almost $C(\lambda)$ manifolds. In the paper [2] the author studied Ricci tensor and quasi-conformal curvature tensor of almost $C(\lambda)$ manifolds. In the paper [1] the authors have studied on quasi-conformally flat spaces. Also in paper [4] the authors have studied on pseudo projective curvature tensor on a Riemannian manifold and in the paper [3] the authors are studied on the Conharmonic and Concircular curvature tensors of almost $C(\lambda)$ manifolds. Our present work is motivated by these works.

2 Preliminaries

Let $M$ be a $n$-dimensional connected differentiable manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, g)$, where $\phi$ is a tensor field of type $(1, 1)$, $\xi$ is a vector field, $\eta$ is an 1-form and $g$ is a Riemannian metric on $M$ such that [5].

\[
\begin{align*}
\eta(\xi) &= 1, \\
\phi^2 &= I + \eta \otimes \xi, \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \\
g(X, \xi) &= \eta(X), \\
\phi \xi &= 0, \\
(\nabla_X \phi) Y &= g(X, Y)\xi - \eta(Y)X.
\end{align*}
\]

If an almost contact Riemannian manifold $M$ satisfies the condition

\[
S = ag + b\eta \otimes \eta
\]
An almost contact manifold is called an almost C(λ) manifold if the Riemannian curvature \( R \) satisfies the following relation \[8\]

\[
R(X, Y)Z = R(\phi X, \phi Y)Z - \lambda [Xg(Y, Z) - g(X, Z)Y - \phi Xg(\phi Y, Z) + g(\phi X, Z)\phi Y]
\]  

(2.8)

where, \( X, Y, Z \in TM \) and \( \lambda \) is a real number.

**Remark 2.1.** A C(\( l \))-curvature tensor is a Sasakian curvature tensor, a C(\( O \))-curvature tensor is a co-Kahler or CK-curvature tensor and a C(\( l \))-curvature tensor is a Kenmotsu curvature tensor.

From [9] we have,

\[
R(X, Y)\xi = R(\phi X, \phi Y)\xi - \lambda [X\eta(Y) - \eta(X)Y]
\]  

(2.9)

On an almost C(\( \lambda \)) manifold, we also have [2]

\[
QX = AX + B\eta(X)\xi.
\]  

(2.10)

wher, \( A = -\lambda(n - 2) \), \( B = -\lambda \) and \( Q \) is the Ricci-operator.

\[
\begin{align*}
\eta(QX) &= (A + B)\eta(X), \\
S(X, Y) &= Ag(X, Y) + B\eta(X)\eta(Y), \\
r &= -\lambda(n - 1)^2, \\
S(\xi, \xi) &= (A + B)\eta(X), \\
g(QX, Y) &= S(X, Y).
\end{align*}
\]  

(2.11-2.16)

### 3 Quasi-conformally flat almost C(\( \lambda \)) manifolds

**Definition 3.1.** The Quasi-conformal curvature tensor \( \tilde{C} \) of type (1, 3) on a Riemannian manifold \((M, g)\) of dimension \( n \) is defined by [11]

\[
\tilde{C}(X, Y)Z = aR(X, Y)Z + b(S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY) - \frac{r}{n} \left( \frac{a}{n - 1} + 2b \right) [g(Y, Z)X - g(X, Z)Y] - b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY].
\]  

(3.17)

For all \( X, Y, Z \in \chi(M) \), where \( Q \) is the Ricci-operator.

If \( \tilde{C} \) vanishes identically then we say that the manifold is Quasi-conformally flat, where \( a, b \neq 0 \) are constants.

Thus for a Quasi-conformally flat C(\( \lambda \)) manifold, we get from [3.17]

\[
aR(X, Y)Z = \frac{r}{n} \left( \frac{a}{n - 1} + 2b \right) [g(Y, Z)X - g(X, Z)Y] - b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY],
\]  

(3.18)

By virtue of (2.10) and (2.12), (3.18) takes the form

\[
aR(X, Y) = \frac{r}{n} \left( \frac{a}{n - 1} + 2b \right) [g(Y, Z)X - g(X, Z)Y] - b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY].
\]  

(3.19)

In view of (2.8) we get from (3.19)

\[
aR(\phi X, \phi Y)Z = \lambda a[Xg(Y, Z) - g(X, Z)Y - \phi Xg(\phi Y, Z) + g(\phi X, Z)\phi Y] + \frac{r}{n} \left( \frac{a}{n - 1} + 2b \right) [g(Y, Z)X - g(X, Z)Y] - b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] - Ag(X, Z)Y + B\eta(Y)g(X, Z)\]
\]  

(3.20)
Putting \( Y = \xi \) and using the value of \( A \) and \( B \) in (3.20) we get

\[
\left[ \lambda a + \lambda b + \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) + 2\lambda b(n-2) \right] \left[ X \eta(Z) - g(X, Z) \xi \right] = 0.
\] (3.21)

Taking inner product of (3.21) with a vector field \( \xi \), we get

\[
\left[ \lambda a + \lambda b + \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) + 2\lambda b(n-2) \right] \left[ \eta(X)\eta(Z) - g(X, Z) \right] = 0.
\] (3.22)

Putting \( X = QX \) in (3.22) we get

\[
\left[ \lambda a + \lambda b + \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) + 2\lambda b(n-2) \right] \left[ \eta(QX)\eta(Z) - g(QX, Z) \right] = 0.
\] (3.23)

Using (2.15), (2.11) and by the virtue of (2.13) in (3.24) we get

\[
\left[ \lambda a + \lambda b - \frac{\lambda(n-1)^2}{n} \left( \frac{a}{n-1} + 2b \right) + 2\lambda b(n-2) \right] \left[ (A + B)\eta(X)\eta(Z) - S(X, Z) \right] = 0.
\] (3.24)

Therefore, either

\[
\lambda = 0 \quad (or) \quad S(X, Z) = A + B \eta(X) \eta(Y)
\] (3.25)

Thus we can state the following:

**Theorem 3.1.** For a Quasi-conformally flat almost \( C(\lambda) \) manifold, either \( \lambda = 0 \) i.e. \( C(\lambda) \) is cosymplectic. or the manifold is special type of \( \eta- \)Einstein.

**Proof.** Follows form (3.25) and remark (2.1). \( \square \)

### 4 \( \xi \)-Quasi-conformally flat almost \( C(\lambda) \) manifolds

**Definition 4.1.** The Quasi-conformal curvature tensor \( \tilde{C} \) of type (1, 3) on a Riemannian manifold \( (M, g) \) of dimension \( n \) will be defined as \( \xi \)-quasi-conformally flat if \( \tilde{C}(X, Y) \xi = 0 \) for all \( X, Y \in TM \).

Thus for a \( \xi \)-quasi-conformally flat almost \( C(\lambda) \) manifolds we get from (3.17)

\[
aR(X, Y) \xi = \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) \left[ \eta(Y)X - \eta(X)Y \right]
- b(S(Y, \xi)X - S(X, \xi)Y + \eta(Y)QX - \eta(X)QY)
\] (4.26)

In the view of (2.9), Taking \( Y = \xi \), by virtue of (2.10), (2.14) and (2.15), putting the value \( A \) and \( B \) taking inner product with respect to vector field \( V \) we get from (4.26).

\[
\left[ \lambda a + \lambda b - \frac{\lambda(n-1)^2}{n} \left( \frac{a}{n-1} + 2b \right) + 2\lambda b(n-2) \right] \left[ g(X, V) - \eta(X)\eta(V) \right] = 0
\] (4.27)

Taking \( X = QX \) in (4.27) we get

\[
\left[ \lambda a + \lambda b - \frac{\lambda(n-1)^2}{n} \left( \frac{a}{n-1} + 2b \right) + 2\lambda b(n-2) \right] \left[ g(QX, V) - \eta(QX)\eta(V) \right] = 0
\] (4.28)

Using (2.11) and (2.16) in (4.28)

\[
\left[ \lambda a + \lambda b - \frac{\lambda(n-1)^2}{n} \left( \frac{a}{n-1} + 2b \right) + 2\lambda b(n-2) \right] \left[ S(X, V) - (A + B)\eta(X)\eta(V) \right] = 0
\] (4.29)

Therefore, either

\[
\lambda = 0 \quad (or) \quad S(X, Z) = A + B \eta(X) \eta(Y)
\] (4.30)

Thus we can state the following:

**Theorem 4.1.** For a \( \xi \)-Quasi-conformally flat almost \( C(\lambda) \) manifold, either \( \lambda = 0 \) i.e. \( C(\lambda) \) is cosymplectic. or the manifold is special type of \( \eta- \)Einstein.

**Proof.** Follows form (4.30) and remark (2.1). \( \square \)
5 Pseudo-projectively curvature flat almost C(λ) manifolds

Definition 5.1. The Pseudo-projective curvature tensor \( \tilde{P} \) of type \((1,3)\) on a Riemannian manifold \((M, g)\) of dimension \(n\) is defined by \([4]\)

\[
\tilde{P}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y]
- \frac{r}{n} \left( \frac{a}{n-1} + b \right) [g(Y, Z)X - g(X, Z)Y].
\] (5.31)

for all \(X, Y, Z \in \chi(M)\). If \( \tilde{P} \) vanishes identically then we say that the manifold is Pseudo-projectively curvature flat, where \(a, b \neq 0\) are constants.

Thus for a Pseudo-projectively curvature flat C(λ) manifold, we get from (5.31)

\[
aR(X, Y)Z = \frac{r}{n} \left( \frac{a}{n-1} + b \right) [g(Y, Z)X - g(X, Z)Y] - b[S(Y, Z)X - S(X, Z)Y].
\] (5.32)

By virtue of (2.12), (5.32) takes the form

\[
aR(X, Y)Z = \frac{r}{n} \left( \frac{a}{n-1} + b \right) [g(Y, Z)X - g(X, Z)Y]
- b[A\eta(Y, Z)X + B\eta(Y)\eta(Z)X - A\eta(X, Z)Y - B\eta(X)\eta(Z)Y].
\] (5.33)

In view of (2.8) we get from (5.33)

\[
aR(\phi X, \phi Y)Z = \lambda a[Xg(Y, Z) - g(X, Z)Y - \phi Xg(\phi Y, Z) + g(\phi X, Z)\phi Y]
+ b[S(Y, Z)X - S(X, Z)Y]
- b[A\eta(Y, Z)X + B\eta(Y)\eta(Z)X - A\eta(X, Z)Y - B\eta(X)\eta(Z)Y].
\] (5.34)

Putting \(Y = \xi\), using the value of \(A\) and \(B\), taking inner product with a vector field \(\xi\) in (5.34)we get

\[
\left[\lambda a + \lambda b + \frac{r}{n} \left( \frac{a}{n-1} + b \right) + b\lambda(n-2) \right] [\eta(X)\eta(Z) - g(X, Z)] = 0.
\] (5.35)

Taking \(X = QX\) and by the virtue of (2.13) in (5.35) we get

\[
\left[\lambda a + \lambda b + \frac{r}{n} \left( \frac{a}{n-1} + b \right) + b\lambda(n-2) \right] [\eta(QX)\eta(Z) - g(QX, Z)] = 0.
\] (5.36)

Using (2.15) and (2.11) in (5.36) we get

\[
\left[\lambda a + \lambda b + \frac{r}{n} \left( \frac{a}{n-1} + b \right) + b\lambda(n-2) \right] [(A + B)\eta(X)\eta(Z) - S(X, Z)] = 0.
\] (5.37)

Therefore, either

\[
\lambda = 0 \quad \text{(or)} \quad S(X, Z) = A + B\eta(X)\eta(Y).
\] (5.38)

Thus we can state the following:

**Theorem 5.1.** For a Pseudo-projectively curvature flat almost C(λ) manifold, either \(\lambda = 0\) i.e. \(C(\lambda)\) is cosymplectic, or the manifold is special type of \(\eta-Einstein\).

**Proof.** Follows from (5.38) and remark (2.1). \(\square\)
6 \(\xi\)-Pseudo-projectively curvature flat almost \(C(\lambda)\) manifolds

**Definition 6.1.** The \(\xi\)-pseudo-projectively curvature tensor \(P\) of type \((1, 3)\) on a Riemannian manifold \((M, g)\) of dimension \(n\) will be defined as \(\xi\)-pseudo-projectively flat \([4]\) if \(P(X, Y)\xi = 0\) for all \(X, Y \in TM\).

Thus for a \(\xi\)-pseudo-projectively flat almost \(C(\lambda)\) manifolds we get from (5.31)

\[
aR(X, Y)\xi = \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) \left[ \eta(Y)X - \eta(X)Y \right] - b(S(Y, \xi)X - S(X, \xi)Y + \eta(Y)QX - \eta(X)QY) \tag{6.39}
\]

In the view of (2.9). Taking \(Y = \xi\), by virtue of (2.10), (2.14) and (2.15), putting the value \(A\) and \(B\) taking inner product with respect to vector field \(V\) we get from (6.39)

\[
\left[ \lambda a + \lambda b + \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) + b\lambda(n-2) \right] \left[ g(X, V) - \eta(X)\eta(V) \right] = 0. \tag{6.40}
\]

Taking \(X = QX\) in (6.40) we get

\[
\left[ \lambda a + \lambda b + \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) + b\lambda(n-2) \right] \left[ g(QX, V) - \eta(QX)\eta(V) \right] = 0. \tag{6.41}
\]

Using (2.11), (2.16) and by the virtue of (2.13) in (6.41)

\[
\left[ \lambda a + \lambda b + \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) + b\lambda(n-2) \right] \left[ S(X, V) - (A + B)\eta(X)\eta(V) \right] = 0. \tag{6.42}
\]

Therefore, either

\(\lambda = 0\) (or) \(S(X, Z) = A + B\eta(X)\eta(Y)\) \(\text{(6.43)}\)

Thus we can state the following:

**Theorem 6.1.** For a \(\xi\)-pseudo-projectively flat almost \(C(\lambda)\) manifold, either \(\lambda = 0\) i.e. \(C(\lambda)\) is cosymplectic. or the manifold is special type of \(\eta\)-Einstein.

**Proof.** Follows form (6.43) and remark (2.1).

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**References**


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