The $b$-chromatic number of some degree splitting graphs

S. K. Vaidya$^a,{}^*$ and Rakhimol V. Isaac$^b$

$^a$Department of Mathematics, Saurashtra University, Rajkot - 360005, Gujarat, India.

$^b$Department of Mathematics, Christ College, Rajkot - 360005, Gujarat, India.

Abstract

A $b$-coloring of a graph $G$ is a variant of proper coloring in which each color class contains a vertex that has a neighbor in all the other color classes. We investigate some results on $b$-coloring in the context of degree splitting graph of $P_n$, $B_n$, $n$, $S_n$ and $G_n$.

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1 Introduction

In this paper we deal with finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. The notations and terminology here are used in the sense of Clark and Holton [1]. A proper $k$-coloring of a graph $G$ is a function $c : V(G) \to \{1, 2, ..., k\}$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. The color class $c_i$ is the subset of vertices of $G$ that is assigned to color $i$. The chromatic number $\chi(G)$ is the minimum number $k$ for which $G$ admits proper $k$-coloring.

A proper $k$-coloring $c$ of a graph $G$ is a $b$-coloring if for every color class $c_i$, there is a vertex with color $i$ which has at least one neighbor in every other color classes. Such vertex is called a $b$-vertex. The $b$-chromatic number of a graph $G$, denoted by $\phi(G)$, is the largest integer $k$ for which $G$ admits a $b$-coloring.

The concept of $b$-coloring was introduced by Irving and Manlove [2]. If $G$ has a $b$-coloring by $k$ colors for every integer $k$ satisfying $\chi(G) \leq k \leq \phi(G)$ then $G$ is called $b$-continuous. The $b$-spectrum $S_b(G)$ of a graph $G$ is the set of integers $k$ such that $G$ has a $b$-coloring by $k$ colors.

The concept of $b$-coloring is explored by many researchers. The bounds for the $b$-chromatic number of a graph is investigated by Kouider and Mahéo [3] while $b$-chromatic number for Peterson graph and power of a cycle is discussed by Chandrakumar and Nicholas [6]. The $b$-continuity of chordal graphs is discussed by Faik [7].

Definition 1.1. ([2], [4]) The $m$-degree of a graph $G$, denoted by $m(G)$, is the largest integer $m$ such that $G$ has $m$ vertices of degree at least $m - 1$.

Proposition 1.2. ([1]) For any graph $G$, $\chi(G) \geq 3$ if and only if $G$ has an odd cycle.

Proposition 1.3. ([2]) If $G$ admits a $b$-coloring with $m$ colors, then $G$ must have at least $m$ vertices with degree at least $m - 1$.

Proposition 1.4. ([3]) $\chi(G) \leq \phi(G) \leq m(G)$.

It is obvious that if $\chi(G) = k$, then every coloring of a graph $G$ by $k$ colors is a $b$-coloring of $G$.

*Corresponding author. 
E-mail address: samirkvaidya@yahoo.co.in (S. K. Vaidya), rakhiiisaac@yahoo.co.in (Rakhimol V. Isaac).
Proposition 1.5. \((\Box)\) If \(P_n, C_n, K_n, K_{m,n}\) and \(W_n : C_n + K_1\) are respectively path, cycle, complete graph, complete bipartite graph and wheel graph, then

1. \(\chi(C_{2n}) = 2, \chi(C_{2n+1}) = 3\).
2. \(\chi(W_n) = 3, \text{if } n \text{ is odd and } \chi(W_n) = 4, \text{if } n \text{ is even.}\)
3. \(\chi(K_{m,n}) = 2\).
4. \(\varphi(P_n) = 2, \text{if } 1 < n < 5 \text{ and } \varphi(P_n) = 3, \text{if } n \geq 5\).
5. \(\varphi(C_n) = 2, \text{if } n = 4 \text{ and } \varphi(C_n) = 3, \text{if } n \neq 4\).
6. \(\varphi(W_n) = 3, \text{if } n = 4 \text{ and } \varphi(W_n) = 4, \text{if } n \neq 4\).
7. \(\chi(K_n) = \varphi(K_n) = n\).

2 Main Results

Definition 2.1. Let \(G = (V(G), E(G))\) be a graph with \(V(G) = S_1 \cup S_2 \cup \ldots \cup S_t \cup T\) where each \(S_i\) is a set of all vertices of the same degree with at least two elements and \(T = V(G) \setminus \bigcup_{i=1}^{t} S_i\). The degree splitting graph of \(G\), denoted by \(DS(G)\), is obtained from \(G\) by adding vertices \(w_1, w_2, \ldots, w_t\) and joining \(w_i\) to each vertex of \(S_i\) for \(1 \leq i \leq t\).

Lemma 2.2. \(\chi(DS(P_n)) = \begin{cases} 2, & n = 3 \\ 3, & n \neq 3. \end{cases}\)

Proof. The path \(P_n\) has two pendant vertices and the remaining \(n - 1\) vertices are of degree two. Thus \(V(P_n) = \{v_i; 1 \leq i \leq n\} = S_1 \cup S_2\) where \(S_1 = \{v_1, v_n\}\) and \(S_2 = \{v_i; 2 \leq i \leq n - 1\}\). For obtaining \(DS(P_n)\) from \(P_n\), add two vertices \(w_1, w_2\) corresponding to \(S_1\) and \(S_2\) respectively. Thus \(V(DS(P_n)) = V(P_n) \cup \{w_1, w_2\}\) and \(E(DS(P_n)) = E(P_n) \cup \{w_1v_1, w_2v_1\} \cup \{v_2v_1, v_{2n}v_1\} \cup \{v_{2k}v_1\} \cup \{v_{2k+1}v_1\} \cup \{v_{2k}v_{2k-1}, v_{2k}v_{2k+1}\}\) where \(k \in \mathbb{N}\). Then by Proposition 1.5, \(\chi(DS(P_n)) = 3\).

Theorem 2.3. \(\varphi(DS(P_n)) = \begin{cases} 2, & n = 3 \\ 3, & n = 2,4 \\ 4, & n \geq 5. \end{cases}\)

Proof. The graphs \(DS(P_2)\) and \(DS(P_3)\) are isomorphic to \(C_3\) and \(C_4\) respectively. Then by Proposition 1.5, \(\varphi(DS(P_2)) = 3\) and \(\varphi(DS(P_3)) = 2\).

In the graph \(DS(P_4)\) there are four vertices of degree 2. Then the m-degree, \(m(DS(P_4)) = 3\). Then by Proposition 1.4, \(\varphi(DS(P_4)) \leq 3\). Moreover \(DS(P_4)\) induces a path of length greater than four, \(\varphi(DS(P_4)) \geq 3\). Hence \(\varphi(DS(P_4)) = 3\).

For \(n \geq 5\), the graph \(DS(P_n)\) has at least four vertices of degree at least 3. Then the m-degree, \(m(DS(P_n)) = 4\). Then by Proposition 1.4, \(\varphi(DS(P_n)) \leq 4\). Moreover \(DS(P_n)\) induces a path of length greater than four, \(\varphi(DS(P_n)) \geq 3\). We suppose that \(DS(P_n)\) has a b-coloring using four colors. By assigning the proper coloring as \(c(w_1) = c(w_2) = 1, c(v_{2k+1}) = 2, c(v_{2k}) = 3; k \in \mathbb{N}\) then \(\chi(DS(P_n)) = 3\).

Definition 2.4. The bistar \(B_{n,n}\) is a graph obtained by joining the center(apex) vertices of two copies of \(K_{1,n}\) by an edge.

Lemma 2.5. For all \(n\), \(\chi(DS(B_{n,n})) = 3\).

Proof. In \(B_{n,n}, V(B_{n,n}) = \{u, v, u_i, v_i; 1 \leq i \leq n\}\) and \(E(B_{n,n}) = \{uu_i, vv_i; 1 \leq i \leq n\} \cup \{uv\}\). The graph bistar \(B_{n,n}\) contains two types of vertices - pendant vertices and vertices of degree \(n + 1\). Thus \(V(B_{n,n}) = S_1 \cup S_2\) where \(S_1 = \{u, v, u, v_i; 1 \leq i \leq n\}\) and \(S_2 = \{u, v\}\). For obtaining \(DS(B_{n,n})\) from \(B_{n,n}\), we add two
vertices \( w_1 \) and \( w_2 \) corresponding to \( S_1 \) and \( S_2 \) respectively. Thus \( V(DS(B_{n,n})) = V(B_{n,n}) \cup \{w_1, w_2\} \) and \( E(DS(B_{n,n})) = E(B_{n,n}) \cup \{u_1w_1, v_1w_1, u_2w_2, v_2w_2\} \). Hence \( |V(DS(B_{n,n}))| = 2n + 4 \) and \( |E(DS(B_{n,n}))| = 4n + 3 \).

As the graph \( DS(B_{n,n}) \) contains a \( C_3 \), \( \chi(DS(B_{n,n})) \geq 3 \). If we assign the proper coloring as \( c(w_2) = 1, c(u) = 2, c(v) = 3, c(v_1) = 2, c(v_i) = c(v_1), \) for \( i = 1, 2, ..., n \), then \( \chi(DS(B_{n,n})) = 3 \) for all \( n \).

**Theorem 2.6.** For all \( n \), \( \varphi(DS(B_{n,n})) = 3 \).

**Proof.** By Lemma 2.5, \( \varphi(DS(B_{n,n})) \geq \chi(DS(B_{n,n})) = 3 \). The graph \( DS(B_{n,n}) \) has at least three vertices of degree at least two. Then \( m(DS(B_{n,n})) = 3 \) and hence by Proposition 1.4, \( \varphi(DS(B_{n,n})) \leq 3 \). Thus \( \varphi(DS(B_{n,n})) = 3 \) for all \( n \).

**Definition 2.7.** A shell \( S_n \) is the graph obtained by taking \( n - 3 \) concurrent chords in cycle \( C_n \). That is, \( S_n = P_{n-1} \cup K_1 \).

**Lemma 2.8.** \( \chi(DS(S_n)) = \begin{cases} 4, & n = 3 \\ 3, & n \neq 3. \end{cases} \)

**Proof.** In the shell graph \( S_n \), \( V(S_n) = \{u, v_1, v_2, ..., v_{n-1}\} \) where \( u \) is the apex vertex and \( E(S_n) = \{uv_i \mid 1 \leq i \leq n - 1\} \cup \{v_iv_{i+1} \mid 1 \leq i \leq n - 2\} \). Clearly \( |V(S_n)| = n \) and \( |E(S_n)| = 2n - 3 \).

There are three types of vertices

(i) vertices of degree 2,

(ii) vertices of degree 3,

(iii) a vertex of degree \( n - 1 \).

Thus \( V(S_n) = \{u, v_1, v_2, ..., v_{n-1}\} = S_1 \cup S_2 \cup T \) where \( S_1 = \{v_1, v_{n-1}\}, S_2 = \{v_i \mid 2 \leq i \leq n - 2\} \) and \( T = \{u\} = V(S_n) \setminus \bigcup_{i=1}^{2} S_i \). For obtaining \( DS(S_n) \) from \( S_n \), we add two vertices \( w_1 \) and \( w_2 \) corresponding to \( S_1 \) and \( S_2 \) respectively. Thus \( V(DS(S_n)) = V(S_n) \cup \{w_1, w_2\} \) and \( E(DS(S_n)) = E(S_n) \cup \{w_1v_1, v_{n-1}w_1\} \cup \{v_iw_2 \mid 2 \leq i \leq n - 2\} \).

When \( n = 3 \), the graph \( DS(DS(S_3)) \) is isomorphic to \( K_4 \). Then by Proposition 1.5, \( \chi(DS(S_3)) = 4 \). But when \( n \neq 3 \), \( DS(S_n) \) contains a \( C_3 \), then by Proposition 1.2, \( \chi(DS(S_n)) \geq 3 \). If we assign the colors as \( c(w_1) = c(w_2) = c(u) = 1, c(v_k) = 2, c(v_{2k}) = 3, k \in \mathbb{N} \), then \( \chi(DS(S_n)) = 3 \).

**Theorem 2.9.** \( \varphi(DS(S_n)) = \begin{cases} 3, & n = 4 \\ 4, & n \neq 4. \end{cases} \)

**Proof.** When \( n = 3 \), the graph \( DS(DS(S_3)) \) is isomorphic to \( K_4 \), by Proposition 1.5, \( \varphi(DS(S_3)) = 4 \).

When \( n = 4 \), the graph \( DS(DS(S_4)) \) has four vertices of degree at least three. Then \( m(DS(DS(S_4))) = 4 \). Then by Proposition 1.4, \( \varphi(DS(DS(S_4))) \leq 4 \). Suppose that \( DS(DS(S_4)) \) does have a \( b \)-chromatic 4-coloring. By assigning the proper coloring as \( c(u) = 1, c(v_1) = 2, c(v_2) = 3, c(v_3) = 4 \) which in turn forces to assign \( c(w_1) \) is either by the color 1 or 3 and \( c(w_2) \) is either by the color 2 or 4. This proper coloring gives the \( b \)-vertices for the color classes 1 and 3 but not for 2 and 4. Similarly all other proper coloring using 4 colors will generate \( b \)-vertices at most for two color classes only. Hence \( \varphi(DS(DS(S_4))) \neq 4 \). Thus \( \varphi(DS(DS(S_4))) \leq 3 \). Also by Lemma 2.8, \( \varphi(DS(DS(S_4))) \geq 3 \). Hence \( \varphi(DS(DS(S_4))) = 3 \).

When \( n = 5 \) and 6, the graph \( DS(DS(S_n)) \) has the \( m \)-degree four. Thus \( \varphi(DS(DS(S_5))) \leq 4 \). Suppose that \( DS(DS(S_5)) \) does have a \( b \)-chromatic 4-coloring. By assigning the proper coloring as \( c(u) = 1, c(v_1) = c(v_4) = 2, c(v_2) = 3, c(v_3) = c(v_1) = 4 \) which gives the \( b \)-vertices \( u, v_1, v_2, v_3 \) for the color classes 1, 2, 3, and 4 respectively. Thus \( \varphi(DS(DS(S_5))) = 4 \).

When \( n \geq 7 \), the graph \( DS(DS(S_n)) \) has the \( m \)-degree five. Thus \( \varphi(DS(DS(S_n))) \leq 5 \). Suppose that \( DS(DS(S_n)) \) does have a \( b \)-chromatic 5-coloring. By assigning the proper coloring as \( c(v_2) = 1, c(v_1) = 2, c(u) = 4, c(v_3) = 5, c(w_2) = 3, c(v_4) = 2 \) which in turn forces to assign \( c(v_5) = 1 \). This proper coloring gives the \( b \)-vertices for the color classes 1, 2 and 5 but not for 3 and 4. Similarly all other proper coloring with 5 colors will generate \( b \)-vertices at most for three color classes only. Hence \( \varphi(DS(DS(S_n))) \neq 5 \). Thus \( \varphi(DS(DS(S_n))) \leq 4 \). If we assign the colors as \( c(w_1) = c(w_2) = 1, c(v_{3k-2}) = 2, c(v_{3k-1}) = 3, c(v_{3k}) = 4, k \in \mathbb{N} \) gives the \( b \)-vertices \( u, v_2, v_3, v_4 \) for the color classes 1, 3, 4 and 2 respectively. Thus \( \varphi(DS(DS(S_n))) = 4 \).
Definition 2.10. The gear Graph, $G_n$, is obtained from the wheel by subdividing each of its rim edge.

That is, let $W_n = C_n + K_1$ be the wheel graph with apex vertex $v$ and the rim vertices $v_1, v_2, ..., v_n$. To obtain the gear graph $G_n$, subdivide each rim edge of wheel $W_n$ by the vertices $u_1, u_2, ..., u_n$ where each $u_i$ subdivides the edge $v_i v_{i+1}$ for $i = 1, 2, ..., n - 1$ and $u_n$ subdivides the edge $v_1 v_n$. Then $|V(G_n)| = 2n + 1$ and $|E(G_n)| = 3n$.

Lemma 2.11. $\chi(DS(G_n)) = \begin{cases} 5, & n = 3 \\ 4, & n \neq 3 \end{cases}$

Proof. The gear graph $G_n$ has three types of vertices

(i) vertices of degree 2

(ii) vertices of degree 3

(iii) a vertex of degree $n$.

Thus $V(G_n) = \{v_i, u_i, v\} = S_1 \cup S_2 \cup T$ where $S_1 = \{v_i\}$, $S_2 = \{u_i\}$, $T = \{v\} = V(G_n) \setminus \bigcup_{i=1}^{2} S_i$. For obtaining $DS(G_n)$ from $G_n$, we add two vertices $w_1$ and $w_2$ corresponding to $S_1$ and $S_2$ respectively. Thus $V(DS(G_n)) = V(G_n) \cup \{w_1, w_2\}$ and $E(DS(G_n)) = E(G_n) \cup \{v_i w_1, u_i w_2\}$.

When $n = 3$, $DS(G_3)$ contains a $K_3$ (formed by the vertices $v, w_1$ and $w_2$), $\chi(DS(G_3)) \geq 3$. If we assign the colors as $c(v) = 1$, $c(w_1) = 2$, $c(w_2) = 3$, $c(u_1) = 2$, $c(u_2) = 3$, $c(v_1) = 3$ for $i = 1, 2, ..., n$ gives the proper coloring using 3 colors. Thus $\chi(DS(G_3)) = 3$. But when $n \neq 3$, $DS(G_n)$ contains no odd cycles and it is a bipartite graph. Hence by Proposition 1.5, $\chi(DS(G_n)) = 2$.

Theorem 2.12. $\varphi(DS(G_n)) = \begin{cases} 5, & n = 3 \\ 4, & n \neq 3 \end{cases}$

Proof. When $n = 3$, the graph $DS(G_3)$ contains five vertices of degree 4. Consequently $m(DS(G_3)) = 5$. Then by Proposition 1.4, $\varphi(DS(G_3)) \leq 5$. Suppose that $DS(G_3)$ does have a $b$-chromatic 5-coloring. By assigning the proper coloring as $c(u_1) = 1$, $c(u_2) = 2$, $c(u_3) = 2$, $c(v_1) = 3$, $c(v_2) = 2$, $c(v_3) = 1$, $c(v) = 4$, $c(w_2) = 4$, $c(w_1) = 5$ then the vertices $v, v_1, v_2, v_3$ and $v_1$ are the $b$-vertices for the color classes 1, 2, 3, 4 and 5 respectively. Thus $\varphi(DS(G_3)) = 5$.

When $n \neq 3$, the graph $DS(G_n)$ contains at least five vertices of degree 4. Then $m(DS(G_n)) = 5$. Then by Proposition 1.4, $\varphi(DS(G_n)) \leq 5$. Suppose that $DS(G_n)$ does have a $b$-chromatic 5-coloring. By assigning the proper coloring as $c(v) = 1$, $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 4$, $c(v_4) = 5$ gives the $b$- vertex $v$ for the color class 1. Again assume that $c(u_1) = 4$ and $c(u_2) = 3$ which in turn forces to assign $c(w_1) = 5$ which is not possible as the adjacent vertices $w_1$ and $v_1$ will receive the same color. Thus $v_1$ is not a $b$-vertex for the color class 2. Similarly we can prove that no $v_i$’s are $b$-vertices when five colors are used for $b$-coloring. Hence $\varphi(DS(G_n)) \neq 5$. But if we assign the colors as $c(v) = 1$, $c(v_{3k-2}) = 2$, $c(v_{3k-1}) = 3$, $c(v_{3k}) = 4$, $k \in \mathbb{N}$ which gives the $b$- vertices $v, v_1, v_2$ and $v_3$ for the color classes 1, 2, 3 and 4 respectively. Thus $\varphi(DS(G_n)) = 4$. Hence the result.

We have the following obvious result stating the $b$-spectrum of $DS(G_n)$ as any proper coloring with $\chi(G)$ colors is a $b$-coloring.

Corollary 2.13. $S_b(DS(G_n)) = \begin{cases} \{3, 4, 5\}, & n = 3 \\ \{2, 3, 4\}, & n \neq 3 \end{cases}$ and $DS(G_n)$ is $b$-continuous.

Proof. When $n = 3$, by assigning the colors as $c(v) = 1$, $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 4$, $c(w_1) = c(w_2) = 4$ and $c(u_i) = 1$ for $i = 1, 2$ and 3, the graph $DS(G_3)$ has the $b$-chromatic 4-coloring. But when $n \neq 3$, by assigning the colors as $c(v) = c(w_1) = c(w_2) = 1$, $c(v_1) = 2$, $c(u_i) = 3$ for $i = 1, 2, ..., n$, $DS(G_n)$ has the $b$-chromatic 3-coloring. Thus by Lemma 2.11 and Theorem 2.12, $DS(G_n)$ is $b$-continuous and the $b$-spectrum $S_b(DS(G_n)) = \begin{cases} \{3, 4, 5\}, & n = 3 \\ \{2, 3, 4\}, & n \neq 3. \end{cases}$
3 Concluding Remarks

The study of $b$-coloring is important due to its applications in many real life problems like scheduling problem, channel assignment problem, routing networks etc. Here we investigate $b$-chromatic number and related parameters for the degree splitting graph of some graphs. We show that the degree splitting graph of $G_n$ is $b$-continuous. The degree splitting graph of $P_n$, $B_{n,n}$ and $S_n$ are obviously $b$-continuous as any proper coloring with $\chi(G)$ colors is a $b$-coloring.

References


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