Some curvature tensors on a generalized Sasakian space form

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Abstract

In the present paper, we have studied the geometry of generalized Sasakian space form with the condition satisfying $W^*(\xi, X)W^* = 0$, $W^*(\xi, X)S = 0$, $W^*(\xi, X)P = 0$ and $P(\xi, X)P = 0$.

Keywords: Generalized Sasakian space form, M-projective curvature tensor, Projective curvature tensor.


1 Introduction

A Sasakian manifold $(M, \phi, \xi, \eta, g)$ is said to be a Sasakian space form [3], if all the $\phi$-sectional curvatures $K(X \wedge \phi X)$ are equal to a constant $C$, where $K(X \wedge \phi X)$ denotes the sectional curvature of the section spanned by the unit vector field $X$, orthogonal to $\xi$ and $\phi X$. In such a case, the Riemannian curvature tensor of $M$ is given by,

$$
R(X, Y)Z = \frac{C + 3}{4} \{g(Y, Z)X - g(X, Z)Y\} + \frac{C - 1}{4} \{g(X, \phi Z)Y - g(Y, \phi Z)X + 2g(X, \phi Y)\phi Z\} + \frac{C - 1}{4} \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}.
$$

(1.1)

As a natural generalization of these manifolds, Alegre P., Blair D. E. and Carriazo A. [1, 3] introduced the notion of generalized Sasakian space form.

Sasakian space form and generalized Sasakian space form have been studied by several authors, viz., [2, 3, 5, 9, 14, 15].


The properties of the $M$-projective curvature tensor in Sasakian and Kaehler manifolds were studied by Ojha R. H. [11, 12]. He showed that it bridges the gap between the conformal curvature tensor, coharmonic curvature tensor and concircular curvature tensor. Chaubey S. K. and Ojha R. H. [17] studied the properties of the $M$-projective curvature tensor in Riemannian and Kenmotsu manifolds. Chaubey S. K. [18] also studied the properties of $M$-projective curvature tensor in LP-Sasakian manifold. Present authors [4] have studied some properties of $M$-projective curvature tensor in a generalized Sasakian space form. Motivated by these ideas, in the present paper we have extended the study of further properties of $M$-projective curvature tensor to generalized Sasakian space form. The present paper is organized as follows:

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In section 2, we review some preliminary results. From section 3 onwards we have obtained necessary and sufficient condition for a generalized Sasakian space form satisfying the derivation conditions \( W^*(\xi, X)W^* = 0, W^*(\xi, X)S = 0, W^*(\xi, X)P = 0 \) and \( P(\xi, X)P = 0 \). We have proved that these conditions are satisfied if and only if \( f_3 = \frac{3f_2}{1-2n} \).

2 Preliminaries

An odd-dimensional Riemannian manifold \((M, g)\) is called an almost contact manifold if there exists on \(M\), a \((1,1)\) tensor field \(\phi\), a vector field \(\xi\) and a 1-form \(\eta\) \([6]\) such that,

\[
\phi^2(X) = -X + \eta(X)\xi, \tag{2.2}
\]

\[
\eta(\phi X) = 0, \tag{2.3}
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}
\]

\[
\phi\xi = 0, \quad \eta(\xi) = 0, \quad g(X, \xi) = \eta(X), \tag{2.5}
\]

for any vector fields \(X, Y\) on \(M\).

If in addition, \(\xi\) is a Killing vector field, then \(M\) is said to be a \(K\)-contact manifold. It is well known that a Contact metric manifold is a \(K\)-contact manifold if and only if \((\nabla_X\xi) = -\phi(X)\) for any vector field \(X\) on \(M\).

Given an almost contact metric manifold \((M, \phi, \xi, \eta, g)\), we say that \(M\) is an generalized Sasakian space form \([1]\), if there exist three functions \(f_1, f_2\) and \(f_3\) on \(M\) such that

\[
R(X, Y)Z = f_1 \{g(Y, Z)X - g(X, Z)Y\} \tag{2.6}
\]

\[
+ f_2 \{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\}
\]

\[
+ f_3 \{\eta(\phi Z)\eta(X)Y - \eta(Y)\eta(Z)X \tag{2.7}
\]

\[
+ g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\},
\]

for any vector fields \(X, Y, Z\) on \(M\), where \(R\) denotes the curvature tensor of \(M\). This kind of manifold appears as a natural generalization of the well-known Sasakian space form \(M(C)\), which can be obtained as particular cases of generalized Sasakian space form by taking \(f_1 = \frac{C+3}{4}\) and \(f_2 = f_3 = \frac{C-1}{4}\). Further in a \((2n + 1)\)-dimensional generalized Sasakian space form, we have \([1]\)

\[
(\nabla_X\phi)(Y) = (f_1 - f_3)\{g(Y, X)\xi - \eta(Y)X\}, \tag{2.8}
\]

\[
(\nabla_X\xi) = -(f_1 - f_3)\phi(X), \tag{2.9}
\]

\[
QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\xi, \tag{2.10}
\]

\[
S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n - 1)f_3)\eta(X)\eta(Y), \tag{2.11}
\]

\[
r = 2n(2n + 1)f_1 + 6nf_2 - 4nf_3, \tag{2.12}
\]

\[
R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \tag{2.13}
\]

\[
R(\xi, X)Y = (f_1 - f_3)\{g(X, Y)\xi - \eta(Y)X\}, \tag{2.14}
\]

\[
\eta(R(X, Y)Z) = (f_1 - f_3)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \tag{2.15}
\]

In 1971, Pokhariyal G. P. and Mishra R. S. \([13]\) defined \(M\)-projective curvature tensor \(W^*\) on a Riemannian manifold as

\[
W^*(X, Y)Z = R(X, Y)Z - \frac{1}{4n}|S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY|, \tag{2.16}
\]

and projective curvature tensor \([16]\) is defined as

\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}|S(Y, Z)X - S(X, Z)Y|. \tag{2.17}
\]
3 Generalized Sasakian space form satisfying $W^*(\xi, X)W^* = 0$

Let us consider a generalized Sasakian space form satisfying

$$W^*(\xi, X)W^* = 0. \quad (3.18)$$

The above equation can be written as


for any vector field $X, Y, Z, U$ on $M$.

In view of (2.5), (2.9), (2.10) and (2.13), (2.16) becomes

$$W^*(\xi, X)Y = \frac{1}{4n}[(1 - 2n)f_3 - 3f_2]g(X, Y)\xi - \eta(Y)X \quad (3.20)$$

and

$$\eta(W^*(X, Y)Z) = \frac{1}{4n}[(1 - 2n)f_3 - 3f_2][g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \quad (3.21)$$

From (2.16) and (3.20), we find

$$W^*(\xi, X)W^*(Y, Z)U = \frac{1}{4n}[(1 - 2n)f_3 - 3f_2][g(X, W^*(Y, Z)U)\xi$$

$$- \frac{1}{4n}[(1 - 2n)f_3 - 3f_2][g(Z, U)\eta(Y)X - g(Y, U)\eta(Z)X]] \quad (3.22)$$

and

$$W^*(W^*(\xi, X)Y, Z)U = \frac{1}{4n}[(1 - 2n)f_3 - 3f_2][\frac{(1 - 2n)f_3 - 3f_2}{4n}g(X, Y)g(Z, U)\xi$$

$$- g(X, Y)\eta(U)Z - \eta(Y)W^*(X, Z)U]. \quad (3.23)$$

Substituting $Z = \xi$ in (2.16), we get

$$W^*(X, Y)\xi = \frac{1}{4n}[(1 - 2n)f_3 - 3f_2][\eta(Y)X - \eta(X)Y], \quad (3.24)$$

Substituting (3.20), (3.22), (3.23) in (3.19), we get

$$\frac{(1 - 2n)f_3 - 3f_2}{4n}[g(W^*(Y, Z)U, X)\xi - (1 - 2n)f_3 - 3f_2]g(Z, U)\eta(Y)X$$

$$- g(Y, U)\eta(Z)X + g(X, Y)g(Z, U)\xi - g(X, Y)\eta(U)Z + g(X, Z)\eta(U)Y$$

$$- g(X, Z)g(U, Y)\xi + g(X, U)\eta(Z)Y - g(X, U)\eta(Y)Z$$

$$+ \eta(Y)W^*(X, Z)U + \eta(Z)W^*(Y, X)U + \eta(U)W^*(Y, Z)X] = 0. \quad (3.25)$$

Taking inner product of (3.25) with respect to $\xi$ and using (2.16) and (3.21), we get

$$\frac{(1 - 2n)f_3 - 3f_2}{4n}[g(R(Y, Z)U, X)$$

$$- 4nf_1 + 3f_2 - (1 + 2n)f_3]g(X, Y)g(Z, U) - g(X, Z)g(Y, U)]$$

$$- \frac{3f_2 + (2n - 1)f_3}{4n}[g(X, Z)g(U, Y)\eta(Y) + g(Y, U)g(Z)\eta(X)$$

$$- g(X, Y)\eta(Z)\eta(U) - g(Z, U)\eta(X)\eta(Y)] = 0. \quad (3.26)$$

This implies either

$$f_3 = \frac{3f_2}{1 - 2n} \quad (3.27)$$
or
\[
g(R(Y, Z)U, X) = \frac{4nf_1 + 3f_2 - (1 + 2n)f_3}{4n}g(X, Y)g(Z, U) \\
- g(X, Z)g(Y, U) + \frac{3f_2 + (2n - 1)f_3}{4n}g(X, Z)\eta(U)\eta(Y) \\
+ g(Y, U)\eta(Z)\eta(X) - g(X, Y)\eta(Z)\eta(U) - g(Z, U)\eta(X)\eta(Y).
\] (3.28)

Let \(\{e_i\}, i = 1, 2, ..., 2n + 1\) be an orthonormal basis of the tangent space at any point of the space form. Then putting \(X = Y = e_i\), in (3.28) and taking summation over \(i, 1 \leq i \leq 2n + 1\), we get
\[
S(Z, U) = \frac{1}{4n}[(2n)(4nf_1 + 3f_2 - (1 + 2n)f_3) \\
- (3f_2 + (2n - 1)f_3)]g(Z, U) \\
- (2n - 1)(3f_2 + (2n - 1)f_3)\eta(U)\eta(Z).
\] (3.29)

Contracting the above equation we get,
\[
r = \frac{1}{2}[(2n + 1)(4nf_1 + 3f_2 - (1 + 2n)f_3) - 2(3f_2 + (2n - 1)f_3)],
\] (3.30)

using (2.11) we get
\[
f_3 = \frac{3f_2}{(1 - 2n)}.
\] (3.31)

This leads us to state the following:

**Theorem 3.1.** A \((2n + 1)\)-dimensional \((n > 1)\) generalized Sasakian space form satisfies the condition \(W^*(\xi, X)W^* = 0\) if and only if \(f_3 = \frac{3f_2}{(1 - 2n)}\).

### 4 Generalized Sasakian space form satisfying \(W^*(\xi, X)S = 0\)

The condition \(W^*(\xi, X)S = 0\) implies that
\[
S(W^*(\xi, X)Y, Z) + S(Y, W^*(\xi, X)Z) = 0.
\] (4.32)

Substituting (3.20) in (4.32), we obtain
\[
\frac{(1 - 2n)f_3 - 3f_2}{4n}g(X, Y)S(\xi, Z) - \eta(Y)S(X, Z) \\
+ S(Y, \xi)g(X, Z) - \eta(Z)S(X, Y)] = 0.
\] (4.33)

Again substituting \(Z = \xi\) in (4.33), we get
\[
\frac{(1 - 2n)f_3 - 3f_2}{4n}[S(X, Y) - 2n(f_1 - f_3)g(X, Y)] = 0.
\] (4.34)

This implies either
\[
f_3 = \frac{3f_2}{(1 - 2n)},
\] (4.35)
or
\[
S(X, Y) = 2n(f_1 - f_3)g(X, Y).
\] (4.36)

On contracting (4.36), we find
\[
r = 2n(2n + 1)(f_1 - f_3) \quad \text{and so} \quad f_3 = \frac{3f_2}{(1 - 2n)}.
\] (4.37)

Thus, we state

**Theorem 4.2.** A \((2n + 1)\)-dimensional \((n > 1)\) generalized Sasakian space form satisfies the condition \(W^*(\xi, X)S = 0\) if and only if \(f_3 = \frac{3f_2}{(1 - 2n)}\).
5 Generalized Sasakian space form satisfying \(W^*(\xi, X)P = 0\)

We know that,
\[
(W^*(\xi, X)P)(Y, Z)U = W^*(\xi, X)P(Y, Z)U - P(W^*(\xi, X)Y, Z)U \tag{5.38}
- P(Y, W^*(\xi, X)Z)U - P(Y, Z)W^*(\xi, X)U.
\]

But as we assume \(W^*(\xi, X)P = 0\), (5.38) takes the form
\[
W^*(\xi, X)P(Y, Z)U - P(W^*(\xi, X)Y, Z)U - P(Y, W^*(\xi, X)Z)U - P(Y, Z)W^*(\xi, X)U = 0. \tag{5.39}
\]

In view of (2.14), we obtain from (2.17) that
\[
\eta(P(X, Y)Z) = \frac{1}{2n}[(1 - 2n)P - 3f_2][g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \tag{5.40}
\]

From (2.17) and (3.20), we find
\[
W^*(\xi, X)P(Y, Z)U = \frac{1}{4n}[(1 - 2n)f_3 - 3f_2][g(X, R(Y, Z)U)\xi
- \frac{1}{2n}[S(U, Z)g(X, Y) - S(Y, U)g(X, Z)]\xi
- \frac{1}{2n}[(1 - 2n)f_3 - 3f_2][g(Z, U)\eta(Y)X - g(Y, U)\eta(Z)X]] \tag{5.41}
\]

and
\[
P(W^*(\xi, X)Y, Z)U = \frac{1}{4n}[(1 - 2n)f_3 - 3f_2][g(X, Y)g(Z, U)\xi
- \frac{1}{2n}S(U, Z)g(X, Y)\xi - \eta(Y)P(X, Z)U]. \tag{5.42}
\]

Also
\[
P(Y, Z)W^*(\xi, X)U = -\frac{1}{4n}[(1 - 2n)f_3 - 3f_2]\eta(U)P(Y, Z)X. \tag{5.43}
\]

Substituting (5.41), (5.42) and (5.43) in (5.39), we get
\[
\frac{(1 - 2n)f_3 - 3f_2}{4n}[g(R(Y, Z)U, X)\xi
- \frac{1}{2n}[S(U, Z)g(X, Y) - S(Y, U)g(X, Z)]\xi
- \frac{1}{2n}[(1 - 2n)f_3 - 3f_2][g(Z, U)\eta(Y)X - g(Y, U)\eta(Z)X]
- (f_1 - f_3)g(X, Y)g(Z, U)\xi + \frac{1}{2n}S(U, Z)g(X, Y)\xi
+ (f_1 - f_3)g(X, Z)g(Y, U)\xi - \frac{1}{2n}S(Y, U)g(X, Z)\xi
+ \eta(Y)P(X, Z)U + \eta(Z)P(Y, X)U + \eta(U)P(Y, Z)X] = 0 \tag{5.44}
\]

Taking inner product of (5.44) with respect to the Riemannian metric \(g\) and then using (2.5) and (5.40), we have
\[
\frac{1}{4n}[(1 - 2n)f_3 - 3f_2][g(R(Y, Z)U, X)
- (f_1 - f_3)\{g(X, Y)g(Z, U) - g(X, Z)g(Y, U)\}
+ \frac{1}{2n}[(1 - 2n)f_3 - 3f_2][g(X, Z)\eta(Y)\eta(U) - g(X, Y)\eta(Z)\eta(U)]] = 0. \tag{5.45}
\]

This implies either
\[
f_3 = \frac{3f_2}{(1 - 2n)}. \tag{5.46}
\]
or
\[
g(R(Y, Z)U, X) = (f_1 - f_3)\{g(X, Y)g(Z, U) - g(X, Z)g(Y, U)\} - \frac{1}{2n}(1 - 2n)f_3 - 3f_2\{g(X, Z)\eta(Y)\eta(U) - g(X, Y)\eta(Z)\eta(U)\}.
\]  
(5.47)

Let \(\{e_i\}, i = 1, 2, ..., 2n + 1\) be an orthonormal basis of the tangent space at any point of the space form. Then putting \(X = Y = e_i\) in (5.47) and taking summation over \(i, 1 \leq i \leq 2n + 1\), we get
\[
S(Z, U) = 2n(f_1 - f_3)g(Z, U) + [(1 - 2n)f_3 - 3f_2]\eta(Z)\eta(U).
\]  
(5.48)

Contracting (5.48), we find
\[
r = 2n(2n + 1)(f_1 - f_3) + (1 - 2n)f_3 - 3f_2.
\]  
(5.49)

Using (2.11), the above equation gives
\[
f_3 = \frac{3f_2}{(1 - 2n)}.
\]  
(5.50)

Thus, we state

**Theorem 5.3.** A \((2n + 1)\)-dimensional \((n > 1)\) generalized Sasakian space form satisfies the condition \(W^*(\xi, X)P = 0\) if and only if \(f_3 = \frac{3f_2}{(1 - 2n)}\).

6. An generalized Sasakian space form satisfying \(P(\xi, X)P = 0\)

The condition \(P(\xi, X)P = 0\) implies that
\[
\]  
(6.51)

In view of (2.5), (2.10) and (2.13), (2.17) becomes
\[
P(\xi, X)Y = (f_1 - f_3)g(X, Y)\xi - \frac{1}{2n}S(X, Y)\xi.
\]  
(6.52)

Using (6.52) in (6.51), we get
\[
(f_1 - f_3)g(P(Y, Z)U, X)\xi - \frac{1}{2n}S(P(Y, Z)U, X)\xi
\]  
(6.53)

\[
- [(f_1 - f_3)g(X, Y) - \frac{1}{2n}S(X, Y)][(f_1 - f_3)g(Z, U) - \frac{1}{2n}S(Z, U)]\xi
\]

\[
- [(f_1 - f_3)g(X, Z) - \frac{1}{2n}S(X, Z)][\frac{1}{2n}S(Y, U) - (f_1 - f_3)g(Y, U)]\xi
\]

\[
- [(f_1 - f_3)g(X, U) - \frac{1}{2n}S(X, U)]P(Y, Z)\xi = 0,
\]

Taking inner product of (6.53) with respect to the Riemannian metric \(g\) and then using (2.10), (2.17) and (5.40), we have
\[
\frac{1}{2n}(1 - 2n)f_3 - 3f_2\{g(R(Y, Z)U, X)
\]

\[
- (f_1 - f_3)\{g(X, Y)g(Z, U) - g(X, Z)g(Y, U)\} = 0.
\]  
(6.54)

\[
\Rightarrow f_3 = \frac{3f_2}{(1 - 2n)}\quad \text{or}
\]

\[
g(R(Y, Z)U, X) = (f_1 - f_3)\{g(X, Y)g(Z, U) - g(X, Z)g(Y, U)\}.
\]  
(6.55)

(6.55) implies
\[
R(Y, Z)U = (f_1 - f_3)\{g(Z, U)Y - g(Y, U)Z\}.
\]  
(6.56)
Contracting (6.56) with respect to the vector field $Y$, we find
\[ S(Z, U) = 2n(f_1 - f_3)g(Z, U). \] (6.57)

On contracting the above equation, we get
\[ r = 2n(2n + 1)(f_1 - f_3) \quad \text{and so} \quad f_3 = \frac{3f_2}{(1 - 2n)}. \] (6.58)

Thus, we state

Theorem 6.4. A $(2n + 1)$-dimensional $(n > 1)$ generalized Sasakian space form satisfies the condition $P(\xi, X) = 0$ if and only if $f_3 = \frac{3f_2}{(1 - 2n)}$.

References


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