Quantum finite automata using quantum logic

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Abstract
Two types of Quantum Finite Automata are, the Measure once quantum finite automata (MO-QFA) proposed by Moore and Crutchfield \cite{moore1996} and the Many measure one-way quantum finite automata (MM-QFA) proposed by Kondacs and Waltrous \cite{kondacs1997}. In both cases it is proved that the language accepted is a subset of regular language. In this paper we define a Quantum Finite Automata using quantum logic. The logic underlying Quantum mechanics is not a Boolean algebra. It is an orthomodular lattice. This logic is called quantum logic. By using this logic we study about various properties of QFA’s.

Keywords
Quantum Logic, Orthomodular lattice, Quantum Finite Automata, Quantum Regular Language.

AMS Subject Classification
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1. Introduction
The quantum logic was first introduced by Birkhoff and von Neumann \cite{birkhoff1936} in connection with Quantum Mechanics. In Von Neumann’s Hilbert space formalism of quantum mechanics the behavior of a quantum mechanical system is described by a closed subspace of a Hilbert space. Since the set of closed subspaces of a Hilbert space is an orthomodular lattice, Birkhoff and Von Neumann suggested to use orthomodular lattice as the logic of quantum mechanics.

The quantum computational model of Finite Automata has been introduced by multiple authors with two different definitions. The Measure once one way quantum finite automata (MO-1QFA) proposed by Moore and Crutchfield \cite{moore1996} and the Many measure one-way quantum finite automata (MM-1QFA) proposed by Kondacs and Waltrous \cite{kondacs1997}. A lot of works were done to study about the power of QFA. In this paper we define a QFA with the help of quantum logic. This logical approach helps to study about the properties of QFA in a different way.

The automata theory based on quantum logic was proposed by Ying in \cite{ying2006}. In his work he introduced an orthomodular lattice valued classical Automata and he discussed about its properties. Many works were done on this line after his work, like \cite{ying2008}.

In our QFA model using quantum logic, we used the concept probability measurement in quantum logic. Detailed study about probability measurement in orthomodular lattice were done in \cite{yin2008} and \cite{yin2009}.

The rest of the paper is organized as follows. In section 2 we recall some definitions that we used in this paper. In section 3 we gave the definition of Quantum Finite Automata using Quantum Logic. Then we give an example of a QFA using Quantum logic. In section 5 we studied about the closure properties of Quantum Regular Languages.

2. Preliminaries
In this section, we recall the definitions of two types of Quantum Finite Automata. Then we discussed about the complete orthomodular lattice which is called the quantum logic.
Definition 2.1. A Measure Once Quantum Finite Automata is defined as a 6-tuple

\[ M = (Q, \Sigma, \delta, q_0, Q_{acc}, Q_{rej}) \]

where,
- \( Q \) is the finite set of quantum states
- \( \Sigma \) is the set of input symbols
- \( q_0 \) is the initial quantum state
- \( Q_{acc} \) is the set of accepting states
- \( Q_{rej} \) is the set of rejecting states
- \( \delta \) is the unitary transformation defined on the Hilbert space spanned by the states in \( Q \)

For each \( \sigma \in \Sigma \), \( \delta_\sigma \) is the unitary transformation defined on the Hilbert space spanned by the states in \( Q \).

For a given input string \( w = \sigma_1 \sigma_2 \cdots \sigma_n \), the procedure is similar to that of Measure Once Quantum Finite Automata except that after every measurement the accepting or rejecting action is performed on the resulting states. Here the projective measurement consists of \( \{P_\sigma, P_r, P_n\} \) where \( P_\sigma \), \( P_r \), and \( P_n \) are the projections onto the subspaces spanned by \( Q_{acc} \), \( Q_{rej} \) and \( Q_{non} \) respectively (\( Q_{non} = Q - (Q_{acc} \cup Q_{rej}) \)). The accepting and rejecting probabilities are given by

\[
\begin{align*}
p(M) &= \sum_{i=0}^{l-1} |(P_\sigma \delta_{\sigma_{i+1}} \cdots \delta_{\sigma_l} q_0)|^2 \\
p(M) &= \sum_{i=0}^{l-1} |(P_r \delta_{\sigma_{i+1}} \cdots \delta_{\sigma_l} q_0)|^2 \\
p(M) &= \sum_{i=0}^{l-1} |(P_n \delta_{\sigma_{i+1}} \cdots \delta_{\sigma_l} q_0)|^2
\end{align*}
\]

Let \( L \) be a complete orthomodular lattice under \( \leq \) and \( \perp \). Then a quantum finite automata is defined using \( L \) as follows.

Definition 3.1. A quantum finite automata using quantum logic is defined as

\[ M = (Q, \Sigma, \delta, q_0, Q_{acc}) \]

where,
- \( Q \) is a finite set of states
- \( \Sigma \) is the set of input alphabets
- \( q_0 \) is the initial state
- \( Q_{acc} \subseteq Q \) is the set of accepting states
- \( \delta \) is the transition function,

\[ \delta : Q \times \Sigma \times Q \to l \]

If \( w = \sigma_1 \sigma_2 \cdots \sigma_n \), then the lattice value of the word \( w \) is defined as

\[ l_M(w) \defeq \bigvee \{ \delta(q_0, \sigma_1, q_1) \land \cdots \land \delta(q_{n-1}, \sigma_n, q_n) : q_0, q_1, \cdots, q_{n-1} \in Q, q_n \in Q_{acc} \} \]

Then we measure this \( l_M(w) \) using a probability measure defined on \( L \) and denote it as \( P(w) \). Quantum finite automata accepted a language \( L \) with probability \( \lambda \) if \( P(w) \geq \lambda \) for all \( w \) in \( L \). A language accepted by a QFA is called Quantum Regular Language (QRL).
4. Example

Let $\otimes^2\mathbb{C}^2$ be the 2 qubit space, where $\mathbb{C}$ denote set of complex numbers. The set of all closed subspaces of the Hilbert space $\otimes^2\mathbb{C}^2$ form an orthomodular lattice $(\mathcal{L}, \leq, \land, \lor, 0, 1)$. $q_0 = |0\rangle \langle 0|$, $q_1 = |1\rangle \langle 1|$, $q_2 = |1\rangle \langle 0| + |0\rangle \langle 1|$ are the basis states in the 2-qubit state space. The automata is defined as $M = (Q, \Sigma, \delta, q_0, Q_{acc})$ where $Q = \{q_0, q_1, q_2\}$, $\Sigma = \{a, b\}$, $Q_{acc} = \{q_2\}$. $a_j = \text{span}\{|i\rangle \langle j|\}$ denote the closed subspace spanned by $|i\rangle \langle j|$, $i, j = 0, 1$.

\[\delta(q_0, a, q_0) = a_00, \delta(q_0, b, q_2) = a_{11}, \delta(q_0, a, q_1) = a_{00}, \delta(q_1, b, q_2) = a_{00}\]

Now $I_M = \begin{cases} a_{00}, \text{ if } w = a^n b, n > 0 \\ a_{11}, \text{ if } w = b \\ 0, \text{ otherwise} \end{cases}$

In this example we use the probability measure $p_\theta : L \rightarrow [0, 1]$ where $p_\theta(S) = ||P^S\phi||^2$ where $P^S$ is the projection operator corresponding to the closed space $S$ and $\phi$ is the initial state of the QFA.

Now the language accepted by the QFA is $L(M) = \{a^n b : n > 0\}$ with probability 1.

5. Closure properties of quantum regular language

Theorem 5.1. If $A$ and $B$ are quantum regular languages with probability $\lambda$ and $\mu$ respectively then $A \cap B$ is also a quantum regular language with probability less than or equal to $\max\{\lambda, \mu\}$.

Proof. Let $A$ be a QRL with accepting probability $\lambda$ and $B$ be a QRL with accepting probability $\mu$. That is there exist two quantum finite automata $M_A$ and $M_B$ such that $L(M_A) = A$ and $L(M_B) = B$. Now we construct a QFA $M_C$ that accepts $A \cup B$. $M_C = (Q_C, \Sigma, \delta_C, r_0, F_C)$ where $Q_C = Q_A \times Q_B$ and $F_C = F_A \times F_B$.

\[\delta_C((p, q), a, (r, s)) = \delta_A(p, a, r) \land \delta_B(q, a, s)\]

Let $x = a_1 a_2 \cdots a_n \in A \cap B$. Then there exist a path $q_0 q_1 \cdots q_n$ in $M_A$ and a path $s_0 s_1 \cdots s_n$ in $M_B$ labeled by $x$ and whose lattice value is greater than zero. Therefore there exist atleast one path $(q_0, s_0)(q_1, s_1) \cdots (q_n, s_n)$ which is labeled by $x$ in $M_C$ whose lattice value is greater than zero. Since $x \in A \cap B$, $p_A(x) \geq \lambda$ and $p_B(x) \geq \mu$.

Now
\[
p_C(x) = p(l_C(x))
\]
\[l_C(x) = \lor\{\delta_C((q_0, s_0), a_1, (q_1, s_1)) \land \delta_C((q_1, s_1), a_2, (q_2, s_2)) \land \cdots \land \delta_C((q_{n-1}, s_{n-1}), a_n, (q_n, s_n))\}\]
\[= \lor\{\delta_A(q_0, a_1, q_1) \land \delta_B(s_0, a_1, s_1) \land \cdots \land \delta_A(q_{n-1}, a_n, q_n) \land \delta_B(s_{n-1}, a_n, s_n)\}\]
\[= l_A(x) \lor l_B(x)\]

Therefore
\[
p_C(x) \leq p(l_A(x) \lor l_B(x))\]

Since
\[
l_A(x) \land l_B(x) \leq l_A(x)\text{ and}
\]
\[l_A(x) \land l_B(x) \leq l_A(x)\text{.}
\]
\[p(l_A(x) \land l_B(x)) \leq \min\{p_A(x), p_B(x)\}\]

\[\Rightarrow p_C(x) \leq \min\{\lambda, \mu\}.\]

Therefore $A \cap B$ is accepted by the QFA $M_C$ with a probability less than or equal to $\min\{\lambda, \mu\}$.

\[\square\]

Theorem 5.2. If $A$ and $B$ are Quantum Regular Languages with accepting probability $\lambda$ and $\mu$ respectively then their union, $A \cup B$ is a QRL with probability greater than or equal to $\max\{\lambda, \mu\}$.

Proof. Let $M_A = (Q_A, \Sigma, \delta_A, q_0, F_A)$ and $M_B = (Q_B, \Sigma, \delta_B, s_0, F_B)$ be the QFA’s accepting $A$ and $B$. To prove the theorem we will construct an automata $M_C$ which will accept the language $A \cup B$. $M_C = (Q_C, \Sigma, \delta_C, r_0, F_C)$ where $Q_C = Q_A \cup Q_B \cup \{r_0\}$, we take the assumption that $Q_A \cap Q_B = \emptyset$.

\[\delta_C((p, a), q) = \begin{cases} \delta_A(p, a, q) & \text{if } p \in Q_A \text{ and } q \in Q_B \\ \delta_B(p, a, q) & \text{if } p \in Q_B \text{ and } q \in Q_B \\ 0 & \text{otherwise} \end{cases}\]

and $\delta_C(r_0, \varepsilon, s_0) = 1$.

\[F_C = F_A \cup F_B\]

Let $x \in A \cup B$. Then there exist a path in $M_A$ or $M_B$ labeled by $x$ and whose lattice value is greater than zero. Therefore the accepting probability of $x$ in $M_C$ is greater than zero.

We know that
\[
p_C(x) = p(l_C(x))
\]
\[l_C(x) = \lor\{\delta_C((q_0, s_0), a_1, (q_1, s_1)) \land \delta_C((q_1, s_1), a_2, (q_2, s_2)) \land \cdots \land \delta_C((q_{n-1}, s_{n-1}), a_n, (q_n, s_n))\}\]

Since $\delta_C((p, a), q) = 0$ if $p \in Q_A$ and $q \in Q_B$,
\[l_C(x) = \lor\{\delta_A(q_0, a_1, q_1) \land \delta_A(q_1, a_2, q_2) \land \cdots \land \delta_A(q_{n-1}, a_n, q_n)\}\]
\[\lor\{\lor\{\delta_B(s_0, a_1, s_1) \land \delta_B(s_1, a_2, s_2) \land \cdots \land \delta_B(s_{n-1}, a_n, s_n)\}\}\]
\[= l_A(x) \lor l_B(x)\]
\[p_C(x) = p(l_A(x) \lor l_B(x)) \geq \max\{\lambda, \mu\}.\]

Therefore $A \cup B$ is a QRL with accepting probability greater than or equal to $\max\{\lambda, \mu\}$.

\[\square\]
Theorem 5.3. If A and B are Quantum Regular languages with accepting probability λ and µ respectively then their concatenation, AB is also a QRL with probability less than or equal to min{λ, µ}.

Proof. Let $M_A = (Q_A, \Sigma, \delta_A, q_0, F_A)$ and $M_B = (Q_B, \Sigma, \delta_B, s_0, F_B)$ be the QFA’s accepting the languages A and B. Now we will construct a QFA, $M_C$ which accepts the language AB. $M_C = (Q_C, \Sigma, \delta_C, r_0, F_C)$ where,

$Q_C = Q_A \cup Q_B$

$r_0 = q_0$

$F_C = F_B$

$\delta_C(p, a, q) = \begin{cases} 
\delta_A(p, a, q) & \text{if } p, q \in Q_A \\
\delta_B(p, a, q) & \text{if } p, q \in Q_B 
\end{cases}$

$\delta_C(p, \varepsilon, s_0) = 1$ for every $p \in F_A$

Let $x \in AB$. Then $x = \sigma_1 \sigma_2$ where, $\sigma_1 \in A$ and $\sigma_2 \in B$. There is a path $p_0 q_0 \cdot \cdot \cdot q_m$ in $M_A$ labeled by $\sigma_1$ and a path $s_0 s_1 \cdot \cdot \cdot s_n$ in $M_B$ labeled by $\sigma_2$ whose lattice values are greater than zero. So $p_0 q_0 \cdot \cdot \cdot q_m s_0 s_1 \cdot \cdot \cdot s_n$ is a path in $M_C$ labeled by x whose lattice value is greater than zero since $\delta_C(q_m, \varepsilon, s_0) = 1$ ($q_m \in F_A$).

Let $\sigma_1 = a_1 \cdot \cdot \cdot a_m$ and $\sigma_2 = b_1 \cdot \cdot \cdot b_n$

$p_C(x) = p(l_C(x))$

$l_C(x) = \lor (\delta_A(q_0, a_1, q_1) \cdot \cdot \cdot \delta_A(q_{m-1}, a_m, q_m) \cdot \cdot \cdot \delta_C(q_m, \varepsilon, s_0) \cdot \cdot \cdot \delta_B(s_{n-1}, b_n, s_n))$

Suppose that the supremum over all paths labeled by x occur along the path $q_0 a_1 \cdot \cdot \cdot a_m s_0 s_1 \cdot \cdot \cdot s_{nk}$.

Then

$l_C(x) = \lor (\delta_A(q_0, a_1, q_1) \cdot \cdot \cdot \delta_A(q_{m-1}, a_m, q_m) \cdot \cdot \cdot \delta_C(q_m, \varepsilon, s_0) \cdot \cdot \cdot \delta_B(s_{n-1}, b_n, s_n))$

Therefore the concatenation of the QRL’s A and B, AB is a QRL with a probability less than or equal to $\min{\lambda, \mu}$.

Now we give a pumping lemma for the Quantum Regular Languages.

6. pumping lemma for Quantum Regular Language

Theorem 6.1. Let L be an infinite Quantum Regular Language. Then there exist some positive integer m such that for any $w \in L$ with $|w| \geq m$ can be decomposed as $w = xyz$ with $|y| \leq m$, $|x| \geq 1$ such that $w_i = xy^iz$ is also in L for all $i = 0, 1, \ldots$. Also $\text{p}(w_i) = \text{p}(w)$.

Proof. The proof is similar to that in classical automata theory.

If L is a QRL then there exist a QFA, M recognizing L. Let $\{q_0, q_1, \cdot \cdot \cdot, q_n\}$ be the set of states of M. Now consider the string $w \in L$ such that $|w| \geq n + 1$. Now consider the path through which M processes the string w. Let it be $p_0, p_1, \cdot \cdot \cdot, p_f$, where $p_f \in \text{acc}$. Since this sequence has exactly $|w| + 1$ states, at least one state must be repeated. Therefore the sequence is of the form $p_0, p_1, \cdot \cdot \cdot, p_r, \cdot \cdot \cdot, p_r, \cdot \cdot \cdot, p_f$. Let $p_0, p_1, \cdot \cdot \cdot, p_r$ be labeled by x; $p_r, \cdot \cdot \cdot, p_f$ be labeled by y and $p_r, \cdot \cdot \cdot, p_f$ be labeled by z. Then $|xy| \leq n + 1$ and $|y| \geq 1$. Let $w = a_1 a_2 \cdot \cdot \cdot a_n$. Then $l(w) = \lor (\delta(q_0, a_1, p_1) \cdot \cdot \cdot \delta(p_{r-1}, a_r, p_n))$

Let $w_i = xy^iz$. Then clearly $l(w_i) = l(w)$ from the above formula. Therefore $\text{p}(w) = \text{p}(w_i)$.

7. Conclusion

In this paper we defined Quantum Finite Automata using quantum logic and give an example of a QFA using quantum logic. We also studied about some closure properties of QRL’s. The quantum logic approach makes it easier to study about the properties of Quantum Finite Automata. Also we introduced a pumping lemma for the Quantum Regular Languages.

References


