Best co-approximation on 2-fuzzy metric space

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Abstract
In this paper the concept of best coapproximation in 2-fuzzy metric space is introduced. Various concepts like coproximinal, co-chebyshev, translation invariant, orthogonal on 2-fuzzy metric linear spaces are established. Using these concepts some related theorems and lemmas are developed.

Keywords
Co-Chebyshev, coproximinal, coapproximation.

AMS Subject Classification
54A40, 54E35, 54H25.

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1. Introduction
The concept of fuzzy sets introduced by L.A.Zadeh in 1965[19] became active in the field of research. Among other fields, a progressive development is made in the field of fuzzy topology. One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space. This problem has been investigated by many different authors in different view point.


A new kind of approximation, called best coapproximation was introduced in normed linear spaces by C.Franchetti and M.Furi[3]. The theory of best coapproximation is much less developed as compared to the theory of best approximation in abstract spaces.

The purpose of this paper is to discuss the existence and uniqueness results on best approximation and best coapproximation in two-fuzzy metric linear spaces.

2. Preliminaries

Definition 2.1. [4] A binary operation ∗: [0, 1] × [0, 1] → [0, 1] is called a t-norm if for all a, b, c, d ∈ [0, 1] the following conditions are satisfied

(i) a ∗ 1 = a.
(ii) a ∗ b = b ∗ a.
(iii) a ∗ b ≤ c ∗ d whenever a ≤ c, b ≤ d.
(iv) a ∗ (b ∗ c) = (a ∗ b) ∗ c.

Definition 2.2. [16] The 3-tuple (X, M, ∗) is called a fuzzy metric space if X is an arbitrary set, ∗ is a continuous t-norm and M is a fuzzy set in X² × [0, ∞] satisfying the conditions

(M1) M(x, y, t) = 0.
(M2) M(x, y, t) = 1, ∀t > 0 if and only if x = y.
(M3) M(x, y, t) = M(y, x, t).
(M4) M(x, y, t) ∗ M(y, z, s) ≤ M(x, z, t + s)
(M5) M(x, y, .) : [0, ∞] → [0, 1] is left continuous.
(M6) lim t→∞ M(x, y, t) = 1.
Definition 2.3. [6] The 3-tuple \((X, M, *)\) is called a 2-fuzzy metric space if \(X\) is an arbitrary set, \(*\) is a continuous \(t\)-norm and \(M\) is a fuzzy set in \(X^2 \times [0, \infty]\) satisfying the conditions

(2-M1) \(M(f, g, t) = 0\).

(2-M2) \(M(f, g, t) = 1\), \(\forall t > 0\) if and only if \(f = g\).

(2-M3) \(M(f, g, t) = M(g, f, t)\).

(2-M4) \(M(f, g, t) \ast M(h, s) \leq M(f, h, t + s)\).

(2-M5) \(M(f, g, t) : [0, \infty] \rightarrow [0, 1]\) is left continuous.

(2-M6) \(\lim_{t \to 0} M(f, g, t) = 1\).

3. Best Coapproximation on 2-Fuzzy Metric space

Definition 3.1. Let \((F(X), M, *)\) be a 2-fuzzy metric space and \(A\) be a nonempty subset of \(F(X)\). An element \(f_0 \in A\) is said to be the coapproximation to \(f \in F(X)\) if

\[ M(f, g, t) \geq M(f_0, g, t) \text{ for all } g \in F(X). \]

The set of all best coapproximations to \(f_0\) in \(A\) is denoted by \(F_R A(f)\).

The set \(A\) is called coproximinal if \(F_R A(f)\) contains at least one \(g \in A\) for every \(f \in F(X)\). If for each \(f \in F(X)\), \(F_R A(f)\) has exactly one \(g \in A\), then \(A\) is said to be co-echyberesh.

A 2-fuzzy metric space \((F(X), M, *)\) is said to be convex if

\[ M(\lambda f_1 + (1 - \lambda) f_2, g, t) \geq \min\{M(f_1, g, t), M(f_2, g, t)\} \quad \text{for all } f_1, f_2 \in F(X) \text{ and } \lambda \in (0, 1). \]

A 2 fuzzy metric \(M\) is said to be translation invariant if

\[ M(f + f_0, g + g_0, t) = M(f, g, t) \quad \text{for all } f, g, f_0, g_0 \in F(X). \]

Definition 3.2. A 2-fuzzy metric linear space is a 2-fuzzy metric space with fuzzy translation invariant 2-fuzzy metric provided

(i) \(M(f + g, t) \geq \min\{M(f, t), M(g, t)\} \quad \text{for all } f, g \in F(X)\).

(ii) \(M(-f, t) \geq M(f, t)\).

(iii) \(M(kx, t) \geq \min\{N(k, t), M(f, t)\} \quad \text{for } k \in F(\text{the field})\).

(iv) \(N(1, t) \geq M(\bar{0}, t) \quad \text{where } N : K \times (0, \infty) \rightarrow [0, 1] \text{ and } K \subset F \text{ the field.}\)

Let \((F(X), M, *, +, \cdot)\) be a 2-fuzzy metric linear space and \(f, g \in F(X)\) we say that \(f\) is orthogonal to \(g\), if \(M(f, \bar{0}, t) \geq M(f, g, t)\) for every scalar \(\alpha\). It is denoted by \(f \perp g\).

Further \(A\) is orthogonal to \(g\) if \(g \perp f\) for every \(f \in F(X)\).

Theorem 3.3. Let \(A\) be a 2-fuzzy linear subspace of the 2-fuzzy metric linear space \((F(X), M, *)\) such that \(A\) is orthogonal to \((f - g_0)\) then \(g_0 \in F_R A(f)\).

Proof. Given \(A\) is orthogonal to \((f - g_0)\) where \(f \in F(X)\) and \(g_0 \in F(R_A(f))\) then \(g\) is orthogonal to \((f - g_0)\) for every \(g \in A\).

\(\text{(ie)} M(g, \bar{0}, t) \geq M(g, \alpha f - g_0, t) \quad \text{for every } \alpha\).

Suppose \(\alpha = 1\) then \(M(g, \bar{0}, t) \geq M(g, f - g_0, t) \quad \text{for every } g \in A\).

As \(M\) is fuzzy translation invariant

\[ M(g + g_0, g + g_0, t) \geq M(g + g_0, f - g_0 + g_0, t) \]

\(\text{(ie) } M(g_0, g + g_0, t) \geq M(f, g + g_0, t) \)

satisfying the requirements that \(g_0 \in F_R A(f)\).

Theorem 3.4. Let \(A\) be a 2-fuzzy linear subspace of a two fuzzy metric linear space \((F(X), M, *, +, \cdot)\) and \(g_0 \in A\). Then \(\alpha g_0 \in F_R A(\alpha f)\) for every scalar \(\alpha\) if and only if \(A\) is orthogonal to \((f - g_0)\).

Proof. Assume \(\alpha g_0 \in F_R A(\alpha f)\).

Then \(M(\alpha g_0, g, t) \geq M(\alpha f, g, t) \quad \text{for every } g \in A\).

\(\text{(ie)} M(\alpha g_0 - \alpha g_0, g - g_0, t) \geq M(\alpha f - \alpha g_0, g - g_0, t) \quad \text{for every } g \in A\).

\(\text{(ie)} M(\bar{0}, g - g_0, t) \geq M(\alpha f - \alpha g_0, g - g_0, t) \quad \text{for every } g \in A\).

\(\text{(ie)} M(\bar{0}, g - g_0, t) \geq M(\alpha f - \alpha g_0, g - g_0, t) \quad \text{for every } g \in A\).

Hence \(A\) is orthogonal to \((f - g_0)\).

Conversely, suppose \(A\) is orthogonal to \((f - g_0)\) then \(g_0\) is orthogonal to \((f - g_0)\) for every \(g \in A\).

\(\text{(ie)} M(g, \alpha f - g_0, t) \geq M(\alpha f - g_0, g, t) \quad \text{for every } g \in A\).

\(\text{(ie)} M(g + g_0, \alpha g_0, t) \geq M(g + g_0, g, t) \quad \text{for every } g \in A\).

\(\text{(ie)} M(\alpha g_0, g + g_0, t) \geq M(\alpha f, g + g_0, t) \quad \text{for every } g \in A\).

\(\text{(ie)} M(\alpha g_0, g + g_0, t) \geq M(\alpha f, g + g_0, t) \quad \text{for every } g \in A\).

Therefore \(\alpha g_0 \in F_R A(\alpha f)\) satisfies the requirement.

Lemma 3.5. Let \(A\) be a 2-fuzzy closed linear subspace of a 2-fuzzy metric linear space \((F(X), M, *, +, \cdot)\). If \(f\) is not an element of \(A\) such that \(\alpha f\) has a best coapproximation in \(A\) then every element of the subspace \(\{x | A\}\) has a best coapproximation in \(A\).

Proof. Consider \(\alpha f + \bar{g} \in f, A\) and \(g_0 \in F_R A(\alpha f)\) then \(M(g_0, g, t) \geq M(\alpha f, g, t)\) for every \(g \in G\).

\(\Rightarrow M(g_0 + \bar{g}, g + \bar{g}, t) \geq M(\alpha f + \bar{g}, g + \bar{g}, t) \quad \text{for every } g \in G\)

\(\Rightarrow M(g_0 + g, g + g, t) \geq M(\alpha f + g, g + g, t) \quad \text{for every } g \in G\).

Therefore \(g_0 + g \in F_R A(\alpha f + g)\).

Thus every element of the subspace \(\{f, A\}\) has a best coapproximation in \(A\).
Lemma 3.6. Let A, B be 2-fuzzy subspaces of 2-fuzzy metric linear space \((\mathcal{F}(X), M, +, .)\) so that A is a subset of B. If an element \(f\) not in B has the best coapproximation in B and if every element of B has the best coapproximation in A then \(f\) has the best coapproximation in A.

Proof. Let \(f \in \mathcal{F}(X)\) such that it does not belong to B, so that \(k_0 \in \mathcal{F}_B(f)\) then \(M(k_0, k, t) \geq M(f, k, t)\) for every \(k \in B\).

Note 3.7. Using lemma 3.5 and lemma 3.6 we deduce the following theorem.

Theorem 3.8. Let A be a 2-fuzzy subspace of a 2-fuzzy metric space \((\mathcal{F}(X), M, +, .)\), if there exists at least one element \(f \in \mathcal{F}(X)\) such that \(f\) has the best coapproximation in A then for any subspace A of \(\mathcal{F}(X)\) every element of \(\mathcal{F}(X)\) has a best approximation in A.

Theorem 3.9. Let A be a 2-fuzzy coproximinal subspace of a 2-fuzzy metric linear space \((\mathcal{F}(X), M, +, .)\) if \(\mathcal{F}_A^{-1}(\bar{0})\) is a convex set then A is co-chesyhev.

Proof. Let \(f \in \mathcal{F}(X)\) and \(g_1, g_2 \in \mathcal{F}_A(f)\). Take \(f - g_1 = h^*\) and \(f - g_2 = h^*\) where \(f - g_1, f - g_2 \in \mathcal{F}_A(\bar{0})\) then \(M(0, g, t) \geq M(f - g_1, g, t)\) for every \(g \in A\).

Since \(\bar{0} \in \mathcal{F}_A(f - g_1)\),

\[ M(g_1 + g, t) \geq M(f, g + g_1, t) \]

(ie)

\[ M(g_1 - f, g_1 - f, t) \geq M(0, g + g_1, t) \]

(ie) \(M(g_1 - f, g, t) \geq M(0, g_1, t)\)

Where \(g = g_1 - f \in A\) therefore \(g_1 - f\) belongs to \(\mathcal{F}_A(\bar{0})\). Suppose \(\mathcal{F}_A^{-1}(\bar{0})\) is convex and \(f - g_2, g_1 - f\) are elements in \(\mathcal{F}_A^{-1}(\bar{0})\).

\[ M(\lambda(f - g_2) + (1 - \lambda)(g_1 - f), g, t) \geq \min\{M(f - g_2, g, t), M(g_1 - f, g, t)\} \]

\[ \geq \min\{M(\bar{0}, g, t), M(0, g, t)\} = M(\bar{0}, g, t) \]

Which implies \(\lambda(f - g_2) + (1 - \lambda)(g_1 - f) \in \mathcal{F}^{-1}_A(\bar{0})\)

If \(\lambda = 1/2\) then \((g_1 - g_2)/2 \in \mathcal{F}_A^{-1}(\bar{0})\) and also \((g_1 - g_2)/2\) belongs to A.

it is obvious that \(\mathcal{F}^{-1}_A(\bar{0}) \cup A = \{\bar{0}\} \) and \(\text{sg}(g_1) = g_2\) and so A is co-chesyhev.

Theorem 3.10. If A is co-chesyhev 2-fuzzy subspace of a 2-fuzzy metric space \((\mathcal{F}(X), M, +, .)\) then the graph of the 2-fuzzy metric coprojection \(\mathcal{F}_A\) is 2-fuzzy closed.

Proof. The 2-fuzzy metric co projection \(\mathcal{F}_A : \mathcal{F}(X) \rightarrow F(2^{\mathcal{F}_A})\) where \(F(2^{\mathcal{F}_A})\) is the set of all fuzzy subsets of \(\mathcal{F}(A)\) and the graph is defined as \(G(\mathcal{F}_A) = \{(f, \mathcal{F}_A(f)) / f \in \mathcal{F}(X)\}\)

Let \((f, h)\) be the limit point of \(G(\mathcal{F}_A) = \{(f, \mathcal{F}_A(f)) / f \in \mathcal{F}(X)\}\) there exists a sequence \((f_n, \mathcal{F}_A(f_n))\) in \(G(\mathcal{F}_A)\) such that \((f_n, \mathcal{F}_A(f_n)) \rightarrow (f, h)\). That is \(f_n \rightarrow f\) and \(\mathcal{F}_A(f_n) \rightarrow h\).

It is obvious that \(M(\mathcal{F}_A(f_n), g, t) \geq M(f_n, g, t)\) for every \(g \in A\).

Then \(M(h, g, t) \geq M(f, g, t)\) for every \(g \in A\) and so \(h \in \mathcal{F}_A(f)\).

Since A is co-chebyshev \(\mathcal{F}_G(f)\) contains only one point \(h\) and so \(\mathcal{F}_G(f) = \{h\}\). Hence \((f, h)\) belongs to \(G(\mathcal{F}_A)\) and so it is 2-fuzzy closed.

4. Conclusion

In the present study, we introduced the concept of best coapproximation in 2-fuzzy metric space, which generalized various concepts like coproximinal, co-Chebyshev translation invariant, orthogonal on 2-fuzzy metric linear spaces. We also proved some related theorems and lemmas.

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