Some inequalities for the $q,k$-Gamma and Beta functions

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Abstract

Using $q$-integral inequalities we establish some new inequalities for the $q,k$-Gamma, Beta and Psi functions.

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1 Introduction

The $q$-analogue $\Gamma_q$ of the well known Gamma function was initially introduced by Thomae [11] and later deeply studied by Jackson [6]. The reader will find in the research literature more about this feature.

In [1], R. Diaz and C. Truel introduced a $q,k$-generalized Gamma and Beta functions and they proved integral representations for $\Gamma_{q,k}$ and $B_{q,k}$ functions.

This work is devoted to establish some inequalities for the generalized $q,k$-Gamma and Beta functions and this has been possible thanks to the inequalities that verify the $q$-Jackson’s integral.

The paper is organized as follows: In section 2, we present some preliminaries and notations that will be useful in the sequel. In section 3, we recall the $q$-Čebyšev’s integral inequality for $q$-synchronous ($q$-asynchronous) functions and in direct consequence, we deduce some inequalities involving $q,k$-Beta and $q,k$-Gamma functions. In section 4, we establish some inequalities for these functions owing to the $q$-Holder’s inequality. Finally section 5 is devoted to some applications of $q$-Grüss integral inequality.

2 Notations and preliminaries

To make this paper self containing, we provide in this section a summary of the mathematical notations and definitions useful. All of these results can be found in [4, 8] or [9].

Throughout this paper, we will fix $q \in [0,1], \ k > 0$ a real number.

For $a \in \mathbb{C}$, we write

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad (a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1,2,\ldots,\infty,$$

$$[n]_q! = [1]_q[2]_q\ldots[n]_q, \quad n \in \mathbb{N}.$$

The $q$-derivative $D_q$ of a function $f$ is given by

$$(D_qf)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad \text{if } x \neq 0,$$

(2.1)

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and \((D_q f)(0) = f'(0)\) provided \(f'(0)\) exists.

The \(q\)-Jackson integrals from 0 to \(b\) and from 0 to \(\infty\) are defined by (see [7])

\[
\int_0^b f(x) d_q x = (1 - q)b \sum_{n=0}^{\infty} f(b q^n) q^n
\]

(2.2)

and

\[
\int_0^{\infty} f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n) q^n,
\]

(2.3)

provided the sums converge absolutely.

The \(q\)-Jackson integral in a generic interval \([a, b]\) is given by (see [7])

\[
\int_a^b f(x) d_q x = \int_a^b f(x) d_q x - \int_0^a f(x) d_q x.
\]

(2.4)

We denote by \(I\) one of the following sets:

\[
\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\},
\]

(2.5)

\[
[0, b]_q = \{b q^n : n \in \mathbb{N}\}, \quad b > 0,
\]

(2.6)

\[
[a, b]_q = \{b q^r : 0 \leq r \leq n\}, \quad b > 0, \quad a = b q^n, n \in \mathbb{N}
\]

(2.7)

and we note \(\int_I f(x) d_q x\) the \(q\)-integral of \(f\) on the correspondent \(I\).

**Definition 2.1.** Let \(x, y, s, t \in \mathbb{R}\) and \(n \in \mathbb{N}\), we note by

1. \((x + y)^n_{q,k} := \prod_{j=0}^{n-1} (x + q^j y)
2. \((1 + x)^n_{q,k} := \prod_{j=0}^{n-1} (1 + q^j x)
3. \((1 + x)^t_{q,k} := \frac{(1+x)^{\infty}}{(1+q^{t}x)^{\infty}}_{q,k}

We have \((1 + x)^{s+t}_{q,k} = (1 + x)^s_{q,k}(1 + q^k x)^t_{q,k}\).

We recall the two \(q, k\)-analogues of the exponential function (see [1]) given by

\[
E_x^{\infty}_{q,k} = \sum_{n=0}^{\infty} q \frac{k^{n-1}}{[n]_{q,k}!} x^n = (1 + (1 - q^k)x)^{\infty}_{q,k}
\]

(2.8)

and

\[
e_x^{\infty}_{q,k} = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q,k}!} = \frac{1}{(1 - (1 - q^k)x)^{\infty}_{q,k}}.
\]

(2.9)

These \(q, k\)-exponential functions satisfy the following relations:

\[
D_q^s e_x^{\infty}_{q,k} = e_x^{s}_{q,k}, \quad D_q^s E_x^{\infty}_{q,k} = E_x^{s}_{q,k}
\]

and

\[
E_x^{-\infty}_{q,k} e_x^{\infty}_{q,k} = E_x^{-\infty}_{q,k} = 1.
\]

The \(q, k\)-Gamma function is defined by (11)

\[
\Gamma_{q,k}(x) = \frac{(1 - q^k)^{\infty}_{q,k}}{(1 - q^x)^{\infty}_{q,k} (1 - q)^{\frac{x}{2} - 1}} \quad x > 0.
\]

(2.10)

When \(k = 1\) it reduces to the known \(q\)-Gamma function \(\Gamma_q\).
It satisfies the following functional equation:
\[ \Gamma_{q,k}(x + k) = [x]_q \Gamma_{q,k}(x), \quad \Gamma_{q,k}(k) = 1 \]  \quad (2.11)

and having the following integral representation (see [1]):
\[ \Gamma_{q,k}(x) = \int_0^{\frac{[x]_q}{(1-q)^{\frac{1}{2}}}} t^{x-1} E_{q,k}^{-\frac{k}{q}} d_q t, \quad x > 0. \]  \quad (2.12)

The previous integral representation, give that \( \Gamma_{q,k} \) is an infinitely differentiable function on \( ]0, +\infty[ \) and
\[ \Gamma_{q,k}^{(i)}(x) = \int_0^{\frac{[x]_q}{(1-q)^{\frac{1}{2}}}} t^{x-1} (\ln t)^i E_{q,k}^{-\frac{k}{q}} d_q t, \quad x > 0, \quad i \in \mathbb{N}. \]  \quad (2.13)

The \( q,k \)-Beta function is defined by (see [1]):
\[ B_{q,k}(t,s) = [k]_q^{-\frac{1}{2}} \int_0^{[k]_q^{\frac{1}{2}}} x^{t-1}(1 - q^k x^k)^{\frac{s}{k_0}} d_q x, \quad s > 0, t > 0. \]  \quad (2.14)

By using the following change of variable \( u = \frac{x}{[k]_q} \), the last equation becomes
\[ B_{q,k}(t,s) = \int_0^1 u^{t-1}(1 - q^k u^k)^{\frac{s}{k_0}} d_q u, \quad s > 0, t > 0. \]  \quad (2.15)

It satisfies
\[ B_{q,k}(t,s) = \frac{\Gamma_{q,k}(t)\Gamma_{q,k}(s)}{\Gamma_{q,k}(t+s)}, \quad s > 0, t > 0. \]  \quad (2.16)

3 \textit{q-Čebyšev’s integral inequality and applications}

We begin this section by recalling the \textit{q-Čebyšev’s} integral inequality for \( q \)-synchronous (\( q \)-asynchronous) mappings [3] and as applications we give some inequalities for the \( q,k \)-Beta and the \( q,k \)-Gamma functions.

\textbf{Definition 3.2.} Let \( f \) and \( g \) be two functions defined on \( I \). The functions \( f \) and \( g \) are said \( q \)-synchronous (\( q \)-asynchronous) on \( I \) if
\[ (f(x) - f(y))(g(x) - g(y)) \geq (\leq) 0 \quad \forall x, y \in I. \]  \quad (3.17)

Note that if \( f \) and \( g \) are both \( q \)-increasing or \( q \)-decreasing on \( I \) then they are \( q \)-synchronous on \( I \).

\textbf{Proposition 3.1.} Let \( f, g \) and \( h \) be three functions defined on \( I \) such that:
1. \( h(x) \geq 0, \quad x \in I, \)
2. \( f \) and \( g \) are \( q \)-synchronous (\( q \)-asynchronous) on \( I \).

Then
\[ \int_I h(x) d_q x \int_I h(x)f(x)g(x) d_q x \geq (\leq) \int_I h(x)d_q x \int_I h(x)g(x)d_q x. \]  \quad (3.18)

\textbf{Proof.} We have
\[ \int_I h(x) d_q x \int_I h(x)f(x)g(x) d_q x - \int_I h(x)f(x)d_q x \int_I h(x)g(x)d_q x = \]
\[ 1/2 \int_I \int_I h(x)h(y) |f(x) - f(y)| |g(x) - g(y)| d_q x d_q y. \]

So, the result follows from the conditions (1) and (2). \( \Box \)

The following theorem is a direct consequence of the previous proposition.
Theorem 3.1. Let \( m, n, p \) and \( p' \) be some positive reals such that
\[
(p - m)(p' - n) \leq (\geq) 0.
\]
Then
\[
B_{q,k}(p, p') B_{q,k}(m, n) \geq (\leq) B_{q,k}(p, n) B_{q,k}(m, p')
\] (3.19)
and
\[
\Gamma_{q,k}(p + n) \Gamma_{q,k}(p' + m) \geq (\leq) \Gamma_{q,k}(p + p') \Gamma_{q,k}(m + n).
\] (3.20)

Proof. Fix \( m, n, p \) and \( p' \) in \([0, +\infty), \) satisfying the condition of the theorem and the functions \( f, g \) and \( h \) defined on \([0, 1],q \) by
\[
f(u) = u^{p-m}, \quad g(u) = (1 - q^n u^k)^{\frac{p'-n}{q,k}} \quad \text{and} \quad h(u) = u^{m-1}(1 - q^n u^k)^{\frac{n-1}{q,k}}.
\]
From the relations
\[
D_q f(u) = (p - m) u^{p-m-1}
\] (3.21)
and
\[
D_q g(u) = [n - p']_q q^n u^{p'-1}(1 - q^n u^k)^{\frac{n-1}{q,k}},
\] (3.22)
on one can see that \( f \) and \( g \) are \( q \)-synchronous (\( q \)-asynchronous) on \( I = [0, 1],q \).
So, by using the relation 2.15 and Proposition 3.1, we obtain
\[
\int_0^1 u^{m-1}(1 - q^n u^k)^{\frac{n}{q,k}} d_q u \int_0^1 u^{p-1}(1 - q^n u^k)^{\frac{n}{q,k}} (1 - q^n u^k)^{\frac{n-1}{q,k}} d_q u \geq (\leq) \int_0^1 u^{p-1}(1 - q^n u^k)^{\frac{n}{q,k}} u^{m-1}(1 - q^n u^k)^{\frac{n-1}{q,k}} (1 - q^n u^k)^{\frac{n-1}{q,k}} d_q u,
\]
which implies that
\[
B_{q,k}(m, n) B_{q,k}(p, p') \geq (\leq) B_{q,k}(p, n) B_{q,k}(m, p').
\] (3.23)

Now, according to the relations 2.16 and 3.19, we obtain
\[
\frac{\Gamma_{q,k}(m) \Gamma_{q,k}(n) \Gamma_{q,k}(p) \Gamma_{q,k}(p')}{\Gamma_{q,k}(m + n) \Gamma_{q,k}(p + p')} \geq (\leq) \frac{\Gamma_{q,k}(p) \Gamma_{q,k}(n) \Gamma_{q,k}(m) \Gamma_{q,k}(p')}{\Gamma_{q,k}(p + n) \Gamma_{q,k}(m + p')}.
\] (3.24)
Therefore
\[
\Gamma_{q,k}(p + n) \Gamma_{q,k}(p' + m) \geq (\leq) \Gamma_{q,k}(p + p') \Gamma_{q,k}(m + n).
\] (3.25)

\[\square\]

Corollary 3.1. For all \( p, m > 0, \) we have
\[
B_{q,k}(p, m) \geq \left[ B_{q,k}(p, p) B_{q,k}(m, m) \right]^{1/2}
\] (3.26)
and
\[
\Gamma_{q,k}(p + m) \leq \left[ \Gamma_{q,k}(2p) \Gamma_{q,k}(2m) \right]^{1/2}.
\] (3.27)

Proof. A direct application of Theorem 3.1 with \( p' = p \) and \( n = m, \) gives the results. \[\square\]

Corollary 3.2. For all \( u, v > 0, \) we have
\[
\Gamma_{q,k}(\frac{u + v}{2}) \leq \sqrt{\Gamma_{q,k}(u) \Gamma_{q,k}(v)}.
\] (3.28)

Proof. The inequality follows from 3.27, by taking \( p = \frac{u}{2} \) and \( m = \frac{v}{2}. \) \[\square\]
Theorem 3.2. Let $m$, $p$ and $r$ be real numbers satisfying $m, p > 0$ and $p > r > -m$ and let $n$ be a nonnegative integer. If
\[ r(p - m - r) \geq (\leq) 0 \] (3.29)
then
\[ \Gamma_{q,k}(p) \Gamma_{q,k}(m) \geq (\leq) \Gamma_{q,k}(p - r) \Gamma_{q,k}(m + r). \] (3.30)

Proof. Let $f$, $g$ and $h$ be the functions defined on $I = [0, (\frac{|k|}{1 - q^k})^\frac{1}{2}]_q$ by
\[ f(x) = x^{p-m-r}, \quad g(x) = x^r \quad \text{and} \quad h(x) = x^{m-1} E_{q,k}^{-\frac{q^k}{q^m}}((\ln x)^2n}. \]
We have
\[ D_q f(x) = [p - m - r] q x^{p-m-r-1} \quad \text{and} \quad D_q g(x) = [r] q x^{r-1}. \]
If the condition (3.29) holds, one can show that the functions $f$ and $g$ are $q$-synchronous ($q$-asynchronous) on $I$ and Proposition 3.1 gives
\[
\int_I x^{m-1} E_{q,k}^{-\frac{q^k}{q^m}}((\ln x)^2n} dx \int_I x^{p-m-r} x^r x^{m-1} E_{q,k}^{-\frac{q^k}{q^m}}((\ln x)^2n} dx
\geq (\leq) \int_I x^{p-m-r} x^r x^{m-1} E_{q,k}^{-\frac{q^k}{q^m}}((\ln x)^2n} dx \int_I x^r x^{m-1} E_{q,k}^{-\frac{q^k}{q^m}}((\ln x)^2n} dx,
\]
which is equivalent to
\[
\int_I x^{m-1} E_{q,k}^{-\frac{q^k}{q^m}}((\ln x)^2n} dx \int_I x^{p-m-r} E_{q,k}^{-\frac{q^k}{q^m}}((\ln x)^2n} dx
\geq (\leq) \int_I x^{p-m-r} E_{q,k}^{-\frac{q^k}{q^m}}((\ln x)^2n} dx \int_I x^{r+m-1} E_{q,k}^{-\frac{q^k}{q^m}}((\ln x)^2n} dx.
\]
Hence, the relation
\[ \Gamma_{q,k}^{(i)}(x) = \int_I t^{i-1} (\ln t)^i E_{q,k}^{-\frac{q^k}{q^m}} dt, \quad x > 0, \quad i \in \mathbb{N}, \]
gives
\[ \Gamma_{q,k}^{(2n)}(m) \Gamma_{q,k}^{(2n)}(p) \geq (\leq) \Gamma_{q,k}^{(2n)}(p - r) \Gamma_{q,k}^{(2n)}(m + r). \] (3.31)

Taking $n = 0$ in the previous theorem, we obtain the following result.

Corollary 3.3. Let $m$, $p$ and $r$ be some real numbers under the conditions of Theorem 3.2, we have
\[ \Gamma_{q,k}(p) \Gamma_{q,k}(m) \geq (\leq) \Gamma_{q,k}(p - r) \Gamma_{q,k}(m + r) \] (3.32)
and
\[ B_{q,k}(p, m) \geq (\leq) B_{q,k}(p - r, m + r). \] (3.33)

Corollary 3.4. Let $n$ be a nonnegative integer, $p > 0$ and $p' \in \mathbb{R}$ such that $|p'| < p$. Then
\[ \left[ \Gamma_{q,k}^{(2n)}(p) \right]^2 \leq \Gamma_{q,k}^{(2n)}(p - p') \Gamma_{q,k}^{(2n)}(p + p'). \] (3.34)

Proof. By choosing $m = p$ and $r = p'$, we obtain
\[ r(p - m - r) = -(p')^2 \leq 0 \]
and the result turns out from Theorem 3.2.
Taking in the previous result \( p = \frac{u+v}{2} \) and \( p' = \frac{u-v}{2} \), we obtain the following result:

**Corollary 3.5.** Let \( u, v \) be two positive real numbers and \( n \) be a nonnegative integer. Then

\[
\Gamma_{q,k}^{(2n)} \left( \frac{u+v}{2} \right) \leq \sqrt{\Gamma_{q,k}^{(2n)}(u) \Gamma_{q,k}^{(2n)}(v)}.
\] (3.35)

**Corollary 3.6.** Let \( p > 0 \) and \( p' \in \mathbb{R} \) such that \( |p'| < p \). Then

\[
\Gamma_{q,k}^{2}(p) \leq \Gamma_{q,k}^{2}(p - p') \Gamma_{q,k}^{2}(p + p')
\] (3.36)

and

\[
B_{q,k}(p,p) \leq B_{q,k}(p-p',p+p').
\] (3.37)

**Proof.** For \( n = 0 \), the inequality (3.34) becomes

\[
\Gamma_{q,k}^{2}(p) \leq \Gamma_{q,k}^{2}(p - p') \Gamma_{q,k}^{2}(p + p').
\]

The inequality (3.37) follows from (2.16). \( \Box \)

**Theorem 3.3.** Let \( a \) and \( b \) be two positive real numbers such

\[
(a-k)(b-k) \geq (\leq) 0
\]

and \( n \) a nonnegative integer. Then

\[
\Gamma_{q,k}^{(2n)} (2k) \Gamma_{q,k}^{(2n)} (a+b) \geq (\leq) \Gamma_{q,k}^{(2n)} (a+k) \Gamma_{q,k}^{(2n)} (b+k).
\] (3.38)

**Proof.** In Theorem 3.2 set \( m = 2k \), \( p = a + b \) and \( r = b - k \). The condition (3.29) becomes

\[
r(p - m - r) = (a-k)(b-k) \geq (\leq) 0.
\] (3.39)

So,

\[
\Gamma_{q,k}^{(2n)} (2k) \Gamma_{q,k}^{(2n)} (a+b) \geq (\leq) \Gamma_{q,k}^{(2n)} (a+k) \Gamma_{q,k}^{(2n)} (b+k).
\] (3.40)

**Corollary 3.7.** If \( a, b > 0 \) such \((a-k)(b-k) \geq (\leq) 0\). Then

\[
\Gamma_{q,k}(a+b) \geq (\leq) \frac{[a]_q [b]_q}{[k]_q} \Gamma_{q,k}(a) \Gamma_{q,k}(b)
\] (3.41)

and

\[
B_{q,k}(a,b) \leq (\geq) \frac{[k]_q}{[a]_q [b]_q}.
\] (3.42)

**Proof.** The inequality (3.41) follows from the previous theorem by taking \( n = 0 \) and using the facts that \( \Gamma_{q,k}(2k) = [k]_q \), \( \Gamma_{q,k}(a+k) = [a]_q \Gamma_{q,k}(a) \) and \( \Gamma_{q,k}(b+k) = [b]_q \Gamma_{q,k}(b) \). (2.16) together with (3.41) give (3.42). \( \Box \)

**Corollary 3.8.** The function \( \ln \Gamma_{q,k} \) is superadditive for \( x \geq k \) and \( k \geq 1 \), in the sense that

\[
\ln \Gamma_{q,k}(a+b) \geq \ln \Gamma_{q,k}(a) + \ln \Gamma_{q,k}(b)
\]

**Proof.** For all \( a, b \geq k \), we have

\[
\ln \Gamma_{q,k}(a+b) \geq \ln \frac{[a]_q [b]_q}{[k]_q} + \ln \Gamma_{q,k}(a) + \ln \Gamma_{q,k}(b)
\]

\[
\geq \ln \Gamma_{q,k}(a) + \ln \Gamma_{q,k}(b),
\]

which completes the proof. \( \Box \)
Corollary 3.9. For a $k \geq n$ and $n = 1, 2, \ldots$, we have

$$\Gamma_{q,k}(n + 1) = \Gamma_{q,k}(n + 1,a) \geq \frac{[n - 1]q^{-1}[a]^{2(n-1)}}{[k]q^{-1}} [\Gamma_{q,k}(a)]^n. \quad (3.43)$$

Proof. We proceed by induction on $n$.

It is clear that the inequality is true for $n = 1$.

Suppose that $(3.43)$ holds for an integer $n \geq 1$ and let us prove it for $n + 1$.

By (3.41), we have

$$\Gamma_{q,k}((n + 1)a) = \Gamma_{q,k}(na + a) \geq \frac{[na]q[a]q^{-1}[a]^{2(n-1)}}{[k]q^{-1}} [\Gamma_{q,k}(a)]^n \Gamma_{q,k}(a) \quad (3.44)$$

and by hypothesis, we have

$$\Gamma_{q,k}(na) \geq \frac{[n - 1]q^{-1}[a]^{2(n-1)}}{[k]q^{-1}} [\Gamma_{q,k}(a)]^n. \quad (3.45)$$

The use of the fact that $[na]q = [n]q^n [a]q$, gives

$$\Gamma_{q,k}((n + 1)a) \geq \frac{[na]q[a]q^{-1}[a]^{2(n-1)}}{[k]q^{-1}} [\Gamma_{q,k}(a)]^n \Gamma_{q,k}(a) \geq \frac{[n]q^n [a]^{2n}}{[k]q^{-1}} [\Gamma_{q,k}(a)]^{n+1}. \quad (3.46)$$

The inequality $(3.43)$ is then true for $n + 1$. \hfill \Box

For a given real $m > 0$ and a nonnegative integer $n$, consider the mapping

$$\Gamma_{q,k,m,n}(x) = \Gamma_{q,k}^{(2n)}(x + m) \frac{\Gamma_{q,k}^{(2n)}(m)}{\Gamma_{q,k}^{(2n)}(m)}. \quad (3.47)$$

We have the following result.

Corollary 3.10. The mapping $\Gamma_{q,k,m,n}(\cdot)$ is suppermultiplicative on $[0, \infty)$, in the sense

$$\Gamma_{q,k,m,n}(x + y) \geq \Gamma_{q,k,m,n}(x) \Gamma_{q,k,m,n}(y).$$

Proof. Fix $x, y$ in $[0, \infty)$ and put $p = x + y + m$ and $r = y$. We have

$$y(x + y + m - m - y) = xy \geq 0.$$

So, the theorem 3.2 leads to

$$\Gamma_{q,k}^{(2n)}(m) \Gamma_{q,k}^{(2n)}(x + y + m) \geq \Gamma_{q,k}^{(2n)}(x + m) \Gamma_{q,k}^{(2n)}(y + m), \quad (3.48)$$

which is equivalent to

$$\Gamma_{q,k,m,n}(x + y) \geq \Gamma_{q,k,m,n}(x) \Gamma_{q,k,m,n}(y). \quad (3.49)$$

This achieves the proof. \hfill \Box

4 Inequalities via the $q$-Hölder’s one

We begin this section by recalling the $q$-analogue of the Hölder’s integral inequality \cite{3}.

Lemma 4.1. Let $p$ and $p'$ be two positive reals satisfying $\frac{1}{p} + \frac{1}{p'} = 1$, $f$ and $g$ be two functions defined on $I$. Then

$$\left| \int_I f(x)g(x)d_qx \right| \leq \left( \int_I |f(x)|^p d_qx \right)^{\frac{1}{p}} \left( \int_I |g(x)|^{p'} d_qx \right)^{\frac{1}{p'}}. \quad (4.48)$$
Owing this lemma, one can establish some new inequalities involving the $q,k$-Gamma and $q,k$-Beta functions.

**Theorem 4.4.** Let $n$ be a nonnegative integer, $x, y$ be two positive real numbers and $a, b$ be two nonnegative real numbers such that $a + b = 1$. Then

$$\Gamma_{q,k}^{(2n)}(ax + by) \leq \left[ \Gamma_{q,k}^{(2n)}(x) \right]^a \left[ \Gamma_{q,k}^{(2n)}(y) \right]^b,$$

(4.49)

that is, the mapping $\Gamma_{q,k}^{(2n)}$ is logarithmically convex on $(0, \infty)$.

**Proof.** Consider the following functions defined on $I = [0, \left( \frac{[k]}{(1-q)^{1/k}} \right)]$,

$$f(t) = t^{a(x-1)} \left( E_{q,k}^{-q^k \frac{k}{nq}}(\text{Int})2^n \right)^a$$

and

$$g(t) = t^{b(y-1)} \left( E_{q,k}^{-q^k \frac{k}{nq}}(\text{Int})2^n \right)^b.$$

By application of the $q$-Hölder’s integral inequality, with $p = \frac{1}{a}$, we get

$$\int_{I} t^{a(x-1)} t^{b(y-1)} E_{q,k}^{-q^k \frac{k}{nq}}(\text{Int})2^n dq t \leq \left[ \int_{I} t^{a(x-1)(1/a)} E_{q,k}^{-q^k \frac{k}{nq}}(\text{Int})2^n dq t \right]^a \times \left[ \int_{I} t^{b(y-1)(1/b)} E_{q,k}^{-q^k \frac{k}{nq}}(\text{Int})2^n dq t \right]^b,$$

which is equivalent to

$$\int_{I} t^{ax+by-1} E_{q,k}^{-q^k \frac{k}{nq}}(\text{Int})2^n dq t \leq \left[ \int_{I} t^{x-1} E_{q,k}^{-q^k \frac{k}{nq}}(\text{Int})2^n dq t \right]^a \left[ \int_{I} t^{y-1} E_{q,k}^{-q^k \frac{k}{nq}}(\text{Int})2^n dq t \right]^b.$$

Then, (4.49) is a direct consequence of (2.13). □

**Corollary 4.11.** Let $(p, p'), (m, m') \in (0, \infty)^2$ such that $p + p' = m + m'$ and $a, b \geq 0$ with $a + b = 1$. Then, we have

$$B_{q,k}(a(p, p') + b(m, m')) \leq \left[ B_{q,k}(p, p') \right]^a \left[ B_{q,k}(m, m') \right]^b.$$  (4.50)

**Proof.** On the one hand, we have

$$B_{q,k}(a(p, p') + b(m, m')) = B_{q,k}(ap + bm, ap' + bm') = \frac{\Gamma_{q,k}(ap + bm)\Gamma_{q,k}(ap' + bm')}{\Gamma_{q,k}(ap + bm + ap' + bm')}$$

$$= \frac{\Gamma_{q,k}(ap + bm)\Gamma_{q,k}(ap' + bm')}{\Gamma_{q,k}(a(p + p') + b(m + m'))}.$$

Since $p + p' = m + m'$ and $a + b = 1$, we have

$$\Gamma_{q,k}(a(p + p') + b(m + m')) = \Gamma_{q,k}(p + p') = \Gamma_{q,k}(m + m').$$

(4.51)

On the other hand, from Theorem 4.4 with $n = 0$, we obtain

$$\Gamma_{q,k}(ap + bm) \leq \left[ \Gamma_{q,k}(p) \right]^a \left[ \Gamma_{q,k}(m) \right]^b.$$  (4.52)

and

$$\Gamma_{q,k}(ap' + bm') \leq \left[ \Gamma_{q,k}(p') \right]^a \left[ \Gamma_{q,k}(m') \right]^b.$$  (4.53)

Thus

$$\Gamma_{q,k}(ap + bm)\Gamma_{q,k}(ap' + bm') \leq \left[ \Gamma_{q,k}(p)\Gamma_{q,k}(p') \right]^a \left[ \Gamma_{q,k}(m)\Gamma_{q,k}(m') \right]^b.$$  (4.54)

From (4.51), we deduce that

$$\frac{\Gamma_{q,k}(ap + bm)\Gamma_{q,k}(ap' + bm')}{\Gamma_{q,k}(a(p + p') + b(m + m'))} \leq \left[ \frac{\Gamma_{q,k}(p)\Gamma_{q,k}(p')}{\Gamma_{q,k}(p + p')} \right]^a \left[ \frac{\Gamma_{q,k}(m)\Gamma_{q,k}(m')}{\Gamma_{q,k}(m + m')} \right]^b,$$

(4.55)

which completes the proof. □
Now, we recall that the logarithmic derivative of the $q,k$-Gamma function is defined on $(0,\infty)$, by
\[ \Psi_{q,k}(x) = \frac{\Gamma_{q,k}'(x)}{\Gamma_{q,k}(x)}. \]

The following result gives some properties of the function $\Psi_{q,k}$.

**Theorem 4.5.** $\Psi_{q,k}$ is monotonic non-decreasing and concave on $(0,\infty)$.

**Proof.** By taking $n = 0$ in Theorem 4.4 we obtain
\[ \Gamma_{q,k}(ax + by) \leq \left[ \Gamma_{q,k}(x) \right]^a \left[ \Gamma_{q,k}(y) \right]^b, \]
for $x, y > 0$ and $a, b \geq 0$ such that $a + b = 1$.
So the function $\ln \Gamma_{q,k}$ is convex. Then the monotonicity of $\Psi_{q,k}$ follows from the relation
\[ \frac{d}{dx} \left[ \ln \Gamma_{q,k}(x) \right] = \frac{\Gamma_{q,k}'(x)}{\Gamma_{q,k}(x)} = \Psi_{q,k}(x), \quad x > 0. \]

On the other hand, since
\[ \Gamma_{q,k}(x) = \frac{(1 - q^x)^{\infty}}{(1 - q^x)^{\infty}(1 - q)^{\frac{x}{k} - 1}}, \quad (4.56) \]
we obtain, for $x > 0$, 
\[ \Psi_{q,k}(x) = \frac{d}{dx} \left[ \ln \Gamma_{q,k}(x) \right] = -\frac{1}{k} \ln(1 - q) + \ln q \sum_{j=0}^{\infty} \frac{q^{x+jk}}{1 - q^{x+jk}}, \]
\[ = -\frac{1}{k} \ln(1 - q) + \ln q \sum_{j=0}^{\infty} q^{x+jk} \sum_{n=0}^{\infty} q^{(x+jk)n} = -\frac{1}{k} \ln(1 - q) + \ln q \sum_{n=0}^{\infty} \frac{q^{(n+1)x}}{1 - q^{(n+1)x}} \]
\[ = -\frac{1}{k} \ln(1 - q) + \ln q \int_0^q \frac{t^{x-1}}{1 - tk} dt. \]

Now, let $x, y > 0$ and $a, b \geq 0$ such that $a + b = 1$. Then
\[ \Psi_{q,k}(ax + by) + \frac{1}{k} \ln(1 - q) = \frac{\ln q}{(1 - q)} \int_0^q \frac{t^{ax+by-1}}{1 - tk} dt = \frac{\ln q}{(1 - q)} \int_0^q \frac{t^{a(x-1)+b(y-1)} dt}{1 - t^k}. \quad (4.57) \]
Since the mapping $x \mapsto t^x$ is convex on $\mathbb{R}$ for $t \in (0,1)$, we have
\[ t^{a(x-1)+b(y-1)} \leq at^{x-1} + bt^{y-1}, \quad \text{for} \ t \in [0,q], \ x, y > 0. \]

Thus,
\[ \frac{\ln q}{(1 - q)} \int_0^q \frac{t^{ax+by-1}}{1 - tk} dt \geq a \left( \frac{\ln q}{(1 - q)} \int_0^q \frac{t^{x-1}}{1 - tk} dt \right) + b \left( \frac{\ln q}{(1 - q)} \int_0^q \frac{t^{y-1}}{1 - tk} dt \right) \quad (4.58) \]
According to the relations (4.57) and (4.58), we have
\[ \Psi_{q,k}(ax + by) + \frac{1}{k} \ln(1 - q) \geq a(\Psi_{q,k}(x) + \frac{1}{k} \ln(1 - q)) + b(\Psi_{q,k}(y) + \frac{1}{k} \ln(1 - q)) \]
\[ \geq a\Psi_{q,k}(x) + b\Psi_{q,k}(y) + \frac{1}{k} \ln(1 - q). \]

This proves the concavity of the function $\Psi_{q,k}$. \qed
5 Inequalities via the $q$-Grüss’s one


Lemma 5.2. Assume that $m \leq f(x) \leq M$, $\varphi \leq g(x) \leq \Phi$, for each $x \in [a,b]$, where $m, M, \varphi, \Phi$ are given real constants. Then

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)d_qx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{1}{4}(M-m)(\Phi-\varphi). \quad (5.59)$$

As application of the previous inequality we state the following result

Theorem 5.6. Let $m, n > 0$, we have

$$\left| B_{q,k}(m+k, n+k) - \frac{1}{[m+1]_q[n+k]_q} \right| \leq \frac{1}{4}. \quad (5.60)$$

Remark that from the relations (2.16) and (2.11), the inequality (5.60) is equivalent to

$$| \Gamma_{q,k}(m+n+2k) - \Gamma_{q,k}(m+2k)\Gamma_{q,k}(m+k)| \leq \frac{1}{4}[m+1]_q[n+k]_q \Gamma_{q,k}(m+n+2k). \quad (5.61)$$

Proof. Consider the functions

$$f(u) = u^m, \quad g(u) = u^{k-1}(1 - q^k u^k)^{\frac{u}{2}}_q, \quad u \in [0,1], \quad m, n > 0.$$  

We have

$$0 \leq f(u) \leq 1 \quad \text{and} \quad 0 \leq g(u) \leq 1 \quad \forall \ u \in [0,1].$$

Then, using the $q$-Grüss’ integral inequality, we obtain

$$\left| \int_0^1 u^{m+k-1}(1 - q^k u^k)^{\frac{u}{2}}_q d_qu - \int_0^1 u^m d_qu \int_0^1 u^{k-1}(1 - q^k u^k)^{\frac{u}{2}}_q d_qu \right| \leq \frac{1}{4}. \quad (5.62)$$

The inequality (5.60) follows from the definition of the $q,k$-Beta function (2.15) and the following facts:

$$\int_0^1 u^m d_qu = \frac{1}{[m+1]_q} \quad \text{and} \quad \int_0^1 u^{k-1}(1 - q^k u^k)^{\frac{u}{2}}_q d_qu = B_{q,k}(k,n+k) = \frac{1}{[n+k]_q}. \quad \Box$$

References


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