Some new Ostrowski type inequalities for functions whose second derivative are h-convexe via Riemann-Liouville fractionnal

B. Meftah\textsuperscript{a} and K. Boukerrioua\textsuperscript{a,*}

\textsuperscript{a}University of Guelma. Guelma, Algeria.

Abstract

A new identity similar to an identity proved in Erhan Set. (2012) \cite{15} for fractional integrals is estab-
lished. By making use of the established identity, some new Ostrowski type inequalities for Riemann–Liouville
fractional integral are obtained.

Keywords: Ostrowski type inequalities, Riemann-Liouville integrals, \((s, m)\) – convex function.

2010 MSC: 26D15, 26D20, 39A12.

1 Introduction

In 1938, A.M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the
derivative of a differentiable function by its integral mean as follows

\textbf{Theorem 1.1.} [13] Let \( f: I \rightarrow \mathbb{R} \), where \( I \subseteq \mathbb{R} \) is an interval, be a mappingin the interior \( I^0 \) of \( I \), and \( a, b \in I^0 \), with \( a < b \).

If \( |f'| \leq M \) for all \( x \in [a, b] \), then

\[ \left| f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq M (b - a) \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right], \quad \forall x \in [a, b] \quad (1.1) \]

This is well-know Ostrowski inequality. In recent years, a number of authors have written about general-
izations, extensions and variants of such inequalities (see \cite{1, 2, 3}).

Let us recall definitions of some kinds of convexity as follows.

\textbf{Definition 1.1.} \( \text{We say that} \ f: I \subset \mathbb{R} \rightarrow \mathbb{R} \ (I \neq \phi) \ \text{is convex function if the inequality} \)

\[ f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1.2) \]

\( \text{holds for all} \ x, y \in I, \ \text{and} \ t \in [0,1] \).

\textbf{Definition 1.2.} \[ \text{We say that} \ f: I \subset \mathbb{R} \rightarrow \mathbb{R} \ (I \neq \phi) \ \text{is P- function if} \ f \ \text{is non-negative and the inequality} \]

\[ f(tx + (1-t)y) \leq f(x) + f(y) \quad (1.3) \]

\( \text{holds for all} \ x, y \in I, \ \text{and} \ t \in [0,1] \).

\textsuperscript{*}Corresponding author.

E-mail address: khaledv2004@yahoo.fr (Khaled Boukerrioua), badrimeftah@yahoo.fr (Badri Meftah).
Definition 1.3. \[8\] We say that \( f : [0, \infty) \rightarrow \mathbb{R} \) is \( s \)-convex function in the second sense, if the inequality
\[
f(t x + (1 - t) y) \leq t^s f(x) + (1 - t)^s f(y)
\]
holds for all \( x, y \in (0, b) \), \( t \in [0, 1] \) and for fixed \( s \in (0, 1] \).

Definition 1.4. \[15\] Let \( h : J \subset \mathbb{R} \rightarrow \mathbb{R} \), be a positive function. We say that \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} (I \neq \emptyset) \) is \( h \)-convex function, if \( f \) is non-negative and
\[
f(t x + (1 - t) y) \leq h(t)f(x) + h(1-t)f(y)
\]
holds for all \( x, y \in I \), and \( t \in [0, 1] \).

Definition 1.5. \[17\] We say that \( f : [0, b] \rightarrow \mathbb{R} \) is said to be \( m \)-convex, where \( m \in (0, 1] \) and \( b > 0 \), if for every \( x, y \in [0, b] \) and \( t \in [0, 1] \), we have
\[
f(t x + m (1 - t) y) \leq t f(x) + m(1-t) f(y)
\]

Definition 1.6. \[12\] We say that \( f : [0, b] \rightarrow \mathbb{R} \) is said to be \((s, m)\)-convex, where \((s, m) \in (0, 1]^2 \) and \( b > 0 \), if for every \( x, y \in [0, b] \) and \( t \in [0, 1] \), we have
\[
f(t x + m (1 - t) y) \leq t^s f(x) + m(1-t)^s f(y)
\]

Definition 1.7. Let \( f \in L_1 [a, b] \). The Riemann-Liouville integrals \( J^a_+ f(x), J^b_- f(x) \) of order \( \alpha > 0 \), with \( a > 0 \) are defined by
\[
J^a_+ f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a
\]
\[
J^b_- f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b
\]
and \( J^0_0 f(x) = J^0_+ f(x) = f(x) \)

where
\[
\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt
\]
noting also
\[
\beta(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\]

Motivated by the recent results given in \[1, 2, 6, 9\], in the present paper, we provide some companions of Ostrowski type inequalities involving Riemann – Liouville fractional integrals for functions whose second derivatives absolute value are \( h \)-convex.

2 OSTROWSKI TYPE INEQUALITIES FOR FRACTIONAL INTEGRALS

In order to prove our main results we need the following identity.

Lemma 2.1. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a twice differentiable mapping on \((a, b)\) with \( a < b \). If \( f'' \in L_1 [a, b] \), then the following equality for fractional integrals holds for any \( x \in [a, b] \)
\[
L_a(x) = (x-a)^{\alpha+1} (b-x)^{\alpha+1} \left[ (a-x) \int_0^1 t^{\alpha+1} f''(tx+(1-t)a) dt + (x-b) \int_0^1 t^{\alpha+1} f''(tx+(1-t)b) dt \right]
\]
\[
= \int_0^1 \frac{d}{dt} \left[ (t-a)^{\alpha+1} (b-t)^{\alpha+1} \right] f''(tx+(1-t)a) dt
\]
where
\[ L_\alpha(x) = (\alpha + 1)(b - x)^\alpha (x - a)^\alpha (b - a) f(x) - \Gamma(\alpha + 2) \left[ (b - x)^{\alpha + 1} \int_a^x f(x) - (x - a)^{\alpha + 1} \int_a^x f(b) \right] \]
(2.13)

**Proof.** We have

\[
\int_a^x f(t) dt = \frac{1}{\Gamma(\alpha)} \int_a^x (t - a)^{\alpha - 1} f(t) dt
\]
(2.14)

\[
= \frac{1}{\Gamma(\alpha)} \left[ (x - a)^\alpha f(x) - \int_a^x (t - a)^\alpha f(t) dt \right]
\]
\[
= \frac{1}{\Gamma(\alpha + 1)} \left[ (x - a)^{\alpha + 1} f(x) - \int_a^x (t - a)^{\alpha + 1} f'(t) dt \right]
\]
\[
= \frac{1}{\Gamma(\alpha + 2)} \left[ (x - a)^{\alpha + 1} f(x) - (x - a)^{\alpha + 1} f'(x) + \int_a^x (t - a)^{\alpha + 1} f''(t) dt \right],
\]

multiplying both side of (2.14) by \(\Gamma(\alpha + 2) (b - x)^{\alpha + 1}\), we get

\[
\Gamma(\alpha + 2) (b - x)^{\alpha + 1} \int_a^x f(t) dt = \left[ (a + 1)(b - x)^{\alpha + 1} (x - a)^\alpha f(x) - (b - x)^{\alpha + 1} (x - a)^{\alpha + 1} f'(x) + (b - x)^{\alpha + 1} (x - a)^{\alpha + 2} \int_0^1 t^{\alpha + 1} f''(tx + (1-t)b) dt \right].
\]
(2.15)

And

\[
\int_a^b (b - t)^{\alpha - 1} f(t) dt = \frac{1}{\Gamma(\alpha)} \int_a^b (b - t)^{\alpha - 1} f(t) dt
\]
(2.16)

\[
= \frac{1}{\Gamma(\alpha)} \left[ (b - x)^\alpha f(x) + \int_a^x (b - t)^\alpha f'(t) dt \right]
\]
\[
= \frac{1}{\Gamma(\alpha + 1)} \left[ (b - x)^{\alpha + 1} f(x) + \int_a^x (b - t)^{\alpha + 1} f'(t) dt \right]
\]
\[
= \frac{1}{\Gamma(\alpha + 2)} \left[ (a + 1)(b - x)^{\alpha + 1} (x - a)^\alpha f(x) - (b - x)^{\alpha + 1} (x - a)^{\alpha + 1} f'(x) + (b - x)^{\alpha + 1} (x - a)^{\alpha + 2} \int_0^1 t^{\alpha + 1} f''(tx + (1-t)b) dt \right],
\]

multiplying both side of (2.16) by \(\Gamma(\alpha + 2) (x - a)^{\alpha + 1}\), we get

\[
\Gamma(\alpha + 2) (x - a)^{\alpha + 1} \int_a^x f(t) dt = \left[ (a + 1)(x - a)^{\alpha + 1} (b - x)^\alpha f(x) + (x - a)^{\alpha + 1} (b - x)^{\alpha + 1} f'(x) + (x - a)^{\alpha + 1} (b - x)^{\alpha + 2} \int_0^1 t^{\alpha + 1} f''(tx + (1-t)b) dt \right].
\]
(2.17)

Summing (2.15) and (2.17), we obtain

\[
\Gamma(\alpha + 2) (b - x)^{\alpha + 1} \int_a^x f(t) dt + \Gamma(\alpha + 2) (x - a)^{\alpha + 1} \int_a^x f(t) dt =
\]
\[
(a + 1)(b - a) (x - a)^\alpha (b - x)^\alpha f(x) +
\]
\[
(x - a)^{\alpha + 1} (b - x)^{\alpha + 1} \left[ (x - a) \int_0^1 t^{\alpha + 1} f''(tx + (1-t)a) dt + (b - x) \int_0^1 t^{\alpha + 1} f''(tx + (1-t)b) dt \right]
\]
(2.18)
we can rewrite (2.18) as follows

\[(a + 1) (b - a) (x - a)^{\alpha} (b - x)^{\alpha} f(x) - \left[ \Gamma(a + 2) (b - x)^{\alpha+1} \int_{x}^{b} f(t) dt + \Gamma(a + 2) (x - a)^{\alpha+1} \int_{a}^{x} f(t) dt \right] \]

and thus (2.19) implies (2.12).

**Theorem 2.1.** Let \( f : I \to \mathbb{R} \) be a twice differentiable mapping on \( I \) such that \( |f''| \in L_{1} [a, b] \), where \( a, b \in I \), with \( a < b \).

If \( |f''| \) is convex function on \( [a, b] \), and \( f'' \) is bounded, i.e., \( \|f''\|_{\infty} = \sup_{x \in [a, b]} |f''| < \infty \) for any \( x \in [a, b] \), then the following inequality holds

\[ |L_{a}(x)| \leq \frac{(x - a)^{\alpha+1} (b - x)^{\alpha+1} (b - a)}{(\alpha + 2)} \|f''\|_{\infty} \]  \hspace{1cm} (2.20)

**Proof.** By lemma 2.1 and Under the given assumptions on \( f'' \) we have

\[
|L_{a}(x)| = \left| (x - a)^{\alpha+1} (b - x)^{\alpha+1} \left[ (a - x) \int_{0}^{1} t^{\alpha+1} f''(tx + (1 - t) a) dt + (x - b) \int_{0}^{1} t^{\alpha+1} f''(tx + (1 - t) b) dt \right] \right|
\]

\[\leq \frac{(x - a)^{\alpha+1} (b - x)^{\alpha+1}}{(\alpha + 2)} \left[ (x - a) \int_{0}^{1} t^{\alpha+1} |f''(x)| dt + (b - x) \int_{0}^{1} t^{\alpha+1} |f''(x)| dt \right] \]

\[\leq \|f''\|_{\infty} (x - a)^{\alpha+1} (b - x)^{\alpha+1} (b - x + x - a) \int_{0}^{1} t^{\alpha+1} dt = \frac{(x - a)^{\alpha+1} (b - x)^{\alpha+1} (b - a)}{(\alpha + 2)} \|f''\|_{\infty} \]

**Remark 2.1.** Under the same hypotheses of Theorem 2.1, at the exception of the convexity of \( f'' \) the inequality (2.20) remains valid.

**Corollary 2.1.** With the assumptions in Theorem 2.1 in the case where \( \alpha = 1 \), one has the inequality

\[ \left| f\left( \frac{a + b}{2} \right) - \frac{1}{(b - a)} \int_{a}^{b} f(t) dt \right| \leq \frac{(b - a)^{2}}{24} \|f''\|_{\infty} \] \hspace{1cm} (2.21)

**Proof.** Choose \( x = \frac{a + b}{2} \) and \( \alpha = 1 \) in (2.9), we get

\[ \frac{(b - a)^{3}}{2} \left| f\left( \frac{a + b}{2} \right) - \frac{1}{(b - a)} \int_{a}^{b} f(t) dt \right| \leq \frac{(b - a)^{5}}{48} \|f''\|_{\infty} \] \hspace{1cm} (2.22)

dividing both side of (2.22) by \( \frac{(b - a)^{3}}{2} \) we obtain (2.21).

**Remark 2.2.** The inequality (2.21) is obtained in [9], choose \( x = \frac{a + b}{2} \) in theorem 2.2.
Corollary 2.2. With the assumptions in Theorem 1.1 in the case where \( \alpha = 1 \), one has the inequality.

\[
\left| f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{96} \|f''\|_\infty
\]  

(2.23)

Proof. Apply Theorem 1.1 a faith on the interval \([a, a+b/2]\), taking \( \alpha = 1 \) in (2.9), and replace \( x \) by \( \frac{3a+b}{4} \), we get

\[
\frac{(b-a)^3}{16} \left| f\left(\frac{3a+b}{4}\right) - \frac{b-a}{2a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^5}{1536} \|f''\|_\infty
\]  

(2.24)

(2.24) implies

\[
\left| f\left(\frac{3a+b}{4}\right) - \frac{b-a}{2a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{96} \|f''\|_\infty
\]  

(2.25)

Apply Theorem 1.1 another faith on the interval \([a+b/2, b]\), taking \( \alpha = 1 \) in (2.9), and replace \( x \) by \( \frac{a+3b}{4} \), we get

\[
\left| f\left(\frac{a+3b}{4}\right) - \frac{b-a}{2a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{96} \|f''\|_\infty
\]  

(2.26)

summing (2.25) and (2.26), dividing the result by 2 we obtain (2.23). \qed

Remark 2.3. The inequality (2.23) is obtained in [9] corollary 2.3

Corollary 2.3. Let \( f : I \rightarrow \mathbb{R} \) be a twice differentiable mapping on \( I^c \) such that \( |f''| \in L_1[a, b] \), where \( a, b \in I \), with \( a < b \).

If \( |f''| \) is convex function on \([a, b]\), \( p, q \geq 1, \frac{1}{p} + \frac{1}{q} = 1 \) and and \( f'' \) is bounded, i.e., \( \|f''\|_\infty = \sup_{x \in [a, b]} |f''| < \infty \), for any \( x \in [a, b] \),

then the following inequality holds

\[
|L_\alpha(x)| \leq (x-a)^{a+1} (b-x)^{a+1} (b-a) \left( \frac{1}{(a+1)^p + 1} \right)^{\frac{1}{p}} \|f''\|_\infty
\]  

(2.27)

Proof. under the assumptions given on \( f'' \) and using the well-known Hölder’s inequality for lemma 2.1 we get

\[
|L_\alpha(x)| = \left| (x-a)^{a+1} (b-x)^{a+1} \left[ (a-x) \int_0^{a+1} f''(tx + (1-t) a) dt + (x-b) \int_0^{a+1} f''(tx + (1-t) b) dt \right] \right|
\]

\[
\leq (x-a)^{a+1} (b-x)^{a+1} \left[ (a-x) \int_0^{a+1} |f''(tx + (1-t) a)| dt + (x-b) \int_0^{a+1} |f''(tx + (1-t) b)| dt \right]
\]

\[
\leq (x-a)^{a+1} (b-x)^{a+1} \left[ (x-a) \left( \frac{1}{0} \int_0^{(a+1)p} f''(tx + (1-t) a) dt \right)^{\frac{1}{p}} + \left( \frac{1}{0} \int_0^{(a+1)p} f''(tx + (1-t) b) dt \right)^{\frac{1}{q}} \right]
\]

\[
\leq (x-a)^{a+1} (b-x)^{a+1} \left[ (b-x) \left( \frac{1}{0} \int_0^{(a+1)p} f''(tx + (1-t) a) dt \right)^{\frac{1}{p}} + \left( \frac{1}{0} \int_0^{(a+1)p} f''(tx + (1-t) b) dt \right)^{\frac{1}{q}} \right]
\]
Corollary 2.5.

Proof.

Corollary 2.6.

Theorem 2.2. Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable mapping on $I^0$ such that $|f''| \in L_1[a, b]$, where $a, b \in I$, with $a < b$.

If $|f''|$ is $P$-convex on $[a, b]$, and and $f''$ is bounded, i.e., $\|f''\|_{\infty} = \sup_{x \in [a, b]} |f''| < \infty$, for any $x \in [a, b]$, then the following inequality holds

$$\|L_\alpha(x)\| \leq \frac{2}{(a + 2)} \|f''\|_{\infty} \tag{2.28}$$

Proof. by lemma 2.1 and Under the given assumptions on $f''$, we have

$$\|L_\alpha(x)\| = \left| (x - a)^{\alpha + 1} (b - x)^{\alpha + 1} \left[ (a - x) \int_0^1 t^{\alpha + 1} f'' (tx + (1 - t)a) dt + (x - b) \int_0^1 t^{\alpha + 1} f'' (tx + (1 - t)b) dt \right] \right|$$

$$\leq (x - a)^{\alpha + 1} (b - x)^{\alpha + 1} \left[ (x - a) \int_0^1 t^{\alpha + 1} (|f''(x)| + |f''(a)|) dt + (x - b) \int_0^1 t^{\alpha + 1} (|f''(x)| + |f''(b)|) dt \right]$$

$$= 2 \|f''\|_{\infty} (x - a)^{\alpha + 1} (b - x)^{\alpha + 1} (b - a) \int_0^1 t^{\alpha + 1} dt = \frac{2}{(a + 2)} \frac{2}{(x - a)^{\alpha + 1} (b - x)^{\alpha + 1} (b - a)} \|f''\|_{\infty} \tag{2.29}$$

Proof. just take in (2.28), $\alpha = 1, x = \frac{a + b}{2}$ and dividing both side of the result by $\frac{(b - a)^3}{2}$ we obtain (2.29).

Corollary 2.4. With the assumptions in Theorem 2.2 in the case where $\alpha = 1$, one has the inequality

$$\left| f\left(\frac{a + b}{2}\right) - \frac{1}{(b - a)} \int_a^b f(t) dt \right| \leq \frac{(b - a)^2}{12} \|f''\|_{\infty} \tag{2.29}$$

Proof. just take in (2.28), $\alpha = 1, x = \frac{a + b}{2}$ and dividing both side of the result by $\frac{(b - a)^3}{2}$ we obtain (2.29).

Corollary 2.5. With the assumptions in Theorem 2.2 in the case where $\alpha = 1$, one has the inequality

$$\left| f\left(\frac{\alpha + b}{4}\right) + f\left(\frac{\alpha + 3b}{4}\right) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{(b - a)^2}{48} \|f''\|_{\infty} \tag{2.30}$$

Proof. The steps of the proof are similar to that of Corollary 2.2, we start by applying Theorem 2.2 a faith on the interval $\left[\frac{\alpha + b}{4}, \frac{\alpha + 3b}{4}\right]$, taking $\alpha = 1$ and $x = \frac{3a + b}{4}$, and a second time on the interval $[\frac{\alpha + b}{2}, b]$ for $\alpha = 1$ and $x = \frac{a + 3b}{4}$, make the sum and dividing the results by 2, we obtain (2.30).

Corollary 2.6. Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable mapping on $I^0$ such that $|f''| \in L_1[a, b]$, where $a, b \in I$, with $a < b$. 
If \( |f''| \) is P-convex on \([a,b] \), \( p, q \geq 1, \frac{1}{p} + \frac{1}{q} = 1 \) and and \( f'' \) is bounded, i.e., \( \|f''\|_{\infty} = \sup_{x \in [a,b]} |f''| < \infty \), for any \( x \in [a,b] \),

then the following inequality holds

\[
|L_a(x)| \leq 2^{\frac{1}{q}} (b - a) (x - a)^{a+1} (b - x)^{a+1} \left( \frac{1}{(a + 1) p + 1} \right)^{\frac{1}{p}} \|f''\|_{\infty} \tag{2.31}
\]

**Proof.** by lemma 2.1, the assumptions given on \( f'' \) and using the well-known Hölder’s inequality, we have

\[
|L_a(x)| \leq (x - a)^{a+1} (b - x)^{a+1} \left[ \int_0^1 |f''(tx + (1 - t) a)| dt + \int_0^1 |f''(tx + (1 - t) b)| dt \right] \leq 2^{\frac{1}{q}} (b - a) (x - a)^{a+1} (b - x)^{a+1} \left( \frac{1}{(a + 1) p + 1} \right)^{\frac{1}{p}} \|f''\|_{\infty}
\]

\[
|L_a(x)| \leq (b - a) (x - a)^{a+1} (b - x)^{a+1} \left[ \alpha + \beta (a + 2, s + 1) \right] \|f''\|_{\infty} \tag{2.32}
\]

**Theorem 2.3.** Let \( f: I \subset [0, \infty) \to \mathbb{R} \) be a twice differentiable mapping on \( I^0 \) such that \( f'' \in L_1 [a,b] \), where \( a, b \in I \), with \( a < b \).

If \( |f''| \) is s-convex on \([a,b] \) with \( s \in (0, 1) \), and and \( f'' \) is bounded, i.e., \( \|f''\|_{\infty} = \sup_{x \in [a,b]} |f''| < \infty \), for any \( x \in [a,b] \),

then the following inequality holds

\[
|L_a(x)| \leq (b - a) (x - a)^{a+1} (b - x)^{a+1} \left[ \frac{1}{a + s + 2} + \beta (a + 2, s + 1) \right] \|f''\|_{\infty}
\]

**Proof.** by lemma 2.1 and since \( |f''| \) is s-convex and \( |f''| \leq M \), then we have

\[
|L_a(x)| \leq (x - a)^{a+1} (b - x)^{a+1} \left[ \int_0^1 |f''(tx + (1 - t) a)| dt + \int_0^1 |f''(tx + (1 - t) b)| dt \right] \leq \|f''\|_{\infty} (x - a)^{a+1} (b - x)^{a+1} \left( \frac{1}{p + 1} \right)^{\frac{1}{q}} \int_0^1 (t^s |f''(x)| + (1 - t)^s |f''(a)|) \ dt + \int_0^1 (b - x)^{a+1} (b - x)^{a+1} \left[ \frac{1}{a + s + 2} + \beta (a + 2, s + 1) \right] \|f''\|_{\infty}
\]

\[
|L_a(x)| \leq \|f''\|_{\infty} \left[ \int_0^1 s dt + \int_0^1 (1 - t)^s dt \right] + (b - x) \left( \int_0^1 s dt + \int_0^1 (1 - t)^s dt \right)
\]

\[
|L_a(x)| \leq \|f''\|_{\infty} \left[ \int_0^1 s dt + \int_0^1 (1 - t)^s dt \right]
\]
\[
= (b - a) (x - a)^{\alpha + 1} (b - x)^{\alpha + 1} \left[ \frac{1}{\alpha + s + 2} + \beta (\alpha + 2, s + 1) \right] \|f''\|_\infty
\]

Corollary 2.7. With the assumptions in Theorem 2.3 in the case where \( \alpha = 1 \), one has the inequality
\[
\left| \frac{f(a + b)}{2} - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{(b - a)^2}{8} \left[ \frac{s^2 + 3s + 4}{(s + 3)(s + 2)(s + 1)} \right] \|f''\|_\infty
\] (2.33)

Proof. The proof is similar to that of Corollary 2.2.

Corollary 2.8. With the assumptions in Theorem 2.3 in the case where \( \alpha = 1 \), one has the inequality
\[
\left| \frac{f(\frac{3a + b}{4}) + f(\frac{a + 3b}{4})}{2} - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{(b - a)^2}{32} \left[ \frac{s^2 + 3s + 4}{(s + 3)(s + 2)(s + 1)} \right] \|f''\|_\infty
\] (2.34)

Proof. The proof is similar to that of Corollary 2.2.

Corollary 2.9. Let \( f : I \subset [0, \infty) \rightarrow \mathbb{R} \) be a twice differentiable mapping on \( I \) such that \( f'' \in L_1 [a, b] \), where \( a, b \in I \), with \( a < b \).

If \( |f''|^q \) is \( s \)-convex on \( [a, b] \) with \( s \in (0, 1) \), \( p, q \geq 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and and \( f'' \) is bounded, i.e., \( \|f''\|_\infty = \sup_{x \in [a, b]} |f''| < \infty \), for any \( x \in [a, b] \),

then the following inequality holds
\[
|L_{\alpha}(x)| \leq 2^{\frac{1}{3}} (b - a) (x - a)^{\alpha + 1} (b - x)^{\alpha + 1} \left[ \frac{1}{(\alpha + 1)p + 1} \right] \left( \frac{1}{s + 1} \right) \|f''\|_\infty
\] (2.35)

Proof. by lemma 2.1 the assumptions given on \( f'' \) and using the well-known Hölder’s inequality, we have
\[
|L_{\alpha}(x)| \leq (x - a)^{\alpha + 1} (b - x)^{\alpha + 1} \left[ (x - a) \int_0^{\frac{1}{p}} |f''(tx + (1 - t)a)| \, dt + (b - x) \int_0^{\frac{1}{p}} |f''(tx + (1 - t)b)| \, dt \right]
\]
\[
\leq (x - a)^{\alpha + 1} (b - x)^{\alpha + 1} \left[ (x - a) \left( \frac{1}{(\alpha + 1)p + 1} \right)^{\frac{1}{q}} \left( \int_0^{\frac{1}{p}} |f''(tx + (1 - t)a)|^q \, dt \right)^{\frac{1}{q}} \right.
\]
\[
+ \left. (b - x) \left( \frac{1}{(\alpha + 1)p + 1} \right)^{\frac{1}{q}} \left( \int_0^{\frac{1}{p}} |f''(tx + (1 - t)b)|^q \, dt \right)^{\frac{1}{q}} \right]
\]
\[
\leq (x - a)^{\alpha + 1} (b - x)^{\alpha + 1} \left[ (x - a) \left( \frac{1}{(\alpha + 1)p + 1} \right)^{\frac{1}{q}} \left( \int_0^{\frac{1}{p}} (t^s |f''(tx)|^q + (1 - t)^s |f''(a)|^q) \, dt \right)^{\frac{1}{q}} \right.
\]
\[
+ \left. (b - x) \left( \frac{1}{(\alpha + 1)p + 1} \right)^{\frac{1}{q}} \left( \int_0^{\frac{1}{p}} (t^s |f''(tx)|^q + (1 - t)^s |f''(b)|^q) \, dt \right)^{\frac{1}{q}} \right]
\]
\[
\leq (b - a) (x - a)^{\alpha + 1} (b - x)^{\alpha + 1} \left( \frac{1}{(\alpha + 1)p + 1} \right)^{\frac{1}{q}} \left( \int_0^{\frac{1}{p}} (t^s + (1 - t)^s) \, dt \right)^{\frac{1}{q}} \|f''\|_\infty
\]
\[
2^{\frac{1}{3}} (b - a) (x - a)^{\alpha + 1} (b - x)^{\alpha + 1} \left( \frac{1}{(\alpha + 1)p + 1} \right)^{\frac{1}{q}} \left( \frac{1}{s + 1} \right)^{\frac{1}{q}} \|f''\|_\infty
\]

\[ \]
Theorem 2.4. Let \( f : I \subset [0, \infty) \to \mathbb{R} \) be a twice differentiable mapping on \( I \) such that \( |f''| \in L_1 [a, b] \), where \( a, b \in I \), with \( a < b \).

If \( |f''| \) is \( h \)-convex on \([a, b] \), and \( f'' \) is bounded, i.e., \( \|f''\|_\infty = \sup_{x \in [a, b]} |f''| < \infty \), for any \( x \in [a, b] \), then the following inequality holds

\[
|L_a(x)| \leq \|f''\|_\infty (b - a) (x - a)^{a+1} (b - x)^{a+1} \int_0^1 (t^{a+1} + (1 - t)^{a+1}) h(t) dt.
\] (2.36)

Proof. by lemma [2.1] and since \( |f''| \) is \( h \)-convex and \( |f''| \leq M \), then we have

\[
|L_a(x)| \leq (x - a)^{a+1} (b - x)^{a+1} \left[ (x - a) \int_0^1 |f''(tx + (1 - t))| dt + (b - x) \int_0^1 |f''(tx + (1 - t))| dt \right]
\]

\[
\leq (x - a)^{a+1} (b - x)^{a+1} \times \left[ (x - a) \int_0^1 (t^{a+1} h(t) |f''(x)| + h(1 - t) |f''(a)|) dt + (b - x) \int_0^1 (t^{a+1} h(t) |f''(x)| + h(1 - t) |f''(b)|) dt \right]
\]

\[
\leq \|f''\|_\infty (b - a) (x - a)^{a+1} (b - x)^{a+1} \int_0^1 t^{a+1} (h(t) + h(1 - t)) dt
\]

\[
= \|f''\|_\infty (b - a) (x - a)^{a+1} (b - x)^{a+1} \int_0^1 (t^{a+1} + (1 - t)^{a+1}) h(t) dt.
\]

Corollary 2.10. With the assumptions in Theorem 2.4 in the case where \( \alpha = 1 \), one has the inequality

\[
\left| f\left(\frac{x + b}{2}\right) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \|f''\|_\infty \frac{(b - a)^2}{8} \int_0^1 (2t^2 - 2t + 1) h(t) dt.
\] (2.37)

Proof. The proof is similar to that of Corollary 2.1

Corollary 2.11. With the assumptions in Theorem 2.4 in the case where \( \alpha = 1 \), one has the inequality

\[
\left| f\left(\frac{3x + a + b}{4}\right) + f\left(\frac{a + 3b}{4}\right) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \|f''\|_\infty \frac{(b - a)^2}{32} \int_0^1 (2t^2 - 2t + 1) h(t) dt.
\] (2.38)

Proof. The proof is similar to that of Corollary 2.2

Corollary 2.12. Let \( f : I \subset [0, \infty) \to \mathbb{R} \) be a twice differentiable mapping on \( I \) such that \( |f''| \in L_1 [a, b] \), where \( a, b \in I \), with \( a < b \).

If \( |f''| \) is \( h \)-convex on \([a, b] \), \( p, q \geq 1, \frac{1}{p} + \frac{1}{q} = 1 \) and and \( f'' \) is bounded, i.e., \( \|f''\|_\infty = \sup_{x \in [a, b]} |f''| < \infty \), for any \( x \in [a, b] \), then the following inequality holds

\[
|L_a(x)| \leq 2^{\frac{1}{p}} (b - a) (x - a)^{a+1} (b - x)^{a+1} \left( \frac{1}{(a + 1) p + 1} \right)^{\frac{1}{q}} \left( \int_0^1 h(t) dt \right)^{\frac{1}{p}} \|f''\|_\infty
\] (2.39)
**Proof.** By lemma 2.1 the assumptions given on \( f'' \) and using the well-known Hölder’s inequality, we have

\[
|L_a(x)| \leq (x-a)^{a+1} (b-x)^{a+1} \left[ (x-a) \int_0^1 |f''(tx + (1-t)a)| \, dt + (b-x) \int_0^1 |f''(tx + (1-t)b)| \, dt \right]
\]

\[
\leq \frac{1}{(a+1)(b-a)} \times \left[ (b-x)(x-a)^2 \int_0^1 |f''(tx + (1-t)a)| \, dt + (a-x)(b-x)^2 \int_0^1 |f''(tx + (1-t)b)| \, dt \right]
\]

\[
\leq (x-a)^{a+1} (b-x)^{a+1} \left[ (x-a) \left( \int_0^1 (t^{a+1})^p \, dt \right)^\frac{1}{p} \left( \int_0^1 |f''(tx + (1-t)a)|^q dt \right)^\frac{1}{q} + (b-x) \left( \int_0^1 (t^{a+1})^p \, dt \right)^\frac{1}{p} \left( \int_0^1 |f''(tx + (1-t)a)|^q dt \right)^\frac{1}{q} \right]
\]

\[
= (b-a)(x-a)^{a+1} (b-x)^{a+1} \left[ \left( \frac{1}{(a+1)p+1} \right)^\frac{1}{p} \left( \int_0^1 (h(t)|f''(x)|^q + h(1-t)|f''(a)|^q dt \right) \right]^{\frac{1}{q}}
\]

\[
\leq 2^{\frac{1}{q}} (b-a)(x-a)^{a+1} (b-x)^{a+1} \left( \frac{1}{(a+1)p+1} \right)^\frac{1}{p} \left( \int_0^1 h(t) \, dt \right) \|f''\|_\infty.
\]

\[\square\]

Now, using the above reasoning we can obtain some new Ostrowski Type inequalities involving Riemann-Liouville fractional integrals for functions whose derivatives are \( m \)-convex.

**Theorem 2.5.** Let \( f : I \rightarrow \mathbb{R} \) be a twice differentiable mapping on \( I \) such that \( |f''| \in L_1 [a, b], \) where \( a, b \in I, \) with \( 0 < a < b. \)

If \( |f''| \) is \( m \)-convex function on \([a, b], m \in (0, 1) \) and \( f'' \) is bounded, i.e., \( \|f''\|_\infty = \sup_{x \in [a, b]} |f''| < \infty \) for any \( x \in [a, b], \) then the following inequality holds

\[
|L_a(x)| \leq (1-m)(x-a)^{a+1}(b-x)^{a+1} (Y_1 + Y_2) \|f''\|_\infty
\]

where \( Y_1 = (x-a) \left[ \frac{1}{a+1} \left( \frac{x-a}{x-ma} \right) + \frac{1}{a+2} \left( \frac{(1-m)a}{x-ma} + \frac{m}{1-m} \right) \right], \)

and \( Y_2 = (b-x) \left[ -\frac{1}{a+1} \left( \frac{b-x}{b-mx} \right) + \frac{1}{a+2} \left( \frac{1}{1-m} \right) \right]. \)

**Proof.** By lemma 2.1 and Under the given assumptions on \( f'' \) we have

\[
|L_a(x)| = \left| (b-x)^{a+1} \int_a^x (y-a)^{a+1} f''(y) \, dy + (b-x)^{a+1} \int_x^b (y-a)^{a+1} f''(y) \, dy \right|
\]

\[
= \left| (b-x)^{a+1} (x-ma) \int_0^{(1-m)a/(x-ma)} (tx + m(1-t)a - a)^{a+1} f''(tx + m(1-t)a) \, dt \right|
\]

\[
+ \left| (x-a)^{a+1} (b-mx) \int_0^{(1-m)x/(b-mx)} (b - (tb + m(1-t)x))^a f''(tb + m(1-t)x) \, dt \right|
\]
Corollary 2.14. Dividing both side of (2.42) by \( (\phi \leq x \leq b) \leq \infty \), we obtain (2.41).

Corollary 2.13. With the assumptions in Theorem 2.5, in the case where \( \alpha = 1 \), one has the inequality

\[
\left| \frac{f(a+b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{(1-m) (b-a)^2}{16} \varphi \|f''\|_\infty
\]

where \( \varphi = \frac{(1-m)(b-a)^2}{4(b+(1-2m)a)(b-(b-a))} + \frac{1}{3} \left[ \frac{1+m}{1-m} + \frac{2(1-m)a}{b+(1-2m)a} \right] \).

Proof. Choose \( x = \frac{a+b}{2} \) and \( \alpha = 1 \) in (2.40), we get

\[
\frac{(b-a)^3}{2} \left| \frac{f(a+b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \varphi \frac{(1-m) (b-a)^5}{32} \|f''\|_\infty
\]

dividing both side of (2.42) by \( \frac{(b-a)^3}{2} \) we obtain (2.41).

Corollary 2.14. With the assumptions in Theorem 2.5, in the case where \( \alpha = 1 \), one has the inequality

\[
\left| \frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq (\Psi_1 + \Psi_2) \frac{(1-m) (b-a)^2}{128} \|f''\|_\infty
\]

where \( \Psi_1 = \frac{b-a}{4} \left[ \frac{1}{b+(3-4m)a} - \frac{1}{(2-m)b+(2-3m)a} \right] + \frac{1}{3} \left[ \frac{1+m}{1-m} + \frac{4(1-m)a}{b+(3-4m)a} \right] \)

and \( \Psi_2 = \frac{b-a}{4} \left[ \frac{1}{(5-2m)b+(1-2m)a} - \frac{1}{(4-3m)b-ma} \right] + \frac{1}{3} \left[ \frac{1+m}{1-m} + \frac{2(1-m)(a+b)}{(3-2m)b+(1-2m)a} \right] \).
Proof. Apply Theorem 2.5 a faith on the interval \([a, \frac{a+b}{2}]\) taking \(\alpha = 1\) in (2.40), and replace \(x\) by \(\frac{3a+b}{4}\), we get

\[
\frac{(b-a)^3}{16} \left[ f\left( \frac{3a+b}{4} \right) - \frac{2}{b-a} \int_a^{b} f(t) dt \right] \leq \frac{(1-m)(b-a)^5}{1024} \|f''\|_\infty \psi_1
\]  

(2.44) implies

\[
\left| f\left( \frac{3a+b}{4} \right) - \frac{2}{b-a} \int_a^{b} f(t) dt \right| \leq \frac{(1-m)(b-a)^2}{64} \|f''\|_\infty \psi_1
\]  

Apply Theorem 2.5 another faith on the interval \([\frac{a+b}{2}, b]\) taking \(\alpha = 1\) in (2.40), and replace \(x\) by \(\frac{a+3b}{4}\), we get

\[
\left| f\left( \frac{a+3b}{4} \right) - \frac{2}{b-a} \int_a^{b} f(t) dt \right| \leq \frac{(1-m)(b-a)^2}{64} \|f''\|_\infty \psi_2
\]  

summing (2.45) and (2.46), dividing the result by 2 we obtain (2.43). \(\square\)

**Theorem 2.6.** Let \(f : I \subset [0, \infty) \to \mathbb{R}\) be a twice differentiable mapping on \(I^0\) such that \(|f''| \in L_1[a,b]\), where \(a, b \in I, a < b\).

If \(|f''|\) is \((s, m)\) - convex on \([a, b]\), where \((s, m) \in (0, 1]^2\), and \(f''\) is bounded, i.e., \(||f''||_\infty = \sup_{x \in [a,b]} |f''| < \infty\), for any \(x \in [a, b]\), then the following inequality holds

\[
|L_a(x)| \leq (1-m) \|f''\|_\infty \left[ \frac{m}{(1-m)(x+2)} (b-x)^{\alpha+1} (x-a)^{\alpha+1} - \chi_1 - \chi_2 \right]
\]  

where \(\chi_1 = (b-x)^{\alpha+1} (x-ma)^{-s} ((1-m)a)^{\alpha+1} \beta (\alpha+2, -s - \alpha - 2),\)

and \(\chi_2 = (x-a)^{\alpha+1} (b-mx)^{\alpha+2} \beta (\alpha+2, s+1)\).

Proof. by lemma 2.1 and Under the given assumptions on \(f''\), we have

\[
|L_a(x)| = \left| (b-x)^{\alpha+1} \int_a^x (y-a)^{\alpha+1} f''(y) dy + (x-a)^{\alpha+1} \int_x^b (b-y)^{\alpha+1} f''(y) dy \right|
\]

\[
= \left| (b-x)^{\alpha+1} (x-ma) \int_0^1 (tx + m (1-t) a - a)^{\alpha+1} f''(tx + m (1-t) a) dt + \right|
\]

\[
\left| (x-a)^{\alpha+1} (b-mx) \int_0^1 (b - (tb + m (1-t) x))^{\alpha+1} f''(tb + m (1-t) x) dt \right|
\]

\[
\leq \left| (b-x)^{\alpha+1} (x-ma)^{\alpha+2} \int_0^1 \left( t - \frac{(1-m)a}{x-ma} \right)^{\alpha+1} f''(tx + m (1-t) a) dt \right| + \right|
\]

\[
\left| (x-a)^{\alpha+1} (b-mx)^{\alpha+2} \int_0^1 (1-t)^{\alpha+1} f''(tb + m (1-t) x) dt \right|
\]
\[
\begin{align*}
\left(\begin{array}{c}
(b - x)^{a+1} (x - ma)^{a+2} \\(1 - m) a x - ma
\end{array}\right) & \int \left( t - \frac{(1-m)a}{x-ma} \right)^{a+1} |f''(tx + m(1-t)a)| \, dt + \\
(x - a)^{a+1} (b - mx)^{a+2} & \int \left( 1 - t \right)^{a+1} |f''(tb + m(1-t)x)| \, dt \\
\end{align*}
\]

\[
\begin{align*}
\left(\begin{array}{c}
(b - x)^{a+1} (x - ma)^{a+2} \\(1 - m) a x - ma
\end{array}\right) & \int \left( t - \frac{(1-m)a}{x-ma} \right)^{a+1} (t^s |f''(x)| + m(1-t^s) |f''(a)|) \, dt + \\
(x - a)^{a+1} (b - mx)^{a+2} & \int \left( 1 - t \right)^{a+1} (t^s |f''(b)| + m(1-t^s) |f''(x)|) \, dt \\
\end{align*}
\]

\[
\leq (1 - m) \| f'' \|_{\infty}
\]

\[
\begin{align*}
\left(\begin{array}{c}
\left(\frac{m}{1-m(a+2)}\right) (b - x)^{a+1} (x - a)^{a+2} \\
\left(\frac{m}{1-m(a+2)}\right) (x - a)^{a+1} (b - x)^{a+2}
\end{array}\right) & \int (1 - t)\left(t^{s+2}\right) dt + \\
\left(\begin{array}{c}
\left(\frac{m}{1-m(a+2)}\right) (x - ma)^{a+2} \\
\left(\frac{m}{1-m(a+2)}\right) (x - ma)^{a+1} (b - ma)^{a+2}
\end{array}\right) & \int (1 - t^{s+3}) dt + \\
\end{align*}
\]

\[
\leq (1 - m) \| f'' \|_{\infty}
\]

\[
\left[\frac{m}{(1-m)(a+2)}\right] (b - x)^{a+1} (x - a)^{a+1} (b - a) + \chi_1 + \chi_2
\]

\]

**Corollary 2.15.** With the assumptions in Theorem 2.6, in the case where \(\alpha = 1\), one has the inequality

\[
\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \Gamma \left(\frac{1-m}{b-a}\right) \| f'' \|_{\infty}
\]

where \(\Gamma = \frac{m(b-a)^3}{2a^m} + \frac{(b+1-2m)a^{-s}(1-m)a^{s+3}}{2^{s-3}} \beta(3, -s - 3) + \frac{(2-m)(b-ma)^3}{2^4} \beta(3, s + 1)\).

**Proof.** Just take in (2.47), \(\alpha = 1\), \(x = \frac{a+b}{2}\) and dividing both sides of the result by \(\frac{(b-a)^3}{2}\) we obtain (2.48). \(\square\)

**Corollary 2.16.** With the assumptions in Theorem 2.6, in the case where \(\alpha = 1\), one has the inequality

\[
\left| f\left(\frac{3a+b}{4}\right) + \frac{f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \zeta_1 + \zeta_2 \left(\frac{1-m}{b-a}\right) \| f'' \|_{\infty},
\]

(2.49)
where $\zeta_1 = \frac{m(b-a)^3}{48(1-m)} + \left(\frac{b+(3-4m)a}{4}\right)^{-s} (1-m) a^{s+3} \beta(3,-s-3) + \frac{(2-m)b-ma)^3}{64} \beta(3, s+1)$, and $\zeta_2 = \frac{m(b-a)^3}{48(1-m)} + \left(\frac{3-2m)b+(1-2m)a}{4}\right)^{-s} (1-m) a^{s+3} \beta(3,-s-3) + \frac{(4-3m)b+(2-3m)a)^3}{64} \beta(3, s+1)$.

Proof. The steps of the proof are similar to that of Corollary 2.2, we start by applying Theorem 2.6 a faith on the interval $[a, a+x_2]$, taking $x = \frac{3a+b}{4}$, and a second time on the interval $[a+x_2, b]$ for $x = \frac{a+3b}{4}$, make the sum and dividing the results by $\frac{(b-a)^3}{x^2}$, we obtain (2.49). $\square$

3 Acknowledgements

The author would like to thank the anonymous referee for his/her valuable suggestions. This work has been supported by CNEPRU–MESRS–B01120120103 project grants.

References


*Received*: June 13, 2014; *Accepted*: July 19, 2014

**UNIVERSITY PRESS**

Website: [http://www.malayajournal.org/](http://www.malayajournal.org/)