Existence of solution of a Coupled system of differential equation with nonlocal conditions

El-Sayed A.M.A\textsuperscript{a,*} Abd-El-Rahman R. O.\textsuperscript{b} and El-Gendy M.\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, Faculty of Science, Alexandria University, Egypt.
\textsuperscript{b,c}Department of Mathematics, Faculty of Science, Damanhour University, Egypt.

Abstract

In this paper, we study the existence of at least one solution of the coupled system of differential equations with nonlocal conditions. Also, a coupled system of differential equations with the nonlocal integral conditions will be considered.

Keywords: Coupled systems, nonlocal conditions, at least one solution, integral conditions.

2010 MSC: 34B18, 34B10.

1 Introduction

Problems with nonlocal conditions have been extensively studied by several authors in the last decades. The reader is referred to\cite{2}-\cite{20} and references therein.

In \cite{13} the authors studied nonlocal cauchy problem

\[ \dot{x} = f(t, x(t)), \ t \in [0, T] \]
\[ \sum_{j=1}^{m} b_j x(\eta_j) = x_1, \ \eta_j \in (0, a) \subset [0, T]. \]

Also, in \cite{7} the authors studied the local and global existence of solutions of the nonlocal problem

\[ \frac{dx}{dt} = f_1(t, y(t)), \ t \in (0, T] \] (1.1)
\[ \frac{dy}{dt} = f_2(t, x(t)), \ t \in (0, T] \] (1.2)

with the nonlocal conditions

\[ x(0) + \sum_{k=1}^{n} a_k x(\tau_k) = x_0, \ a_k > 0, \ \tau_k \in (0, T) \] (1.3)
\[ y(0) + \sum_{j=1}^{m} b_j y(\eta_j) = y_0, \ b_j > 0, \ \eta_j \in (0, T) \] (1.4)

*Corresponding author.
E-mail address: amasyed5@yahoo.com (El-Sayed A.M.A).
Here we are studied the existence of at least one solution of the nonlocal problem (1.1)-(1.4), the problem with nonlocal integral conditions

\[ x(0) + \int_0^T x(s)\,ds = x_0, \quad (1.5) \]

\[ y(0) + \int_0^T y(s)\,ds = y_0. \quad (1.6) \]

are studied.

2 Preliminaries

we need the following definitions.

**Definition 2.1.** \([19]\) Let \( F = \{ f_i : X \to Y, i \in I \} \) be a family of functions with \( Y \) being a set of real (or complex) numbers, then we call \( F \) uniformly bounded if there exists a real number \( c \) such that

\[ |f_i(x)| \leq c \quad \forall i \in I, x \in X. \]

**Definition 2.2.** \([19]\) Let \( F = \{ f(x) \} \) is the class of functions defined on \( A \) where \( A = [a, b] \subset \mathbb{R} \), the class of functions \( F = \{ f(x) \} \) is equicontinuous if \( \forall \varepsilon > 0 \), \( \exists \delta(\varepsilon) \) such that

\[ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \quad \forall f \in F, x, y \in A. \]

**Theorem 2.1.** \([1]\) The function \( f(x) = (f_1(x), f_2(x), \ldots, f_n(x)) \) is uniformly continuous in \( I = [a, b] \) if and only if each \( f_i \) is uniformly continuous in \([a, b]\).

**Theorem 2.2.** \([19]\) (Lebesgue Dominated Convergence Theorem) let \( f_n \) be a sequence of functions converging to a limit \( f \) of \( A \), and suppose that

\[ |f_n(t)| \leq \phi(t), \quad t \in A, \quad n = 1, 2, 3, \ldots \]

where \( \phi \) is integrable on \( A \). Then

1. \( f \) is integrable on \( A \)
2. \( \lim_{n \to \infty} \int_A f_n(t)\,d\mu = \int_A f(t)\,d\mu. \)

**Theorem 2.3.** \([18]\) (Schauder) Let \( Q \) be a convex subset of a Banach space \( X, T : Q \to Q \) be a compact and continuous map, then \( T \) has at least one fixed point in \( Q \).

3 Integral Representation

Let \( X \) be the class of all columns vectors \( \left( \begin{array}{c} x \\ y \end{array} \right) \), \( x, y \in C([0, T]) \) with the norm

\[ \left\| \left( \begin{array}{c} x \\ y \end{array} \right) \right\|_X = \|x\| + \|y\| = \sup_{t \in [0, T]} |x(t)| + \sup_{t \in [0, T]} |y(t)|. \]

Throughout the paper we assume that the following assumptions hold:

i. \( f_i : [0, T] \times R \to R \) satisfies Caratheodory conditions, that is \( f_i \) is

1. measurable in \( t \in (0, T) \), for any \( x \in R \).
2. continuous in \( x \in R \), for almost all \( t \in (0, T) \).

ii. There exist two integrable functions \( m_i \in L_1[0, T], i = 1, 2 \) such that

\[ |f_i(t, x)| \leq m_i(t), \]

\[ \int_0^t m_i(s)\,ds < k_i, i = 1, 2 \quad \forall t \in [0, T]. \]
Lemma 3.1. The solution of the nonlocal problem (1.1)-(1.4) can be expressed by the system of the integral equations

\[
\begin{pmatrix}
  x(t) \\
  y(t)
\end{pmatrix} = \begin{pmatrix}
  a x_0 + \int_0^t f_1(s, y(s)) \, ds - a \sum_{k=1}^n a_k \int_0^{\tau_k} f_1(s, y(s)) \, ds \\
  b y_0 + \int_0^t f_2(s, x(s)) \, ds - b \sum_{j=1}^m b_j \int_0^{\eta_j} f_2(s, x(s)) \, ds
\end{pmatrix},
\]

where \((1 + \sum_{k=1}^n a_k)^{-1} = a, \quad (1 + \sum_{j=1}^m b_j)^{-1} = b.\)

3.1 Existence of solution

Here, we study the existence of at least one solution of the nonlocal problem (1.1)-(1.4). Define the superposition operator \(F\) by

\[
F \begin{pmatrix}
  x(t) \\
  y(t)
\end{pmatrix} = \begin{pmatrix}
  a x_0 + \int_0^t f_1(s, y(s)) \, ds - a \sum_{k=1}^n a_k \int_0^{\tau_k} f_1(s, y(s)) \, ds \\
  b y_0 + \int_0^t f_2(s, x(s)) \, ds - b \sum_{j=1}^m b_j \int_0^{\eta_j} f_2(s, x(s)) \, ds
\end{pmatrix} = \begin{pmatrix}
  F_1 y \\
  F_2 x
\end{pmatrix}.
\]

Now we have the following theorem.

Theorem 3.4. Consider the assumptions (i)-(ii) are satisfied, then there exists at least one solution of the nonlocal problem (1.1)-(1.4).

Proof. Define the operator \(F (x, y) = (F_1 x, F_2 y)\), where

\[
F_1 y = a x_0 + \int_0^t f_1(s, y(s)) \, ds - a \sum_{k=1}^n a_k \int_0^{\tau_k} f_1(s, y(s)) \, ds,
\]

\[
F_2 x = b y_0 + \int_0^t f_2(s, x(s)) \, ds - b \sum_{j=1}^m b_j \int_0^{\eta_j} f_2(s, x(s)) \, ds.
\]

Now

\[
| F_1 y | = \left| a x_0 + \int_0^t f_1(s, y(s)) \, ds - a \sum_{k=1}^n a_k \int_0^{\tau_k} f_1(s, y(s)) \, ds \right|
\]

\[
\leq | a x_0 | + \int_0^t | f_1(s, y(s)) | \, ds + | a | \sum_{k=1}^n | a_k | \int_0^{\tau_k} | f_1(s, y(s)) | \, ds
\]

\[
\leq a | x_0 | + \int_0^t m_1(s) \, ds + a \sum_{k=1}^n | a_k | \int_0^{\tau_k} m_1(s) \, ds
\]

\[
\leq a | x_0 | + K_1 + a \sum_{k=1}^n a_k \leq a | x_0 | + K_1 (1 + a \sum_{k=1}^n a_k)
\]

\[
\leq a | x_0 | + K_1 \left( 1 + \sum_{k=1}^n a_k \right) \leq a | x_0 | + 2K_1 = M_1.
\]
then $F_1$ is uniformly bounded. Similarly

$$|F_2x| \leq b |y_0| + 2K_2 = M_2,$$

then $F_2$ is uniformly bounded.

Hence $\|F(x,y)\|_X = \|F_1y\| + \|F_2x\| \leq M_1 + M_2 = M$, and then $F$ is uniformly bounded.

For $t_1, t_2 \in (0, T], t_1 < t_2$, let $|t_2 - t_1| < \delta$, then

$$|F(x(t_2)) - F(x(t_1))| = |F_1y(t_2) - F_1y(t_1)|$$

$$= \left| \int_0^{t_2} f_1(s, y(s)) \, ds - \int_0^{t_1} f_1(s, y(s)) \, ds \right|$$

$$= \left| \int_{t_1}^{t_2} f_1(s, y(s)) \, ds \right|$$

$$\leq \int_{t_1}^{t_2} |f_1(s, y(s))| \, ds$$

$$\leq \int_{t_1}^{t_2} m_1(s) \, ds \leq \epsilon,$$

then $\{F_1y\}$ is a class of equicontinuous functions.

Similarly

$$|F(y(t_2)) - F(y(t_1))| = |F_2x(t_2) - F_2x(t_1)| \leq \int_{t_1}^{t_2} m_2(s) \, ds \leq \epsilon,$$

then $\{F_2x\}$ is a class of equicontinuous functions.

Therefore the operator $F$ is equicontinuous and uniformly bounded.

Let

$$\{y_N(t)\} \in C[0, T], y_N(t) \to y(t), \{x_N(t)\} \in C[0, T], x_N(t) \to x(t),$$

So,

$$\lim_{N \to \infty} F_1(y_N) = \lim_{N \to \infty} \left( a x_0 + \int_0^t f_1(s, y_N(s)) \, ds - a \sum_{k=1}^n a_k \int_0^{\tau_k} f_1(s, y_N(s)) \, ds \right),$$

but $|f_i(s, y_N(s))| \leq m_i$, and $f_i(s, y_N(s)) \to f_i(s, y(s))$

applying Lebesgue dominated convergence theorem [19], then we deduce that

$$\lim_{N \to \infty} \int_0^t f_1(s, y_N(s)) \, ds = \int_0^t \lim_{N \to \infty} f_1(s, y_N(s)) \, ds = \int_0^t f_1(s, \lim_{N \to \infty} y_N(s)) \, ds = \int_0^t f_1(s, y(s)) \, ds,$$
Then the nonlocal conditions (1.3)-(1.4) will be in the form

\[ \lambda x \in [0, \tau] \]

Let

\[ F : X \to X \]

be a continuous operator.

Then \( F \) is a convex, continuous, and compact operator.

Now we show that \( X \) is convex,

let \((x_1, y_1), (x_2, y_2) \in X\)

\[ || (x_i, y_i) ||_X = || x_i || + || y_i || < M, \quad i = 1, 2. \]

For \( \lambda \in [0, T] \)

\[ || \lambda (x_1, y_1) + (1 - \lambda) (x_2, y_2) || = || \lambda x_1 + (1 - \lambda) x_2, \lambda y_1 + (1 - \lambda) y_2 || \]

\[ \leq \lambda || x_1 || + ((1 - \lambda) || x_2 || + \lambda || y_1 || + (1 - \lambda) || y_2 || \]

\[ \leq \lambda \] \[ \leq \lambda M + (1 - \lambda) M = M, \]

this means that \( X \) is convex.

Then \( F \) has a fixed point \((x, y) \in X\) which proves that there exists at least one solution of the nonlocal problem (1.1)-(1.4).

4 Nonlocal Integral Condition

Let \( a_k = (t_k - t_{k-1}), \tau_k \in (t_{k-1}, t_k), \) and \( b_j = (t_j - t_{j-1}), \eta_j \in (t_{j-1}, t_j), \)

where \( 0 < t_1 < t_2 < t_3 < \ldots < 1. \)

Then, the nonlocal conditions (1.3)-(1.4) will be in the form

\[ x(0) + \sum_{k=1}^{n} (t_k - t_{k-1}) x(\tau_k) = x_0, \quad y(0) + \sum_{j=1}^{m} (t_j - t_{j-1}) y(\eta_j) = y_0. \]
From the continuity of the solution of the nonlocal problem (1.1)-(1.4), we obtain

\[
\lim_{n \to \infty} \sum_{k=1}^{n} (t_k - t_{k-1}) x(t_k) = \int_0^T x(s)ds,
\]
\[
\lim_{m \to \infty} \sum_{j=1}^{m} (t_j - t_{j-1}) y(t_j) = \int_0^T y(s)ds,
\]
that is, the nonlocal conditions (1.3)-(1.4) is transformed to the integral condition

\[
x(0) + \int_0^T x(s)ds = x_0, \quad y(0) + \int_0^T y(s)ds = y_0.
\]

Now, we have the following theorem.

**Theorem 4.5.** Let the assumption (i)-(ii) be satisfied, then the coupled system of differential equations (1.1) and (1.4) with the nonlocal integral condition (1.5) and (1.6) has at least one solution represented in the form

\[
U = \begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix}
= \begin{pmatrix}
a^* x_0 + \int_0^T f_1 (\theta, y(\theta)) \, d\theta - a^* \int_0^s \int_0^T f_1 (\theta, y(\theta)) \, d\theta \, ds \\
a^* y_0 + \int_0^T f_2 (\theta, x(\theta)) \, d\theta - a^* \int_0^s \int_0^T f_2 (\theta, x(\theta)) \, d\theta \, ds
\end{pmatrix},
\]

where \( a^* = (1 + T)^{-1} \).

**References**


Received: ?, 2014; Accepted: ?, 2014

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Website: http://www.malayajournal.org/