Some new general integral inequalities for $P$-functions

İmdat İŞCAN$^a$, Erhan SET$^b$* and M. Emin ÖZDEMİR$^c$

$^a$Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28100, Giresun, Turkey.
$^b$Department of Mathematics, Faculty of Arts and Sciences, Ordu University, 52200, Ordu, Turkey.
$^c$Atatürk University, K.K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey.

Abstract

In this paper, we derive new estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae for functions whose derivatives in absolute value at certain power are $P$-functions. Some applications to special means of real numbers are also given.

Keywords: Convex function, $P$-function, Simpson’s inequality, Hermite-Hadamard’s inequality.

2010 MSC: 26A33, 26D10, 26D15,41A55.

1 Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following inequality holds:

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \quad (1.1)$$

This double inequality is well known as Hermite-Hadamard integral inequality for convex functions in the literature.

In [2] Dragomir et al. defined the concept of $P$-function as the following:

**Definition 1.1.** We say that $f : I \rightarrow \mathbb{R}$ is a $P$-function, or that $f$ belongs to the class $P(I)$, if $f$ is a non-negative function and for all $x, y \in I$, $\alpha \in [0, 1]$, we have

$$f(\alpha x + (1 - \alpha)y) \leq f(x) + f(y).$$

$P(I)$ contain all nonnegative monotone convex and quasi convex functions.

In [2], Dragomir et al., proved following inequalities of Hadamard’s type for $P$-function

**Theorem 1.1.** Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L \left[ a, b \right]$. Then the following inequality holds

$$f \left( \frac{a + b}{2} \right) \leq \frac{2}{b - a} \int_{a}^{b} f(x) \, dx \leq 2 \left[ f(a) + f(b) \right]. \quad (1.2)$$
The following inequality is well known in the literature as Simpson’s inequality.

Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on $(a, b)$ and $\|f^{(4)}\|_{\infty} = \sup_{x \in (a, b)} \left| f^{(4)}(x) \right| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3} \left( f(a) + f(b) \right) - \frac{1}{b-a} \int_{a}^{b} f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^4.$$  

In recent years many authors have studied error estimations for Simpson’s inequality and Hermite-Hadamard inequality; for refinements, counterparts, generalizations, see [11-10].

In [3], Iscan obtained a new generalization of some integral inequalities for differentiable convex mapping which are connected Simpson and Hadamard type inequalities by using the following lemma.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^o$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $a, \lambda \in [0, 1]$. Then the following equality holds:

$$\lambda \left( af(a) + (1-\alpha) f(b) \right) + (1-\lambda) f(aa + (1-\alpha) b) - \frac{1}{b-a} \int_{a}^{b} f(x)dx$$

$$= (b-a) \left[ \int_{0}^{1-a} (t-a\lambda) f' (tb + (1-t)a) dt \right.$$

$$\left. + \int_{1-a}^{1} (t-1 + \lambda (1-\alpha)) f' (tb + (1-t)a) dt \right].$$

The aim of this paper is to establish some new general integral inequalities for functions whose derivatives in absolute value at certain power are $P$-functions. Some applications of these results to special means is to give as well.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^o$, the interior of $I$. Throughout this section we will take

$$I_f (\lambda, a, a, b) = \lambda \left( af(a) + (1-\alpha) f(b) \right) + (1-\lambda) f(aa + (1-\alpha) b) - \frac{1}{b-a} \int_{a}^{b} f(x)dx$$

where $a, b \in I^o$ with $a < b$ and $a, \lambda \in [0, 1]$.

**Theorem 1.2.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^o$ such that $f' \in L[a, b]$, where $a, b \in I^o$ with $a < b$ and $a, \lambda \in [0, 1]$. If $|f'|^q$ is $P$-function on $[a, b]$, $q \geq 1$, then the following inequality holds:

$$\left| I_f (\lambda, a, a, b) \right| \leq (b-a) \left( |f'(b)|^q + |f'(a)|^q \right)^{\frac{1}{q}}$$

$$\times \left\{ \begin{array}{ll}
\gamma_2 (a, \lambda) + \gamma_2 (1-a, \lambda) & a\lambda \leq 1-a \leq 1-\lambda (1-a) \\
\gamma_2 (a, \lambda) + \gamma_1 (1-a, \lambda) & a\lambda \leq 1-\lambda (1-a) \leq 1-a \\
\gamma_1 (a, \lambda) + \gamma_2 (1-a, \lambda) & 1-a \leq a\lambda \leq 1-\lambda (1-a) 
\end{array} \right\}$$

where

$$\gamma_1 (a, \lambda) = (1-a) \left[ a\lambda - \frac{1-a}{2} \right],$$

$$\gamma_2 (a, \lambda) = (a\lambda)^2 - \gamma_1 (a, \lambda).$$

**Proof.** Suppose that $q \geq 1$. Since $|f'|^q$ is $P$-function on $[a, b]$, from Lemma 1 and using the well known power mean inequality, we have

$$\left| I_f (\lambda, a, a, b) \right|$$

$$\leq (b-a) \left[ \int_{0}^{1-a} \left| t-a\lambda \right| |f' (tb + (1-t)a)| dt + \int_{1-a}^{1} \left| t+1 + \lambda (1-a) \right| |f' (tb + (1-t)a)| dt \right].$$
\[ \leq (b-a) \left( \int_0^{1-a} |t-a\lambda| \, dt \right)^{1-\frac{1}{q}} \left( \int_{1-a}^1 |t-a\lambda| |f'(tb+(1-t)a)|^q \, dt \right)^{\frac{1}{q}} \]

\[ + \left( \int_{1-a}^{1} |t-1+\lambda(1-a)| \, dt \right)^{1-\frac{1}{q}} \left( \int_{1-a}^1 |t-1+\lambda(1-a)| |f'(tb+(1-t)a)|^q \, dt \right)^{\frac{1}{q}} \]

\[ \leq (b-a) \left( |f'(b)|^q + |f'(a)|^q \right)^{\frac{1}{q}} \left\{ \int_0^{1-a} |t-a\lambda| \, dt + \int_{1-a}^1 |t-1+\lambda(1-a)| \, dt \right\} \] (1.5)

Additionally, by simple computation

\[ \int_0^{1-a} |t-a\lambda| \, dt = \begin{cases} \gamma_2(a, \lambda), & a\lambda \leq 1-a \\ \gamma_1(a, \lambda), & a\lambda \geq 1-a \end{cases}, \] (1.6)

\[ \gamma_1(a, \lambda) = (1-a) \left[ a\lambda - \frac{1-a}{2} \right], \gamma_2(a, \lambda) = (a\lambda)^2 - \gamma_1(a, \lambda), \]

\[ \int_{1-a}^1 |t-1+\lambda(1-a)| \, dt = \begin{cases} \gamma_1(1-a, \lambda), & 1-\lambda(1-a) \leq 1-a \\ \gamma_2(1-a, \lambda), & 1-\lambda(1-a) \geq 1-a \end{cases} \] (1.7)

Thus, using (1.6) and (1.7) in (1.5), we obtain the inequality (1.3). This completes the proof. \( \square \)

**Corollary 1.1.** Under the assumptions of Theorem 1.2 with \( q = 1 \), we have

\[ \left| f\right|_{\gamma}(a, b) \leq (b-a) \left( |f'(b)| + |f'(a)| \right) \]

\[ \times \begin{cases} \gamma_2(a, \lambda) + \gamma_2(1-a, \lambda) & a\lambda \leq 1-a \leq 1-\lambda(1-a) \\ \gamma_2(a, \lambda) + \gamma_1(1-a, \lambda) & a\lambda \leq 1-\lambda(1-a) \leq 1-a \end{cases}, \gamma_1(1-a, \lambda) + \gamma_2(1-a, \lambda) & 1-a \leq a\lambda \leq 1-\lambda(1-a) \]

**Corollary 1.2.** In Theorem 1.2, if we take \( \alpha = \frac{1}{2} \) and \( \lambda = \frac{1}{2} \), then we have the following Simpson type inequality

\[ \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{5 (b-a)^3}{36} \left( |f'(b)|^q + |f'(a)|^q \right)^{\frac{1}{q}} \]

**Corollary 1.3.** In Theorem 1.2, if we take \( \alpha = \frac{1}{2} \) and \( \lambda = 0 \), then we have following midpoint inequality

\[ \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( |f'(b)|^q + |f'(a)|^q \right)^{\frac{1}{q}} \]

**Corollary 1.4.** In Theorem 1.2, if we take \( \alpha = \frac{1}{2} \) and \( \lambda = 1 \), then we get the following trapezoid inequality which is identical to the inequality in [1, Theorem 2.3].

\[ \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( |f'(b)|^q + |f'(a)|^q \right)^{\frac{1}{q}} \]

Using Lemma 1 we shall give another result for convex functions as follows.
Theorem 1.3. Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^c \) such that \( f' \in L[a, b] \), where \( a, b \in I^c \) with \( a < b \) and \( a, \lambda \in [0,1] \). If \( |f'|^q \) is \( P \)-function on \([a, b], q > 1\), then the following inequality holds:

\[
\left| I_f (\lambda, \alpha, a, b) \right| \leq (b-a) \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \tag{1.8}
\]

\[
\times \left\{ \begin{array}{l}
\epsilon_1^{1/p} (\alpha, \lambda, p)c_f^{1/q} (a, q) + \epsilon_1^{1/p} (1-\alpha, \lambda, p)k_f^{1/q} (a, q), \quad a\lambda \leq 1-\alpha \leq 1-\lambda (1-\alpha) \\
\epsilon_2^{1/p} (\alpha, \lambda, p)c_f^{1/q} (a, q) + \epsilon_2^{1/p} (1-\alpha, \lambda, p)k_f^{1/q} (a, q), \quad a\lambda \leq 1-\lambda (1-\alpha) \leq 1-\alpha , \\
\epsilon_1^{1/p} (\alpha, \lambda, p)c_f^{1/q} (a, q) + \epsilon_1^{1/p} (1-\alpha, \lambda, p)k_f^{1/q} (a, q), \quad 1-\alpha \leq a\lambda \leq 1-\lambda (1-\alpha) \end{array} \right.
\]

where

\[
\epsilon_f (a, q) = (1-a) \left[ |f' ((1-\alpha) b + aa)|^q + |f' (a)|^q \right], \tag{1.9}
\]

\[
k_f (a, q) = a \left[ |f' ((1-\alpha) b + aa)|^q + |f' (b)|^q \right],
\]

\[
\epsilon_1 (\alpha, \lambda, p) = (a\lambda)^{p+1} + (1-\alpha - a\lambda)^{p+1}, \quad \epsilon_2 (\alpha, \lambda, p) = (a\lambda)^{p+1} - (a\lambda - 1 + a)^{p+1},
\]

and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** Since \( |f'|^q \) is \( P \)-function on \([a, b], \) from Lemma [1] and by Hölder’s integral inequality, we have

\[
\left| I_f (\lambda, \alpha, a, b) \right| \leq (b-a) \left\{ \left( \int_0^{1-a} \left| t - \alpha \lambda \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^{1-a} \left| f' (tb + (1-t)a) \right|^q dt \right)^{\frac{1}{q}} \right\}.
\]

By the inequality (1.12), we get

\[
\int_0^{1-a} \left| f' (tb + (1-t)a) \right|^q dt = (1-a) \left( \frac{1}{(1-a)(b-a)} \int_a^b \left| f' (x) \right|^q dx \right).
\]

The inequality (1.11) also holds for \( \alpha = 1 \). Similarly, for \( \alpha \in (0, 1] \) by the inequality (1.12), we have

\[
\int_{1-a}^1 \left| f' (tb + (1-t)a) \right|^q dt = \alpha \left( \frac{1}{\alpha (b-a)} \int_{(1-a)b+aa}^b \left| f' (x) \right|^q dx \right).
\]

The inequality (1.12) also holds for \( \alpha = 0 \). By simple computation

\[
\int_0^{1-a} \left| t - \alpha \lambda \right|^p dt = \begin{cases} \frac{(a\lambda)^{p+1} + (1-\alpha - a\lambda)^{p+1}}{p+1}, & a\lambda \leq 1-\alpha \\
\frac{(a\lambda)^{p+1} - (a\lambda - 1 + a)^{p+1}}{p+1}, & a\lambda \geq 1-\alpha , \end{cases}
\]
and
\[
\int_{1-a}^{1} |t-1+\lambda(1-\alpha)|^p \, dt = \begin{cases} 
\frac{[\lambda(1-\alpha)]^{p+1}+|\alpha-\lambda(1-\alpha)|^{p+1}}{p+1}, & 1-\alpha \leq 1-\lambda(1-\alpha), \\
\frac{[\lambda(1-\alpha)]^{p+1}+|\lambda(1-\alpha)-\alpha|^{p+1}}{p+1}, & 1-\alpha \geq 1-\lambda(1-\alpha),
\end{cases}
\] (1.14)

thus, using (1.11)-(1.14) in (1.10), we obtain the inequality (1.8). This completes the proof.

\[\square\]

Corollary 1.5. In Theorem 1.3 if we take \( \alpha = \frac{1}{2} \) and \( \lambda = \frac{1}{3} \), then we have the following Simpson type inequality
\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{b-a}{12} \left( \frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}}
\]

\[
\times \left\{ \left( \left| f' \left( \frac{a+b}{2} \right) \right|^q + \left| f' (a) \right|^q \right)^{\frac{1}{q}} + \left( \left| f' \left( \frac{a+b}{2} \right) \right|^q + \left| f' (b) \right|^q \right)^{\frac{1}{q}} \right\}.
\]

Corollary 1.6. In Theorem 1.3 if we take \( \alpha = \frac{1}{2} \) and \( \lambda = 0 \), then we have the following midpoint inequality
\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}}
\]

\[
\times \left\{ \left( \left| f' \left( \frac{a+b}{2} \right) \right|^q + \left| f' (a) \right|^q \right)^{\frac{1}{q}} + \left( \left| f' \left( \frac{a+b}{2} \right) \right|^q + \left| f' (b) \right|^q \right)^{\frac{1}{q}} \right\}.
\]

We note that by inequality
\[
\left| f' \left( \frac{a+b}{2} \right) \right|^q \leq \left| f' (a) \right|^q + \left| f' (b) \right|^q
\]

we have
\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left( \left| f' (b) \right|^q + 2 \left| f' (a) \right|^q \right)^{\frac{1}{q}} + \left( \left| f' (a) \right|^q + 2 \left| f' (b) \right|^q \right)^{\frac{1}{q}} \right\}.
\]

Corollary 1.7. In Theorem 1.3 if we take \( \alpha = \frac{1}{2} \) and \( \lambda = 1 \), then we have the following trapezoid inequality
\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}}
\]

\[
\times \left\{ \left( \left| f' \left( \frac{a+b}{2} \right) \right|^q + \left| f' (a) \right|^q \right)^{\frac{1}{q}} + \left( \left| f' \left( \frac{a+b}{2} \right) \right|^q + \left| f' (b) \right|^q \right)^{\frac{1}{q}} \right\}.
\]

2 Some applications for special means

We now recall the following well-known concepts. For arbitrary real numbers \( a, b, a \neq b \), we define

1. The unweighted arithmetic mean
\[
A(a,b) := \frac{a+b}{2}, \ a, b \in \mathbb{R}.
\]

2. Then \( n \)-Logarithmic mean
\[
L_n(a, b) := \left( \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right)^{\frac{1}{n}}, \ n \in \mathbb{N}, \ n \geq 1, \ a, b \in \mathbb{R}, \ a < b.
\]
Now we give some applications of the new results derived in section 2 to special means of real numbers.

**Proposition 2.1.** Let \( a, b \in \mathbb{R} \) with \( a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then
\[
\left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A(a^n, b) - L_n(a, b) \right| \leq \frac{5n}{36} \frac{(b-a)}{(n-1)q} \left( |b|^{(n-1)q} + |a|^{(n-1)q} \right)^{\frac{1}{q}}
\]

**Proof.** The assertion follows from Corollary 1.2 applied to the function \( f(x) = x^n, x \in \mathbb{R} \), because \( |f'|^q \) is a \( P \)-function.

**Proposition 2.2.** Let \( a, b \in \mathbb{R} \) with \( a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then
\[
|A^n(a, b) - L_n^a(a, b)| \leq \frac{n}{4} \frac{(b-a)}{(n-1)q} \left( |b|^{(n-1)q} + |a|^{(n-1)q} \right)^{\frac{1}{q}}
\]
and
\[
|A(a^n, b^n) - L_n^a(a, b)| \leq \frac{n}{4} \frac{(b-a)}{(n-1)q} \left( |b|^{(n-1)q} + |a|^{(n-1)q} \right)^{\frac{1}{q}}
\]

**Proof.** The assertion follows from Corollary 1.3 and Corollary 1.4 applied to the function \( f(x) = x^n, x \in \mathbb{R} \), because \( |f'|^q \) is a \( P \)-function.

**Proposition 2.3.** Let \( a, b \in \mathbb{R} \) with \( a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then
\[
\left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A(a^n, b) - L_n^a(a, b) \right| \leq \frac{n}{12} \frac{(b-a)}{(p+1)} \left( \frac{1}{p+1} \right)^{\frac{1}{q}}
\]
\[
\times \left\{ \left( |A(a, b)|^{(n-1)q} + |a|^{(n-1)q} \right)^{\frac{1}{q}} + \left( |A(a, b)|^{(n-1)q} + |b|^{(n-1)q} \right)^{\frac{1}{q}} \right\}
\]

**Proof.** The assertion follows from Corollary 1.5 applied to the function \( f(x) = x^n, x \in \mathbb{R} \), because \( |f'|^q \) is a \( P \)-function.

**Proposition 2.4.** Let \( a, b \in \mathbb{R} \) with \( a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then
\[
|A^n(a, b) - L_n^a(a, b)| \leq \frac{n}{4} \frac{(b-a)}{(p+1)} \left( \frac{1}{p+1} \right)^{\frac{1}{q}}
\]
\[
\times \left\{ \left( |A(a, b)|^{(n-1)q} + |a|^{(n-1)q} \right)^{\frac{1}{q}} + \left( |A(a, b)|^{(n-1)q} + |b|^{(n-1)q} \right)^{\frac{1}{q}} \right\}
\]
and
\[
|A(a^n, b^n) - L_n^a(a, b)| \leq \frac{b-a}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{q}}
\]
\[
\times \left\{ \left( |A(a, b)|^{(n-1)q} + |a|^{(n-1)q} \right)^{\frac{1}{q}} + \left( |A(a, b)|^{(n-1)q} + |b|^{(n-1)q} \right)^{\frac{1}{q}} \right\}
\]

**Proof.** The assertion follows from Corollary 1.6 and Corollary 1.7 applied to the function \( f(x) = x^n, x \in \mathbb{R} \), because \( |f'|^q \) is a \( P \)-function.

**References**


Received: April 25, 2014; Accepted: Julu 31, 2014

UNIVERSITY PRESS
Website: http://www.malayajournal.org/