About m-domination number of graphs

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Abstract
In this paper, we have defined the concept of m-dominating set in graphs. In order to define this concept we have used the notion of m-adjacent vertices. We have also defined the concepts of minimal m-dominating set, minimum m-dominating set and m-domination number which is the minimum cardinality of an m-dominating set. We prove that the complement of a minimal m-dominating set is an m-dominating set. Also we prove a necessary and sufficient condition under which the m-domination number increases or decreases when a vertex is removed from the graph. Further we have also studied the concept of m-removing a vertex from the graph and we prove that the m-removal of a vertex from the graph always increases or does not change the m-domination number. Some examples have also been given.

Keywords
m-dominating set, minimal m-dominating set, minimum m-dominating set, private m-neighbourhood of a vertex, m-removal of a vertex.

AMS Subject Classification
05C69

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Article History: Received 12 October 2018; Accepted 17 March 2019
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1. Introduction
In the area of mixed domination several new concepts have been appeared. The concept of a vertex which m-dominates an edge and the concept of an edge which m-dominates a vertex have been defined and studied by some authors like R. Laskar, K. Peters, E. Sampathkumar, S. S. Kamath and others [3–5]. The above concepts can be used to define m-adjacent vertices and m-adjacent edges. In fact, we have defined m-adjacent vertices and m-adjacent edges in [1]. We observe that these concepts give rise to new concept called m-dominating set using m-adjacent vertices.

We also introduce the concepts of minimal m-dominating set, minimum m-dominating set and m-domination number which is the minimum cardinality of an m-dominating set.

We have also a concept called m-removal of a vertex in graphs which has been introduced in [2]. We proved the effect of m-removing a vertex on m-domination number.

2. Preliminaries and Notations

If G is a graph then E(G) denotes the edge set and V(G) denotes the vertex set of the graph. If v is a vertex of G then G\v denotes the subgraph of G obtained by removing the vertex v and all the edges incident to v. N(v) denotes the set of vertices which are adjacent to v. N[v] = N(v) ∪ v. If x is any vertex then d(x) denotes the degree of x and is the number of edges incident at x.

Definition 2.1. [1] Let u and v be two vertices of G. Then u and v are said to be m-adjacent vertices in G if there is an edge of G which m-dominates both u and v in G.

Definition 2.2. [2] Let G be a graph and v ∈ V(G). We obtain a subgraph of G by removing vertex v and certain edges which is called the subgraph obtained by m-removing the vertex v from the graph G.

Definition 2.3. [2] Let G be a graph and v ∈ V(G). The subgraph obtained by m-removing vertex v from G has the vertex set V(G) \ {v} and by removing all the edges of G which m-dominate vertex v. This subgraph is denoted as G\"m v}.
3. Main Results

**Definition 3.1.** Let $G$ be a graph and $S \subset V(G)$. Then $S$ is said to be an $m$-dominating set if for every vertex $v$ in $V(G) \setminus S$, there is a vertex $u$ in $S$ such that $u$ and $v$ are $m$-adjacent.

Note that every dominating set is an $m$-dominating set but $m$-dominating set need not be a dominating set.

**Example 3.2.** Consider the path graph $P_5$ with vertices $\{v_1, v_2, v_3, v_4, v_5\}$

\[ \text{Figure 1. } P_5 \]

Let $S = \{v_3\}$ then $S$ is an $m$-dominating set but not dominating set.

**Definition 3.3.** Let $G$ be a graph and $S \subset V(G)$ be an $m$-dominating set. Then $S$ is said to be a minimal $m$-dominating set if $S \setminus \{v\}$ is not an $m$-dominating set for every $v$ in $S$.

**Definition 3.4.** An $m$-dominating set with minimum cardinality is called a minimum $m$-dominating set. The cardinality of minimum $m$-dominating set is the $m$-domination number of the graph $G$ and it is denoted as $\gamma_m(G)$.

**Definition 3.5.** Let $G$ be a graph and $v \in V(G)$. Then $v$ is said to be an $m$-isolated vertex of $G$ if for every other vertex $u$ of $G$, $u$ is not $m$-adjacent to $v$.

Obviously, a vertex $v$ is isolated if and only if it is $m$-isolated.

**Theorem 3.6.** Let $G$ be a graph and $S \subset V(G)$ be an $m$-dominating set of $G$. Then $S$ is a minimal $m$-dominating set of $G$ if and only if for every $u \in S$ atleast one of the following two conditions holds.

(i) $u$ is not $m$-adjacent to any other vertex of $S$.

(ii) There exist a vertex $v \in V(G) \setminus S$ such that $v$ is $m$-adjacent to only one vertex of $S$ namely $u$.

**Proof.** Suppose $S$ is a minimal $m$-dominating set. Let $u \in S$. Now $S \setminus \{u\}$ is not an $m$-dominating set. Therefore, there is a vertex $v$ outside $S \setminus \{u\}$ such that $v$ is not $m$-adjacent to any vertex of $S \setminus \{u\}$.

Case (i): $v = u$

Then $u$ is not $m$-adjacent to any other vertex of $S$.

Case (ii): $v \neq u$

Then $v \notin S$.

Subcase (i): $v$ is not $m$-adjacent to any vertex of $S \setminus \{u\}$.

Subcase (ii): $v$ is $m$-adjacent to some vertex of $S$.

Therefore, $v$ is $m$-adjacent to only one vertex of $S$ namely $u$.

Conversely, suppose any of condition (i) and (ii) is satisfied for any $u \in S$.

Let $u \in S$.

Case (i): Suppose condition (i) is satisfied.

Therefore, $u$ is not $m$-adjacent to any vertex of $S \setminus \{u\}$ and also $u \notin S \setminus \{u\}$.

Case (ii): Suppose condition (ii) is satisfied.

Let $v \in V(G) \setminus S$ such that $v$ is $m$-adjacent to only one vertex of $S$ namely $u$. Then $v$ is not $m$-adjacent to any vertex of $S \setminus \{u\}$. Thus it follows that $S \setminus \{u\}$ is not an $m$-dominating set of $G$ for any $u \in S$.

Therefore, $S$ is a minimal $m$-dominating set.

**Theorem 3.7.** Let $G$ be a graph without $m$-isolated vertices and $S$ be a minimal $m$-dominating set of $G$. Then $V(G) \setminus S$ is an $m$-dominating set of $G$.

**Proof.** Let $v \in S$. Since $S$ is a minimal $m$-dominating set, (i) or (ii) of theorem (3.6) is satisfied.

Suppose (i) is satisfied. Then $v$ is not $m$-adjacent with any other vertex of $S$. Since $v$ is not an $m$-isolated vertex of $G$, $v$ is $m$-adjacent to some vertex $u$ of $G$. Then $u \in V(G) \setminus S$.

Suppose (ii) is satisfied and suppose $v$ is $m$-adjacent to some vertex of $S$. Now, there is a vertex $u$ in $V(G) \setminus S$ such that $u$ is $m$-adjacent to $v$ and $u$ is not $m$-adjacent to any other vertex of $S$.

Thus in both the cases $v$ is $m$-adjacent to some vertex of $V(G) \setminus S$. Therefore, $V(G) \setminus S$ is an $m$-dominating set of $G$.

**Corollary 3.8.** Let $G$ be a graph without $m$-isolated vertices. Then $\gamma_m(G) \leq n/2$.

**Proof.** Let $S$ be a minimum $m$-dominating set of $G$. Then $\gamma_m(G) = |S|$. Now by the theorem (3.7), $V(G) \setminus S$ is also an $m$-dominating set.

Therefore, $\gamma_m(G) \leq |V(G) \setminus S|$. Therefore, $\gamma_m(G) = \min(|S|, |V(G) \setminus S|)$. If $|S| \leq n/2$ then $\gamma_m(G) \leq n/2$. If $|V(G) \setminus S| > n/2$ then $|S| < n/2$ and therefore $\gamma_m(G) \leq n/2$.

**Definition 3.9.** Let $G$ be a graph and $x \in V(G)$. The $m$-vertex open neighbourhood of $x$ (or simply $m$-open neighbourhood of $x$) is the set $N_{mv}(x) = \{u \in V(G) \text{ such that } u \text{ is } m\text{-adjacent to } x\}$.

Also the $m$-vertex closed neighbourhood of $x$ is the set $N_{m}[x] = N_{m}(x) \cup \{x\}$.

Now we state and prove a necessary and sufficient condition under which the $m$-domination number of a graph increases when a vertex is removed from the graph.

**Theorem 3.10.** Let $G$ be a graph and $v \in V(G)$. Then $\gamma_m(G \setminus \{v\}) > \gamma_m(G)$ if and only if following conditions are satisfied

(i) $v$ is not an $m$-isolated vertex of $G$.
(ii) If $S$ is a minimum $m$-dominating set of $G$ and $v \notin S$ then there is a vertex $x$ in $V(G) \setminus S$ such that $x \notin v$ and $d(x,S) > 3$ in the subgraph $G \setminus v$.

(iii) There is no subset $S$ of $V(G) \setminus N_{mn}[v]$ such that $|S| \leq \gamma_{mn}(G)$ and it is an $m$-dominating set of $G \setminus v$.

Proof. Suppose $\gamma_{mn}(G \setminus v) > \gamma_{mn}(G)$.

(i) Suppose $v$ is an $m$-isolated vertex of $G$. Let $S$ be any minimum $m$-dominating set of $G$. Then $v \notin S$. Let $S_1 = S \setminus \{v\}$. Let $x$ be any vertex of $G \setminus v$ such that $x \notin S_1$. Then $x \notin S$. Since $S$ is an $m$-dominating set of $G$, $d(x,S) \leq 3$ in $G$. Now $v$ is an $m$-isolated vertex, $d(x,S_1) = d(x,S)$ in $G \setminus v$. Therefore, $d(x,S_1)$ in $G \setminus v \leq 3$. Thus, $x$ is $m$-adjacent to some member of $S_1$ in $G \setminus v$. This proves that $S_1$ is an $m$-dominating set in $G \setminus v$. Therefore $\gamma_{mn}(G \setminus v) \leq |S_1| < |S| = \gamma_{mn}(G)$, which is a contradiction. Therefore, $v$ cannot be an $m$-isolated vertex of $G$.

(ii) Suppose, there is a minimum $m$-dominating set $S$ of $G$ such that $v \notin S$. Suppose for every vertex $x$ which is not in $S$ and $x \notin v$, $d(x,S) \leq 3$ in $G \setminus v$. Then $S$ is an $m$-dominating set of $G \setminus v$. This implies that $\gamma_{mn}(G \setminus v) \leq |S| = \gamma_{mn}(G)$ which is a contradiction. Therefore, (ii) is satisfied.

(iii) Suppose, there is a subset $S$ of $V(G) \setminus N_{mn}[v]$ such that $|S| \leq \gamma_{mn}(G)$ and $S$ is an $m$-dominating set of $G \setminus v$. Then $\gamma_{mn}(G \setminus v) \leq |S| \leq \gamma_{mn}(G)$ which is again a contradiction. Therefore, (iii) holds.

Conversely, suppose condition (i), (ii) and (iii) are satisfied. First suppose that $\gamma_{mn}(G \setminus v) = \gamma_{mn}(G)$. Let $S$ be a minimum $m$-dominating set of $G \setminus v$. Let $x$ be any vertex of $G$ such that $x \notin S$ and $x \notin v$. Then $d(x,S) \leq 3$ in $G \setminus v$ which is $\leq 3$. Now suppose $v$ is $m$-adjacent to some vertex of $S$. Then $S$ is a minimum $m$-dominating set of $G$ and $v \notin S$. If $x \in V(G) \setminus S$ such that $x \notin v$ then $d(x,S) \leq 3$ in $G \setminus v$. This contradicts condition (ii). Therefore, $v$ cannot be an $m$-adjacent to any vertex of $S$. Then $S$ is a subset of $V(G) \setminus N_{mn}[v]$. Also, $|S| \leq \gamma_{mn}(G)$. Also, $S$ is an $m$-dominating set of $G \setminus v$. This contradicts condition (iii). Thus, $\gamma_{mn}(G \setminus v) = \gamma_{mn}(G)$ is not possible.

Suppose, $\gamma_{mn}(G \setminus v) < \gamma_{mn}(G)$. Let $S$ be a minimum $m$-dominating set of $G \setminus v$. Since $|S| < \gamma_{mn}(G)$, $S$ cannot be an $m$-dominating set of $G$. Therefore, $v$ cannot be $m$-adjacent to any vertex of $G$. Therefore, $S$ is a subset of $V(G) \setminus N_{mn}[v]$. Also $|S| \leq \gamma_{mn}(G)$. Also $S$ is an $m$-dominating set of $G \setminus v$. This again contradicts (iii). Therefore, $\gamma_{mn}(G \setminus v) < \gamma_{mn}(G)$ is also not possible. Thus, $\gamma_{mn}(G \setminus v) > \gamma_{mn}(G)$.

Corollary 3.11. Let $G$ be a graph and $v \in V(G)$ be such that $\gamma_{mn}(G \setminus v) > \gamma_{mn}(G)$ then $d(v,S) \leq 2$ for every minimum $m$-dominating set $S$ of $G$.

Proof. Let $S$ be any minimum $m$-dominating set of $G$. Suppose $v \notin S$. By (ii) of theorem (3.10), there is a vertex $x$ in $V(G) \setminus S$ such that $d(x,S) > 3$ in $G \setminus v$. However, $d(x,S) \leq 3$ in $G$. Therefore, there is a vertex $y$ in $S$ such that $d(x,y) \leq 3$. Any path from $x$ to $y$ in $G$ must contain $v$ as an internal vertex (otherwise $v$ does not appear in the path and therefore there is a path of length less than or equal to 3 between $x$ and $y$ in $G \setminus v$). Obviously, there is a path from $v$ to $y$ of length $\leq 2$. Therefore, $d(v,S) \leq 2$.

Definition 3.12. Let $G$ be a graph, $v \in V(G)$ and $S \subseteq V(G)$ such that $v \notin S$. Then private $m$-neighbourhood of $v$ with respect to $S$ is defined as $P_{mn}[v,S] = \{u \in V(G) \text{ such that } N_{mn}[u] \cap S = \{v\}\}$.

Remark 3.13. Note that if $v \in S$ and $v$ is not $m$-adjacent to any other vertex of $S$ then $v \in P_{mn}[v,S]$. If $u \in V(G) \setminus S$ then $u \in P_{mn}[v,S]$ if and only if $u$ is $m$-adjacent to only one vertex of $S$ namely $v$.

Now we state and prove a necessary and sufficient condition under which the $m$-domination number of a graph decreases when a vertex is removed from the graph.

Theorem 3.14. Let $G$ be a graph and $v \in V(G)$. Then $\gamma_{mn}(G \setminus v) < \gamma_{mn}(G)$ if and only if there is a minimum $m$-dominating set $S$ of $G$ such that $v \in S$ and $P_{mn}[v,S] = \{v\}$.

Proof. Suppose $\gamma_{mn}(G \setminus v) < \gamma_{mn}(G)$. Let $S_1$ be a minimum $m$-dominating set of $G \setminus v$. Then $S_1$ cannot be a minimum $m$-dominating set of $G$. Therefore, $d(v,S_1) > 3$. Let $S = S_1 \cup \{v\}$. Let $x \in V(G) \setminus S$ then $x \notin S_1$. Since $S_1$ is an $m$-dominating set of $G \setminus v$, $x$ is $m$-adjacent to some vertex $z$ of $S_1$ in $G \setminus v$. Then $x$ is $m$-adjacent to $z$ in $G$ also. Thus $S$ is an $m$-dominating set of $G$ and $v \notin S$. Note that as mentioned above $v$ is not $m$-adjacent to any other vertex of $S$ in $G$. Therefore, $v \in P_{mn}[v,S]$. Let $x \in V(G) \setminus S$ such that $x$ is $m$-adjacent to $v$ in $G$. Now, $x$ is $m$-adjacent to $y$ in $S$ in $G \setminus v$ such that $y \neq v$. Then $x$ is also $m$-adjacent to $y$ in $G$. Thus $x$ is $m$-adjacent to two distinct vertices of $S$. Therefore, $x \notin P_{mn}[v,S]$ if $x \in V(G) \setminus S$. Thus $P_{mn}[v,S] = \{v\}$.

Conversely, suppose there is a minimum $m$-dominating set $S$ of $G$ such that $v \in S$ and $P_{mn}[v,S] = \{v\}$. Let $S_1 = S \setminus \{v\}$. Let $x$ be a vertex of $S \setminus v$ such that $x \notin S_1$. Then $x \notin S$. Since $S$ is an $m$-dominating set of $G$, $x$ is $m$-adjacent to some vertex $z$ of $S_1$ in $G \setminus v$. Then $x$ is $m$-adjacent to $z$ in $G$ also. Suppose $y = v$. Now $x \notin P_{mn}[v,S]$. Therefore, $x$ is $m$-adjacent to some vertex $z$ of $S$ in $G$ such that $z \neq v$. Therefore, $d(x,z) \leq 3$. In $G$ let $P$ be a path in $G$ joining $x$ to $z$. If $v$ is a vertex in this path then it will imply that $d(v,z) \leq 3$ and this implies that $v$ is $m$-adjacent to $z$ and $z \in S$. This contradicts the fact that $v \notin P_{mn}[v,S]$. Thus, $v$ does not appear in this path. Thus $P$ is a path in $G \setminus v$ joining $x$ to $z$. Therefore, $x$ is $m$-adjacent to $z$ in $G \setminus v$ and $z \in S_1$. Thus $S_1$ is an $m$-dominating set in $G \setminus v$. Thus, $\gamma_{mn}(G \setminus v) \leq |S_1| < |S| = \gamma_{mn}(G)$.

Corollary 3.15. Let $G$ be a graph and $v \in V(G)$ be such that $v$ is not $m$-isolated vertex of $G$. If $\gamma_{mn}(G \setminus v) < \gamma_{mn}(G)$ then there is a minimum $m$-dominating set $S$ such that $v \notin S$.
Theorem 3.16. Let \( G \) be a graph and \( v \in V(G) \) such that \( v \) is not an m-isolated vertex in \( G \). Then \( \gamma_{mn}(G \setminus v) < \gamma_{mn}(G) \) and if only if there is a minimum m-dominating set \( S \) not containing \( v \) and a vertex \( x \) in \( S \) such that \( P_{mn}[x,S] = \{v\} \).

Proof. Suppose \( \gamma_{mn}(G \setminus v) < \gamma_{mn}(G) \). By theorem 3.14, there is a minimum m-dominating set \( S_1 \) such that \( v \in S_1 \) and \( P_{mn}[v,S_1] = \{v\} \). Let \( x \in V(G) \setminus S_1 \), which is adjacent to \( v \). Let \( S = (S_1 \setminus \{v\}) \cup \{x\} \). Then \( x \in S \) and by the corollary 3.15, \( S \) is a minimum m-dominating set of \( G \) not containing \( v \). Note that \( v \) is not m-adjacent to any vertex of \( S_1 \) because \( v \in P_{mn}[v,S_1] \). Therefore, \( v \) is adjacent to only one vertex of \( S \) namely \( x \). Thus \( v \in P_{mn}[x,S] \). Again \( x \) is m-adjacent to \( v \) and since \( v \notin P_{mn}[v,S_1] \), \( x \) is m-adjacent to some vertex of \( S \) where \( y \neq v \). Therefore, \( x \) is m-adjacent to some vertex of \( S \) and therefore \( x \notin P_{mn}[x,S] \). Let \( z \) be a vertex of \( V(G) \setminus S \) such that \( z \) is m-adjacent to \( x \). Since \( z \notin S_1 \), \( z \) is m-adjacent to some vertex \( w \) of \( S \) because \( S_1 \) is a minimum dominating set of \( G \). Thus, \( z \) is m-adjacent to two distinct vertices of \( S \) namely \( x \) and \( w \). Therefore, \( z \notin P_{mn}[x,S] \). Hence, \( P_{mn}[x,S] = \{v\} \).

Conversely, suppose there is a minimum m-dominating set \( S \) such that \( v \notin S \) and for some vertex \( x \) in \( S \), \( P_{mn}[x,S] = \{v\} \). Let \( S_1 = S \setminus \{x\} \). Now, \( x \notin P_{mn}[x,S] \). Therefore, \( x \) is m-adjacent to some vertex of \( S \) in \( G \). Note that \( v \) is not m-adjacent to any vertex of \( S \) except \( x \). Let \( P \) be a path in \( G \) from \( x \) to \( y \) whose length is \( \leq 3 \). If \( v \) is an internal vertex in this path then it would imply that \( d(v,y) \leq 3 \) in \( G \) and this means that \( v \) is m-adjacent to \( y \) in \( G \) and \( y 
eq x \). This is a contradiction. Thus \( v \) cannot appear as an internal vertex in the path from \( x \) to \( y \). Therefore, this is a path in \( G \setminus v \) from \( x \) to \( y \) having length \( \leq 3 \). Thus \( x \) is m-adjacent to \( y \) in \( G \setminus v \) and \( v \in S_1 \). Let \( z \) be any vertex of \( G \setminus v \) such that \( z \notin S_1 \) and \( z \neq x \). Then \( z \notin S \). Now, \( z \) is m-adjacent to some vertex of \( S \) in \( G \). If \( w = x \) then there is another vertex \( w' \) in \( S \) such that \( z \) is m-adjacent to \( w' \) in \( G \). By the same reasoning as given above we say that \( z \) is m-adjacent to \( w' \) in \( G \) also. Also \( w' \in S_1 \). Thus, we have proved that \( S_1 \) is an m-dominating set of \( G \). Therefore, \( \gamma_{mn}(G \setminus v) \leq |S_1| < |S| = \gamma_{mn}(G) \). Hence, \( \gamma_{mn}(G \setminus v) < \gamma_{mn}(G) \).

Example 3.17. Consider the path graph \( P_8 \) with vertices \( \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \)

![Figure 2. P_8](image)

Here, \( \gamma_{mn}(G) = 2 \) and \( \gamma_{mn}(G \setminus \{v_8\}) = 1 \). Let \( S = \{v_4, v_5\} \). Then \( P_{mn}[v_8, S] = \{v_8\} \).

Corollary 3.18. Let \( G \) be a graph and \( v \in V(G) \) be such that \( d(v,S) = 3 \) for every minimum m-dominating set \( S \) of \( G \). Then \( \gamma_{mn}(G \setminus v) < \gamma_{mn}(G) \).

Proof. If \( \gamma_{mn}(G \setminus v) > \gamma_{mn}(G) \) then \( d(v,S) \leq 2 \) for every minimum m-dominating set \( S \) of \( G \) which is a contradiction. If \( \gamma_{mn}(G \setminus v) < \gamma_{mn}(G) \) then there is a minimum m-dominating set \( S \) of \( G \) such that \( d(v,S) = 0 \) which is again a contradiction. Therefore, \( \gamma_{mn}(G \setminus v) = \gamma_{mn}(G) \).

Proposition 3.19. Let \( G \) be a graph and \( F \) be a set of edges of \( G \). Then \( \gamma_{mn}(G \setminus F) \geq \gamma_{mn}(G) \).

Proof. Let \( S \) be a minimum m-dominating set of \( G \setminus F \). Let \( x \in V(G) \setminus S \). Now, \( x \) is m-adjacent to some vertex of \( S \) in \( G \setminus F \). Therefore, there is an edge \( e \) in the graph \( G \setminus F \) which m-dominates both \( x \) and \( y \). Therefore, \( e \) m-dominates \( x \) and \( y \) in \( G \) also. Therefore, \( x \) and \( y \) are m-adjacent in \( G \) also. Thus, \( x \) is m-adjacent to some vertex \( y \) of \( S \) in \( G \). Therefore, \( \gamma_{mn}(G \setminus F) \geq |S| = \gamma_{mn}(G) \).

Proposition 3.20. Let \( G \) be a graph and \( v \in V(G) \). Then \( \gamma_{mn}(G \setminus \{v\}) \geq \gamma_{mn}(G) \).

Proof. Note that \( G \setminus \{v\} \) is obtained by removing those edges of \( G \) which m-dominate \( v \) but which are not incident to \( v \). These are the edges of \( G \setminus v \). Let \( F \) be the set of these edges. Then by the proposition 3.19, \( \gamma_{mn}(G \setminus \{v\}) = \gamma_{mn}(G \setminus F) \geq \gamma_{mn}(G) \).

Proposition 3.21. Let \( G \) be a graph and \( v \in V(G) \) be a non-isolated vertex of \( G \). Then \( \gamma_{mn}(G \setminus \{v\}) \geq \gamma_{mn}(G) \).

Proof. Let \( T \) be a minimum m-dominating set of \( G \setminus \{v\} \). Then \( T \) contains all m-isolated vertices of \( G \setminus \{v\} \). Now every neighbour of \( v \) is an m-isolated vertex of \( G \setminus \{v\} \). Therefore, every neighbour of \( v \) is an element of \( T \). Thus \( T \) is a minimum dominating set of \( G \). Therefore, \( \gamma_{mn}(G) \leq |T| = \gamma_{mn}(G \setminus \{v\}) \).

Theorem 3.22. Let \( G \) be a graph and \( v \in V(G) \) be such that \( d(v) \geq 2 \). Then \( \gamma_{mn}(G \setminus \{v\}) \geq \gamma_{mn}(G) \).

Proof. Suppose \( S \) is a minimum m-dominating set of \( G \setminus \{v\} \). Let \( S_1 = S \setminus (N(v) \setminus \{v\}) \). Then \( |S_1| < |S| \). Let \( x \) be any vertex of \( G \) such that \( x \notin S_1 \). If \( x \in N(v) \) then \( x \) is adjacent to \( v \) and of course \( v \in S_1 \). Suppose, \( x \notin N(v) \). Then \( x \notin S \) and also \( x \neq v \). Thus \( x \) is a vertex of \( G \setminus \{v\} \) and \( x \notin S \). Therefore, \( x \) is m-adjacent to some vertex \( y \) of \( S \). Therefore, \( d(x,y) \leq 3 \).
in \( G \setminus \{v\} \). Since elements of \( N(v) \) are isolated vertices in \( G \setminus \{v\} \), \( y \notin N(v) \) and hence \( y \in S_1 \). Also \( d(x,y) \leq 3 \) in \( G \). Thus, \( x \) is \( m \)-adjacent to \( y \) where \( y \in S_1 \). Thus, \( S_1 \) is an \( m \)-dominating set in \( G \). Therefore, \( \gamma_{mv}(G) \leq |S_1| < |S| = \gamma_{mv}(G \setminus \{v\}) \).

**Definition 3.23.** Let \( G \) be a graph, \( S \subseteq V(G) \) and \( v \in S \). Then the external private \( m \)-neighbourhood of \( v \) with respect to \( S \) is \( E_P[m, n][v, S] = \{w \in V(G) \setminus S \mid w \text{ is } m \text{-adjacent to } v \text{ in } G \text{ but } w \text{ is not } m \text{-adjacent to any other member of } S\} \).

**Theorem 3.24.** Let \( G \) be a graph, \( v \) be a pendant vertex of \( G \) and \( u \) be its neighbour. Then \( \gamma_{mv}(G \setminus \{v\}) = \gamma_{mv}(G) \) if and only if there is a minimum \( m \)-dominating set \( S \) of \( G \) such that \( u \in S, v \notin S \) and \( E_P[m, n][u, S] \subseteq \{v\} \).

**Proof.** It is already true that \( \gamma_{mv}(G \setminus \{v\}) \geq \gamma_{mv}(G) \). Suppose there is a minimum \( m \)-dominating set \( S \) of \( G \) such that \( u \in S, v \notin S \) and the condition is satisfied. Let \( x \) be a vertex of \( G \setminus \{v\} \) such that \( x \notin S \). Now \( x \) is \( m \)-adjacent to some vertex \( y \) of \( S \) in \( G \). If \( y = u \) then \( x \) is not \( m \)-adjacent to \( u \) in \( G \setminus \{v\} \). Since the condition is satisfied, \( x \) is \( m \)-adjacent to some vertex \( z \) of \( S \) such that \( z \neq u \). If \( x \) is not \( m \)-adjacent to \( u \) then \( x \) is \( m \)-adjacent in \( G \) to some vertex \( w \) in \( S \) such that \( w \neq u \). Then \( x \) is \( m \)-adjacent to \( w \) in \( G \setminus \{v\} \) also. The path joining \( x \) and \( w \) cannot contain \( u \) as \( x \) is not \( m \)-adjacent to \( u \). Thus from both the above cases it follows that \( S \) is an \( m \)-dominating set in \( G \setminus \{v\} \). Thus, \( \gamma_{mv}(G \setminus \{v\}) \leq |S| = \gamma_{mv}(G) \). Hence, \( \gamma_{mv}(G \setminus \{v\}) = \gamma_{mv}(G) \).

Conversely, suppose \( \gamma_{mv}(G \setminus \{v\}) = \gamma_{mv}(G) \). Let \( S \) be a minimum \( m \)-dominating set of \( G \setminus \{v\} \). Since \( u \) is an isolated vertex in \( G \setminus \{v\} \), \( u \in S \). Obviously, \( v \notin S \). Let \( z \) be a vertex such that \( z \notin S \) and \( z \neq v \). Suppose, \( z \) is \( m \)-adjacent to \( u \) in \( G \). Since \( S \) is an \( m \)-dominating set of \( G \setminus \{v\} \), \( z \) is \( m \)-adjacent in \( G \setminus \{v\} \) to some vertex \( u' \) of \( S \). Note that \( u' \neq u \) because \( u \) is an isolated vertex in \( G \setminus \{v\} \). Now \( d(z,u') \leq 3 \) in \( G \setminus \{v\} \). Therefore, \( d(z,u') \leq 3 \) in \( G \). Thus we have proved that \( z \in V(G) \setminus S, z \neq v \) and if \( z \) is \( m \)-adjacent to \( u \) in \( G \) then \( z \) is also \( m \)-adjacent to some other vertex \( u' \) of \( S \) in \( G \setminus \{v\} \). Note that \( S \) is an \( m \)-dominating set in \( G \) also. Since \( \gamma_{mv}(G \setminus \{v\}) = \gamma_{mv}(G), S \) is a minimum \( m \)-dominating set of \( G \) and the condition is satisfied. \( \square \)

**Acknowledgment**

The work for the second author is financially supported by INSPIRE Fellowship of the “Department of Science and Technology” of Government of India.

**References**