$C_m$-$E$- super magic graceful labeling of graphs

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Abstract
A simple graph $G$ admits an $H$-covering if every edge in $E(G)$ belongs to a subgraph of $G$ isomorphic to $H$. The graph $G$ is said to be $H$-magic if there exists a total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ such that for every subgraph $H'$ of $G$ isomorphic to $H$, $\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = M$ for some positive integer $M$. An $H$-$E$-super magic graceful labeling ($H$-$E$-SMGL) is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ with the property $f(E(G)) = \{1, 2, \ldots, q\}$ such that $\sum_{v \in V(H')} f(v) - \sum_{e \in E(H')} f(e) = M$ for some positive integer $M$. In this paper, we introduce $H$-$E$-SMGL and study $C_m$-$E$-super magic graceful labeling of generalized book graph.

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1. Introduction

Throughout this paper, we consider only finite, simple and undirected graphs. The set of vertices and edges of a graph $G(p, q)$ will be denoted by $V(G)$ and $E(G)$ respectively, $p = |V(G)|$ and $q = |E(G)|$.

A labeling of a graph $G$ is a mapping that carries a set of graph elements, usually vertices or edges into a set of numbers, usually integers. Many kinds of labelings have been defined and studied by many authors and an excellent survey of graph labelings can be found in [1].

In 1963, Sedláček [7] introduced the concept of magic labeling in graphs. A graph $G$ is magic if the edges of $G$ can be labeled by the set of numbers $\{1, 2, \ldots, q\}$ so that the sum of labels of all the edges incident with any vertex is the same [5].

A covering of $G$ is a family of subgraphs $H_1, H_2, \ldots, H_h$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i, 1 \leq i \leq h$. Then, it is said that $G$ admits an $(H_1, H_2, \ldots, H_h)$ covering. If every $H_i$ is isomorphic to a given graph $H$, then $G$ admits an $H$-covering. Suppose $G$ admits an $H$-covering. A total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, \frac{|V(G)| + |E(G)|}{M}\}$ is called an $H$-magic labeling of $G$ if there exists a positive integer $M$ (called the magic constant) such that for every subgraph $H'$ of $G$ isomorphic to $H$, $\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = M$. A graph that admits such a labeling is called $H$-magic. The function $f$ is said to be $H$-$E$-super magic labeling if $f(E(G)) = \{1, 2, \ldots, q\}$.

The notion of $H$-magic labeling was introduced by Gutierrez and Llad [2] in 2005.


Golomb [3] called such labeling as graceful. An injection $f$ from the vertices of $G$ to $\{0, 1, 2, \ldots, q\}$ is called a graceful labeling of $G$ if when we assign each edge $uv$ the label $|f(u) - f(v)|$, the resulting edge labels are distinct.

For further information about $H$-$E$-super magic graphs, see [8].

By using definitions of $H$-$E$-super magic labeling and graceful labeling, we define a new labeling called $H$-$E$-super magic graceful. An $H$-$E$-super magic graceful labeling ($H$-
E-SMGL) is a bijection $f: V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ with the property $f(E(G)) = \{1, 2, \ldots, q\}$ and $\sum_{v \in V(H')} f(v) = \sum_{e \in E(H')} f(e) = M$ for some positive integer $M$. In this paper, we introduce H-E-SMGL and study $C_m$-E-super magic graceful labeling of generalized book graph.

2. $C_m$-E-Super magic graceful graphs

A book graph $B_n = K_{1,n} \times K_2$, is defined as follows: The vertex set $V(B_n) = \{u_1, u_2\} \cup \{v_i, w_i : 1 \leq i \leq n\}$ and the edge set $E(B_n) = \{u_1, u_2\} \cup \{u_iw_i, u_iw_{i+1}, w_i : 1 \leq i \leq n\}$. The generalized book graph $B_{m,n}$ with vertex and edge sets by $V(B_{m,n}) = \{u_1, u_2\} \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(B_{m,n}) = \{u_iu_{i+1} : 1 \leq i \leq m-3\} \cup \{v_iw_j : 1 \leq j \leq n, i = m - 2\}$ $\cup \{u_iw_i : 1 \leq i \leq n\} \cup \{v_iw_j : 1 \leq i \leq n\}$. Number of vertices and edges of $B_{m,n}$ is $|V(B_{m,n})| = 2n + m - 2$ and $|E(B_{m,n})| = 3n + m - 3$. The following theorems shows that the generalised book graph $B_{m,n}$ is $C_m$-E-super magic graceful (SMG).

\textbf{Theorem 2.1.} For an odd integer $n \geq 3$ and $m \geq 4$ the generalized book graph $B_{m,n}$ is $C_m$-E-SMG.

Proof. We define a total labeling $f : V(B_{m,n}) \cup E(B_{m,n}) \to \{1, 2, 3, \ldots, 2m + 5n - (n - 1)\}$ as follows:

$f(u_i) = \begin{cases} 3m + i - 1 & \text{if } i \text{ is odd} \\ 3m - i & \text{if } i \text{ is even} \end{cases}$

$f(v_i) = \begin{cases} m - 1 & \text{if } i \leq m - 1 \\ 3m + i - 3 & \text{if } i = m + 1 \\ 3m - i & \text{if } i = m + 2 \\ 3m + i - 5 & \text{if } i = m + 3 \\ 5m + 2i - 4 & \text{if } i = m + 4 \\ 5m + 2i - 6 & \text{if } i = m + 5 \\ 5m + 2i - 8 & \text{if } i = m + 6 \\ 5m + 2i - 10 & \text{if } i = m + 7 \end{cases}$

$f(w_i) = \begin{cases} m - 1 & \text{if } i \leq m - 1 \\ 3m + i - 3 & \text{if } i = m + 1 \\ 3m - i & \text{if } i = m + 2 \\ 3m + i - 5 & \text{if } i = m + 3 \\ 5m + 2i - 4 & \text{if } i = m + 4 \\ 5m + 2i - 6 & \text{if } i = m + 5 \\ 5m + 2i - 8 & \text{if } i = m + 6 \\ 5m + 2i - 10 & \text{if } i = m + 7 \end{cases}$

Now we prove $f$ is a $C_m$-E-super magic graceful labeling. For $1 \leq i \leq n$, let $C_{m,n}(i)$ be the sub cycle of the graph $B_{m,n}$ with $V(C_{m,n}(i)) = \{u_j : 1 \leq j \leq m - 2\} \cup \{v_i, w_i\}$ and

$E(C_{m,n}(i)) = \{u_iu_{i+1} : 1 \leq j \leq m - 3\} \cup \{v_iw_j : 1 \leq j \leq n - 2\} \cup \{u_{m-1}w_i : 1 \leq i \leq n - 1\} \cup \{u_{m-1}w_m : i \leq n\}$

\textbf{Case 1:} Suppose $1 \leq i \leq \frac{m-1}{2}$. Then $\sum_{v \in V(C_m)} f(v) - \sum_{e \in E(C_m)} f(e) = \frac{m-2}{2} \sum_{j=1}^{m-2} f(u_j) + \sum_{j=1}^{m-2} f(v_j) = \frac{m-2}{2} \sum_{j=1}^{m-2} f(u_j) + \sum_{j=1}^{m-2} f(v_j)$

$= \frac{1}{2}(2m^2 + 6mn - 6m - 5n + 5)$, which is an integer since $n$ is odd.

\textbf{Case 2:} Suppose $\frac{m+1}{2} \leq i \leq n$. Then $\sum_{v \in V(C_m)} f(v) - \sum_{e \in E(C_m)} f(e) = \left[m - 2 \sum_{j=1}^{m-2} f(u_j) + \sum_{j=1}^{m-2} f(v_j) \right] - \left[m - 3 \sum_{j=1}^{m-3} f(u_ju_{j+1}) + f(u_{m-2}v_i) + f(u_{m-1}v_i) \right]$

$= \left[m - 2 \sum_{j=1}^{m-2} f(u_j) + \sum_{j=1}^{m-2} f(v_j) \right] - \left[m - 3 \sum_{j=1}^{m-3} f(u_ju_{j+1}) + f(u_{m-2}v_i) + f(u_{m-1}v_i) \right]$

$\leq \left[\frac{1}{2}(2m^2 + 6mn - 6m - 5n + 5)\right] - \left[\frac{1}{2}(2m^2 - 2m + 2m - 2)\right] = 124 - 42 = 82$.
Theorem 2.3. For an even integer \( n \geq 3 \) and \( m \geq 4 \) the generalized book graph \( B_{n,m} \) is \( C_m \)-E-SMG.

Proof. We define a total labeling \( f : V(B_{n,m}) \cup E(B_{n,m}) \rightarrow \{1, 2, 3, \ldots, 2m + 5(n - 1)\} \) as follows:

\[
\begin{align*}
f(u) = \begin{cases}
3n + m - i & \text{if } u = u_i \text{ for } 1 \leq i \leq m - 2, \\
3n + 2m - i - 2 & \text{if } u = u_i \text{ for } 1 \leq i \leq n,
\end{cases}
\end{align*}
\]

Now we prove \( f \) is a \( C_m \)-E-super magic labeling. 

**Case 1:** Suppose \( 1 \leq i \leq \frac{n}{2} \). Then

\[
\sum_{v \in V(C_m)} f(v) - \sum_{e \in E(C_m)} f(e) = \left[ \sum_{j=1}^{m-2} f(u_j) + f(v_i) + f(w_i) \right] - \left[ f(u_1u_2) + \sum_{j=2}^{m-3} f(u_ju_{j+1}) + f(u_1w_i) + f(v_iw_i) \right]
\]

\[
= [f(u_1) + f(u_2) + \cdots + f(u_{m-2}) + f(v_i) + f(w_i)] - [f(u_1u_2) + f(u_2u_3) + \cdots + f(u_{m-3}u_{m-2}) + f(u_{m-2}w_i) + f(v_iw_i) + f(u_1w_i)]
\]

\[
= [(3n + m - 3 + 1) + (3n + 5 - 3 + 2) + (3n + 5 - 3 + 3) + (3n + 6 - 5 + 4) + (5n + 2(5) - 4 - i)] - (1 + 2) + (2 + i) + \frac{1}{2}(7(2) + 7(2)) + (4(7) + 5 - 2i)
\]

\[
= 142 - 42 = 100.
\]

The graph \( B_7,5 \) is \( C_m \)-E-super magic graceful.

**Case 2:** Suppose \( 5 \leq i \leq 7 \). Then

\[
\sum_{v \in V(C_m)} f(v) - \sum_{e \in E(C_m)} f(e) = \left[ \sum_{j=1}^{3} f(u_j) + f(v_i) + f(w_i) \right] - \left[ \sum_{j=1}^{2} f(u_ju_{j+1}) + f(u_3w_i) + f(v_iw_i) \right]
\]

\[
= [f(u_1) + f(u_2) + f(u_3) + f(v_i) + f(w_i)] - [f(u_1u_2) + f(u_2u_3) + f(u_3w_i) + f(v_iw_i) + f(u_3w_i)]
\]

\[
= [(3n + 5 - 3 + 1) + (3n + 5 - 3 + 2) + (3n + 5 - 3 + 3) + (3n + 6 - 5 + 4) + (5n + 2(5) - 4 - i)] - (1 + 2) + (2 + i) + \frac{1}{2}(7(2) + 7(2)) + (4(7) + 5 - 2i)
\]

\[
= 142 - 42 = 100.
\]

The graph \( B_{7,5} \) is \( C_m \)-E-SMG when \( n \) is even.

**Example 2.4.** Consider the following graph \( G = B_{8,5} \).

Here, \( V(G) = \{u_1 \leq i \leq 3\} \cup \{v_i \leq i \leq 8\} \cup \{w_i \leq i \leq 8\} \) and \( E(G) = \{u_{i+1} : \text{for } 1 \leq i \leq 2\} \cup \{u_3v_i \leq i \leq 8\} \cup \{u_7w_i \leq i \leq 8\} \).

As discussed in Theorem 2.3, we define a total labeling \( f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, 45\} \) as follows:

\[
f(u) = \begin{cases}
26 + i & \text{if } u = u_i \leq i \leq 3; [27, 28, 29], \\
29 + i & \text{if } u = v_i \leq i \leq 8; [30, 31, \ldots, 37], \\
46 - i & \text{if } u = w_i \leq i \leq 8; [44, 45, \ldots, 48].
\end{cases}
\]

\[
f(v) = \begin{cases}
0 & \text{if } e = u_{i+1} \leq i \leq 2, \\
5 & \text{if } e = u_3v_i \leq i \leq 8, [6, 7, 8, 9], \\
8 + i & \text{if } e = u_{i+1}v_i \leq i \leq 2; [10], \\
16 + i & \text{if } e = v_iw_i \leq i \leq 8; [11, 12, 13, 14], \\
14 + i & \text{if } e = u_7w_i \leq i \leq 8; [15, 16, 17, 18], \\
35 - 2i & \text{if } e = u_3w_i \leq i \leq 8; [25, 23, 21, 19], \\
28 - 2i & \text{if } e = w_iw_i \leq i \leq 8; [26, 24, 22, 20].
\end{cases}
\]
Case 2: Suppose $1 \leq i \leq 8$ be the sub cycle of the graph $B_{8,5}$ with $V(C_m^{(i)}) = \{u_j : 1 \leq j \leq 3\} \cup \{v_i, w_i\}$ and $E(C_m^{(i)}) = \{u_{j\ 1}u_{j+1} : 1 \leq j \leq 2\} \cup \{u_1w_j, v_jw_i\} \cup \{u_3w_j\}$.

Case 1: Suppose $1 \leq i \leq 4$. Then
\[
\sum_{v \in V(C_m^{(i)})} f(v) - \sum_{e \in E(C_m^{(i)})} f(e) = \left[3 \sum_{j=1}^{3} f(u_j) + f(v_i) + f(w_i) \right] - \left[f(u_1u_2) + f(u_2u_3) + f(u_3u_1) + f(w_1w_2) \right] = [f(u_1) + f(u_2) + f(u_3) + f(v_i) + f(w_i)] - [f(u_1u_2) + f(u_2u_3) + f(u_3u_1) + f(w_1w_2)] = [(3(8) + 5 + 1 - 3) + (3(8) + 5 + 2 - 3) + (3(8) + 5 + 3 - 3) + (3(8) + 5 + 1) + (5(8) + 2(5) - 4 - i)] - \frac{3 + i}{2} + (n + 2) + (4(8) + 5 - 2 - 2i) + \frac{1}{2}(8 + 2(5) - 6 + 2i)] = 159 - 57 = 102.
\]

The graph $B_{8,5}$ is $C_m - E$-super magic graceful.

Corollary 2.5. For any integer $n \geq 2$, the book graph $B_n$ is $C_4$-supermagic.

References