On semi-invariant submanifolds of a nearly trans-hyperbolic Sasakian manifold

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Abstract

Semi-invariant submanifold of a trans Sasakian manifold has been studied. In the present paper we study semi invariant submanifolds of a nearly trans hyperbolic Sasakian manifold. Nejenhuis tensor in a nearly trans hyperbolic Sasakian manifold is calculated. Integrability conditions for some distributions on a semi invariant submanifold of a nearly trans hyperbolic Sasakian manifold are investigated.

Keywords: Semi-invariant submanifolds, nearly trans hyperbolic Sasakian manifold, Gauss and Weingarten equations, integrability conditions, distributions.

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1 Introduction

The study of geometry of semi invariant submanifold of a Sasakian manifold has been studied by Bejancu [1] and Bejancu and Papaghiuc [4]. After that a number of authors have studied these submanifolds ([3],[5],[12]). Latter on, Oubina [8] introduced a new class of almost contact Riemannian manifold known as trans Sasakian manifold. Upadhyay and Dube [13] have studied almost contact hyperbolic $(f,g,\eta,\xi)$-structure. Shahid studied on semi invariant submanifolds of a nearly Sasakian manifold [14]. Matsumoto, Shahid, and Mihai [10] have also worked on semi invariant submanifolds of certain almost contact manifolds. Joshi and Dube [15] studied on Semi-invariant submanifold of an almost $r$-contact hyperbolic metric manifold. Gill and Dube have worked on CR submanifolds of trans-hyperbolic Sasakian manifolds [7].

2 Preliminaries

Nearly trans hyperbolic Sasakian Manifolds: Let $\bar{M}$ be an $n$ dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure $(\phi,\xi,\eta, g)$ where a tensor $\phi$ of type (1, 1), a vector field $\xi$, called structure vector field and $\eta$, the dual 1-form of is a 1-form $\xi$ satisfying the following

$$\phi^2X = X - \eta(X)\xi, \quad g(X,\xi) = \eta(X),$$

$$\phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = -1$$

$$g(\phi X,\phi Y) = -g(X,Y) - \eta(X)\eta(Y)$$

for any $X,Y$ tangents to $\bar{M}$ [6]. In this case

$$g(\phi X,Y) = -g(X,\phi Y)$$

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An almost hyperbolic contact metric structure $(\phi, \xi, \eta, g)$ on $\tilde{M}$ is called trans-hyperbolic Sasakian [7] if and only if
\[
(\bar{\nabla}_X \phi) Y = a[g(X, Y)\xi - \eta(Y)\phi X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X]
\]
for all $X, Y$ tangents to $\tilde{M}$ and $a, \beta$ are functions on $\tilde{M}$. On a trans-hyperbolic Sasakian manifold $M$, we have
\[
\bar{\nabla}_X \xi = -a(\phi X) + \beta[X - \eta(X)\xi]
\]
a Riemannian metric $g$ and Riemannian connection $\bar{\nabla}$. Further, an almost contact metric manifold $\tilde{M}$ on $(\phi, \xi, \eta, g)$ is called nearly trans-hyperbolic Sasakian if [9]
\[
(\nabla_X \phi) Y + (\nabla_Y \phi) X = a[2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y] - \beta[\eta(X)\phi Y + \eta(Y)\phi X]
\]
Semi-invariant submanifolds: Let $M$ be a submanifold of a Riemannian manifold $\tilde{M}$ endowed with a Riemannian metric $g$. Then Gauss and Wiengarten formulae are given respectively by
\[
\begin{align*}
\bar{\nabla}_X Y &= \nabla_X Y + h(X, Y) (X, YeTM) \\
\bar{\nabla}_X N &= -A_N X + \nabla^x_X N (NeT^+ M)
\end{align*}
\]
where $\nabla, \bar{\nabla}$ and $\nabla^\perp$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}, M$ and the normal bundle of $T^+ M$ of $M$ respectively, and $h$ is the second fundamental form related to $A$ by
\[
g(h(X, Y), N) = g(A_N X, Y)
\]
Moreover, if $\phi$ is a $(1, 1)$ tensor field on $\tilde{M}$, for $X \in TM$ and $N \in T^+ M$ we have
\[
\begin{align*}
(\bar{\nabla}_X \phi) Y &= ((\nabla_X P) Y - A_{FY} X - th(X, Y)) + ((\nabla_X F) Y + h(X, PY) - fh(X, Y)) \\
(\bar{\nabla}_X \phi) N &= ((\nabla_X t) Y - A_{fN} X - PA_N X) + ((\nabla_X f) N + h(X, tN) - FA_N X))
\end{align*}
\]
where
\[
\begin{align*}
\phi X &\equiv PX + FX (PXeTM, FXeT^+ M) \\
\phi N &\equiv tN + fN (tNeTM, fNeT^+ M) \\
(\nabla_X P) Y &\equiv \nabla_X PY - P\nabla_X Y, (\nabla_X F) Y \equiv \nabla^\perp_X FY - F\nabla_X Y \\
(\nabla_X t) N &\equiv \nabla_X tN - t\nabla^\perp_X N, (\nabla_X f) N \equiv \nabla^\perp_X fN - f\nabla^\perp_X N
\end{align*}
\]
The submanifold $M$ is known to be totally geodesic in $\tilde{M}$ if $h = 0$, minimal in $\tilde{M}$ if $H = \text{trace}(h) / \dim(M) = 0$, and totally umbilical in $\tilde{M}$ if $h(X, Y) = g(X, Y)H$.

For a distribution $D$ on $M$, $M$ is said to be $D$-totally geodesic if for all $X, YeD$ we have $h(X, Y) = 0$. If for all $X, YeD$ we have $h(X, Y) = g(X, Y)K$ for some normal vector $K$, then $M$ is called $D$-totally umbilical. For two distributions $D$ and $\epsilon$ defined on $M$, $M$ is said to be $(D, \epsilon)$-mixed totally geodesic if for all $X eD$ and $Ye\epsilon$ we have $h(X, Y) = 0$.

Let $D$ and $\epsilon$ be two distributions defined on a manifold $M$. We say that $D$ is $\epsilon$-parallel if for all $X e\epsilon$ and $YeD$ we have $\nabla_X YeD$. If $D$ is $D$-parallel then it is called autoparallel. $D$ is called $X$-parallel for some $X eTM$ if for all $YeD$ we have $\nabla_X YeD$. $D$ is said to be parallel if for all $X eTM$ and $YeD, \nabla_X YeD$.

If a distribution $D$ on $M$ is autoparallel, then it is clearly integrable, and by Gauss formula $D$ is totally geodesic in $M$. If $D$ is parallel then the orthogonal complementary distribution $D^\perp$ is also parallel, which implies that $D$ is parallel if and only if $D^\perp$ is parallel. In this case $M$ is locally the product of the leaves of $D$ and $D^\perp$.

Let $M$ be a submanifold of an almost contact metric manifold. If $\xi eTM$ then we write $TM = \{\xi\} \oplus \{\xi\}^\perp$, where $\{\xi\}$ is the distribution spanned by $\xi$ and $\{\xi\}^\perp$ is the complementary orthogonal distribution of $\{\xi\}$ in $M$. Then one gets
\[
\begin{align*}
P F = 0 &= 0, \quad \eta o P = 0 = \eta o F, \\
P^2 + tF &= -I + \eta \otimes \xi, \quad FP + FF = 0, \\
f^2 + tF &= -I, \quad tf + Pt = 0
\end{align*}
\]
A submanifold $M$ of an almost contact metric manifold $\tilde{M}$ with $\xi\epsilon TM$ is called a semi-invariant submanifold (Bejancu, [1]) of $\tilde{M}$ if there exists two differentiable distributions $D^1$ and $D^0$ on $M$ such that

1. $TM = D^1 \oplus D^0 \oplus \{\xi\}$,
2. the distribution $D^1$ is invariant by $\phi$, that is, $\phi(D^1) = D^1$ and
3. the distribution $D^0$ is anti-invariant by $\phi$, that is, $\phi(D^0) \subseteq T^\perp M$.

For $X \epsilon TM$ we can write

$$X = D^1 X + D^0 X + \eta(X)\xi$$

(2.18)

where $D^1$ and $D^0$ are the projection operators of $TM$ on $D^1$ and $D^0$, respectively. A semi-invariant submanifold of an almost contact metric manifold becomes an invariant submanifold ([2], [11]) (resp. anti-invariant submanifold ([2], [11]) if $D^0 = \{0\}$ (resp. $D^1 = \{0\}$).

### 3 The Nijenhuis tensor

A hyperbolic contact metric manifold is said to be normal ([6]) if the torsion tensor $N^1$ vanishes:

$$N^1 \equiv [\phi, \phi] + d\eta \otimes \xi = 0$$

(3.19)

where $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$ and $d$ denotes the exterior derivatives operator. In this section we obtain expression for Nijenhuis tensor $[\phi, \phi]$ of the structure tensor field $\phi$ given by

$$[\phi, \phi](X, Y) = ((\tilde{\nabla}_\phi X)Y - (\tilde{\nabla}_\phi Y)X) - \phi((\tilde{\nabla}_X \phi)Y - (\tilde{\nabla}_Y \phi)X)$$

(3.20)

in a nearly trans hyperbolic Sasakian manifold. First, we need the following lemma.

**Lemma 3.1.** In an almost hyperbolic contact metric manifold we have

$$(\tilde{\nabla}_Y \phi)X = -\phi(\tilde{\nabla}_Y \phi)X - ((\tilde{\nabla}_Y \eta)X)\xi - \eta(X)\tilde{\nabla}_Y \xi$$

(3.21)

**Proof.** For $X, Y \epsilon T\tilde{M}$, we have

$$(\nabla_Y \phi)X = -\phi^2 \nabla_Y X - \phi(\nabla_Y \phi)X + \nabla_Y X - ((\nabla_Y \eta)X)\xi - \eta(\nabla_Y X)\xi - \eta(X)\nabla_Y \xi$$

$$= -\nabla_Y X + \eta(\nabla_Y X)\xi - \phi(\nabla_Y \phi)X + \nabla_Y X - ((\nabla_Y \eta)X)\xi - \eta(\nabla_Y X)\xi - \eta(X)\nabla_Y \xi$$

which gives the equation (3.21). □

Now, we prove the following theorem.

**Theorem 3.1.** In a nearly trans-hyperbolic Sasakian manifold the Nijenhuis tensor $[\phi, \phi]$ of $\phi$ is given by

$$[\phi, \phi](X, Y) = 4\phi(\tilde{\nabla}_Y \phi)X + 2d\eta(X, Y)\xi + \eta(X)\tilde{\nabla}_Y \xi - \eta(Y)\tilde{\nabla}_X \xi + 4\alpha\gamma(\phi X, Y)\xi + (\alpha + \beta)\eta(Y)\phi^2 X + 3(\alpha + \beta)\eta(X)\phi^2 Y$$

(3.22)

**Proof.** Using Lemma 3.1 and $\eta \phi = 0$ in (2.7) we get

$$(\tilde{\nabla}_\phi X)Y = \phi(\tilde{\nabla}_Y \phi)X + ((\tilde{\nabla}_Y \eta)X)\xi + \eta(X)\tilde{\nabla}_Y \xi + 2\alpha\gamma(\phi X, Y)\xi - (\alpha + \beta)\eta(Y)\phi^2 X$$

(3.23)
Thus

\[
[\phi, \phi](X, Y) = ((\nabla_\phi X)\phi Y - (\nabla_\phi Y)\phi X) - \phi((\nabla_\phi X)Y - (\nabla_\phi Y)X)
\]

\[
= 2\phi(\nabla_\phi Y)X - 2\phi(\nabla_\phi Y)Y + [(\nabla_\phi Y)\phi Y - ((\nabla_\phi Y)\phi X)] + \eta(X)\nabla_\phi \xi
\]

\[
- \eta(\nabla_\phi X)\phi Y + 4\alpha g(\phi X, Y)\phi Y - (\alpha + \beta)[\eta(Y)\phi X - \eta(X)\phi Y]
\]

\[
= 4\phi(\nabla_\phi Y)X + 2\alpha \eta(\phi Y)\phi X + 2\alpha \eta(\phi X)\phi Y - \beta[\eta(Y)\phi X + \eta(X)\phi Y]
\]

\[
+ 2d\eta(X, Y)\xi + \eta(X)\nabla_\phi \xi - \eta(\nabla_\phi Y)\phi X
\]

\[
+ 4\alpha g(\phi X, Y)\phi Y - (\alpha + \beta)[\eta(Y)\phi X - \eta(X)\phi Y]
\]

\[
\]

which implies the equation (3.22). From Equation (3.22), we get

\[
\eta(N^1(X, Y)) = 3d\eta(X, Y) - 4\alpha g(\phi X, \phi Y)
\] (3.24)

In particular, if X and Y are perpendicular to \(\xi\), then (3.22) gives

\[
[\phi, \phi](X, Y) = 4\phi(\nabla_\phi Y)X - 2(\eta[X, Y])\xi
\] (3.25)

\[
\square
\]

4 Some basic results

Let \(M\) be a submanifold of a nearly trans-hyperbolic Sasakian manifold. Using (2.11), (2.13) in (2.7) for \(X, Y\in TM\), we get

\[
(\nabla_X P)Y + (\nabla_Y P)X - A_{FY}X - A_{FX}Y - 2th(X, Y) + (\nabla_X F)Y
\]

\[
+ (\nabla_Y F)X + h(X, PY) + h(Y, PX) - 2fh(X, Y)
\]

\[
= \alpha[2g(X, Y)\xi - \eta(Y)PX - \eta(Y)FX - \eta(X)PY - \eta(X)FY]
\]

\[
- \beta[\eta(X)PY + \eta(X)FY + \eta(Y)PX + \eta(Y)FX]
\]

Consequently, we have

Proposition 4.1. Let \(M\) be a submanifold of a nearly trans-hyperbolic Sasakian manifold. Then for all \(X, Y\in TM\) we have

\[
(\nabla_X P)Y + (\nabla_Y P)X - A_{FY}X - A_{FX}Y - 2th(X, Y)
\]

\[
= 2\alpha g(X, Y)\xi - (\alpha + \beta)(\eta(Y)PX + \eta(X)PY)
\] (4.27)

and

\[
(\nabla_X F)Y + (\nabla_Y F)X + h(X, PY) + h(Y, PX) - 2fh(X, Y)
\]

\[
= -(\alpha + \beta)[\eta(X)FY + \eta(Y)FX]
\] (4.28)

for all \(X, Y\in TM\).
Now we state the following proposition.

**Proposition 4.2.** Let $M$ be a submanifold of a nearly trans-hyperbolic Sasakian manifold. Then

$$
\nabla_X\phi Y + \nabla_Y\phi X - \phi[X, Y] = 2((\nabla_X P)Y - A_{FY}X - th(X, Y))
$$

(4.29)

$$
+ 2((\nabla_X F)Y + h(X, PY) - fh(X, Y)) + 2\sigma(X, Y)\xi
$$

$$
- (\alpha + \beta)(\eta(Y)PX + \eta(X)PY) - (\alpha + \beta)(\eta(Y)FX + \eta(X)FY)
$$

Consequently,

$$
P[X, Y] = A_{FY}X + A_{FX}Y + 2th(X, Y) - 2\sigma(X, Y)\xi
$$

(4.30)

$$
- (\alpha + \beta)(\eta(Y)PX + \eta(X)PY - \nabla_X PY - \nabla_Y PX + 2P\nabla_X Y)
$$

$$
F[X, Y] = -\nabla_\xi FY - \nabla_Y FX - h(X, PY) - h(Y, PY) + 2fh(X, Y)
$$

(4.31)

$$
- (\alpha + \beta)(\eta(Y)FX + \eta(X)FY) + 2F\nabla_X Y
$$

The proof is straightforward and hence omitted.

**Proposition 4.3.** Let $M$ be a semi invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. Then $(P, \xi, \eta, g)$ is a nearly trans-hyperbolic Sasakian structure on the distribution $D^1 \oplus \{\xi\}$ if $th(X, Y) = 0$ for all $X, YeD^1 \oplus \{\xi\}$.

Proof. From $D^1 \oplus \{\xi\} = ker(F)$ and (2.16) we have $P^2 = I - \eta \otimes \xi$ on $D^1 \oplus \{\xi\}$. We also get $P\xi = 0, \eta(\xi) = 2$ and $P = 0$. Using $D^1 \oplus \{\xi\} = ker(F)$ and $th(X, Y) = 0$ in 4.27 we get

$$
(\nabla_X P)Y + (\nabla_Y P)X = 2\sigma(X, Y)\xi - (\alpha + \beta)(\eta(Y)PX + \eta(X)PY),
$$

(4.32)

for all $X, YeD^1 \oplus \{\xi\}$.

This completes the proof.

**Theorem 4.2.** Let $M$ be a semi invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. We have (i) if $D^0 \oplus \{\xi\}$ is autoparallel then

$$
A_{FX}Y + A_{FY}X + 2th(X, Y) = 0, \forall X, YeD^0 \oplus \{\xi\}
$$

(4.33)

(ii) if $D^1 \oplus \{\xi\}$ is autoparallel then

$$
h(X, PY) + h(PX, Y) = 2fh(X, Y) \forall X, YeD^1 \oplus \{\xi\}.
$$

(4.34)

Proof. In view of (4.27) and autoparallelness of $D^0 \oplus \{\xi\}$ we get (1), while in view of (4.28) and appropriateness of $D^1 \oplus \{\xi\}$ we get (ii). In view of Proposition 4.3 and Theorem 4.2(ii), we get

**Theorem 4.3.** Let $M$ be a submanifold of a nearly trans-hyperbolic Sasakian manifold with $\xi e TM$. If $M$ is invariant then $M$ is nearly trans-hyperbolic Sasakian. Moreover

$$
h(X, PY) + h(PX, Y) - 2fh(X, Y) = 0, \forall X, YeTM.
$$

5 Integrability Conditions

Integrability of the distribution $D^1 \oplus \{\xi\}$: We begin with a lemma

**Lemma 5.2.** Let $M$ be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. For $X, YeD^1 \oplus \{\xi\}$ we get

$$
F[X, Y] = -h(X, PY) - h(PX, Y) + 2F\nabla_X Y + 2fh(X, Y)
$$

(5.35)

or equivalently

$$
- h(X, PX) + F\nabla_X X + fh(X, X) = 0
$$

(5.36)
Proof. Equation (5.1) follows from \( D^1 \oplus \{ \xi \} = \ker(F) \) and (4.6). Equivalence of (5.1) and (5.2) is obvious. In view of (5.1) and \( D^1 \oplus \{ \xi \} = \ker(F) \) we can state the following theorem.

**Theorem 5.4.** The distribution \( D^1 \oplus \{ \xi \} \) on a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold is integrable if and only if

\[
h(X, PY) + h(PX, Y) = 2(F \nabla_{X} Y + fh(X, Y))
\]  

(5.37)

Now, we need the following

**Definition 5.1.** (16) Let \( M \) be a Riemannian manifold with the Riemannian connection \( \nabla \). A distribution \( D \) on \( M \) will be called nearly autoparallel if for all \( X, Y \in D \) we have \( (\nabla_X Y + \nabla_Y X) \in D \) or equivalently \( \nabla_X Y \in D \).

Thus, we have the following flow chart ((16)):

- Parallel \( \Rightarrow \) Autoparallel \( \Rightarrow \) Nearly autoparallel,
- Parallel \( \Rightarrow \) Integrable,
- Autoparallel \( \Rightarrow \) Integrable, and
- Nearly autoparallel + Integrable \( \Rightarrow \) Autoparallel.

**Theorem 5.5.** Let \( M \) be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. Then the following four statements

(a) the distribution \( D^1 \oplus \{ \xi \} \) is autoparallel,
(b) \( h(X, PY) + h(PX, Y) = 2fh(X, Y) \), \( X, Y \in D^1 \oplus \{ \xi \} \),
(c) \( h(X, PX) = fh(X, X) \), \( X \in D \oplus \{ \xi \} \),
(d) the distribution \( D^1 \oplus \{ \xi \} \) is nearly autoparallel,

are related by \( (a) \Rightarrow (b) \Leftrightarrow (c) \Rightarrow (d) \). In particular, if \( D^1 \oplus \{ \xi \} \) is integrable then the above four statements are equivalent.

The proof is similar to that Theorem 4.4 of [16].

Let \( X, Y \in D^1 \oplus \{ \xi \} \). Using (2.1) and (2.13) in (3.19) and we get

\[
N^{(1)}(X, Y) = [\phi X, \phi Y] - P[\phi X, Y] - F[\phi X, Y] - P[X, \phi Y] - F[X, \phi Y] + [X, Y] + \eta([X, Y])\xi + 2d\eta \otimes \xi
\]  

(5.38)

On the other hand from equation (3.23) we have

\[
(\tilde{\nabla}_\phi X)Y = \phi(\tilde{\nabla}_Y X) + ([\tilde{\nabla}_Y \eta] X)\xi + \eta(X) \tilde{\nabla}_Y \xi + 2\alpha\tilde{\nabla}X Y = (\alpha + \beta)\eta(X) \phi^2 X
\]

which implies that

\[
(\nabla_{\phi X} Y - (\nabla_{Y \phi} X) = \phi((\nabla_{Y \phi} X) - (\nabla_{\phi X} Y) + 2d\eta(X, Y)\xi + \eta(X) U^1 \nabla_Y \xi
\]

(5.39)

Next we easily can get

\[
\phi((\nabla_{Y \phi} X - (\nabla_{\phi X} Y) = (\nabla_{Y \phi} X - \nabla_{\phi X} Y) + [X, Y] - \eta([X, Y])\xi
\]

(5.41)

so that

\[
\phi((\nabla_{Y \phi} X - (\nabla_{\phi X} Y) = (\nabla_{Y \phi} X - \nabla_{\phi X} Y) + [X, Y] - \eta([X, Y])\xi + F(\nabla_{Y \phi} X - \nabla_{\phi X} Y) + \phi(h(Y, \phi X) - h(X, \phi Y))
\]

(5.42)

In view of (5.39) and (5.41) we get

\[
N^{(1)}(X, Y) = 4d\eta \otimes \xi + 2[X, Y] - 2\eta([X, Y])\xi + 2P(\nabla_{Y \phi} X - \nabla_{\phi X} Y)
\]

(5.42)
Theorem 5.6. The distribution \( D^1 \oplus \{ \xi \} \) is integrable on a semi-invariant submanifold \( M \) of a nearly trans-hyperbolic Sasakian manifold if and only if for all \( X, Y \in D^1 \oplus \{ \xi \} \)

\[
N^1(X, Y) e^1 \oplus (\xi) = 2(h(Y, \phi X) - h(X, \phi Y)) - \eta(X)(\phi U^0 \nabla Y \xi + f h(Y, \xi)) + \eta(Y)(\phi U^0 \nabla X \xi + f h(X, \xi))
\]

(5.43)

\[
0 = 2F(\nabla Y \phi X - \nabla X \phi Y) + 2\phi(h(Y, \phi X) - h(X, \phi Y) + \eta(X)U^0 \nabla Y \xi + \eta(Y)U^0 \nabla X \xi - \eta(Y)h(X, \xi)
\]

Proof. Let \( X, Y e^1 \oplus \{ \xi \} \). If \( D^1 \oplus \{ \xi \} \) is integrable, then (5.43) is true and from (5.42) we get

\[
0 = 2U^0[X, Y] + 2F(\nabla Y \phi X - \nabla X \phi Y) + 2\phi(h(Y, \phi X) - h(X, \phi Y) + \eta(X)U^0 \nabla Y \xi + \eta(Y)U^0 \nabla X \xi - \eta(Y)h(X, \xi)
\]

Hence taking the normal part we get (5.44).

Conversely, let (5.43) and (5.44) be true. Using (5.44) in (5.42) we get

\[
0 = -2U^0(\nabla Y \phi X - \nabla X \phi Y) + 2\phi(h(Y, \phi X) - h(X, \phi Y) + \eta(X)U^0 \nabla Y \xi + \eta(Y)U^0 \nabla X \xi - \eta(Y)h(X, \xi)
\]

Applying \( \phi \) to the above equation, we get

\[
0 = -2U^0(\nabla Y \phi X - \nabla X \phi Y) + 2\phi(h(Y, \phi X) - h(X, \phi Y) + \eta(X)U^0 \nabla Y \xi + \eta(Y)U^0 \nabla X \xi - \eta(Y)h(X, \xi)
\]

Integrability of the distribution \( D^0 \oplus \{ \xi \} \):

Corollary 5.1. . If \( M \) is a semi-invariant submanifold of a trans-hyperbolic Sasakian manifold, then the distribution \( D^1 \oplus \{ \xi \} \) is integrable if and only if for all \( X, Y e^1 \oplus \{ \xi \} \) it is known that \( h(X, \xi) = 0 \) and \( U^0 \nabla X \xi = 0 \). Hence in view of the previous theorem we have

\[
\eta(X, \phi Y) = h(Y, \phi X)
\]

Lemma 5.3. Let \( M \) be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. Then

\[
3(\phi A_{XY} - A_{YX}) = P[X, Y], \quad X, Y e^0 \oplus (\xi)
\]

(5.45)

Proof. Let \( X, Y e^0 \oplus (\xi) \) and \( Z e^TM \). We have

\[
-\phi A_{XY} + \nabla^2_X \phi X = \nabla^2_Z \phi X = (\nabla_Z \phi)X + \phi(\nabla_Z X)
\]

so that

\[
\phi h(Z, X) = -\phi A_{XY} + \nabla^2_X \phi X + (\nabla_X \phi)Z + \phi(\nabla_X Z) + \phi A_{XY} - \phi \nabla_X Z
\]

and hence we have

\[
g(\phi h(Z, X), Y) = -g(\phi A_{XY} Y, Z) - g((\nabla_X \phi)Y, Z)
\]

On the other hand

\[
g(\phi h(Z, X), Y) = -g(h(Z, X), \phi Y) = -g(\phi A_{XY}, Z)
\]

Thus from the above two relations we get

\[
g(\phi A_{XY}, Z) = g(\phi A_{XY} Y, Z) + g((\nabla_X \phi)Y, Z)
\]

(5.46)
For \( X, Y \in D^0 \oplus \{ \xi \} \) we calculate \((\nabla_X \phi)Y\) as follows. In view of
\[
\nabla_X \phi Y - \nabla_Y \phi X = A_{\phi X} Y - A_{\phi Y} X + \nabla^{\perp}_{\nabla X} \phi Y - \nabla^{\perp}_{\nabla Y} \phi X
\]
and
\[
\nabla_X \phi Y - \nabla_Y \phi X = (\nabla_X \phi) Y - (\nabla_Y \phi) X + \phi[X, Y]
\]
we have
\[
(\nabla_X \phi) Y - (\nabla_Y \phi) X = A_{\phi X} Y - A_{\phi Y} X + \nabla^{\perp}_{\nabla X} \phi Y - \nabla^{\perp}_{\nabla Y} \phi X - \phi[X, Y]
\]
which gives
\[
(\nabla_X \phi) Y = 1/2(A_{\phi X} Y - A_{\phi Y} X + \nabla^{\perp}_{\nabla X} \phi Y - \nabla^{\perp}_{\nabla Y} \phi X - \phi[X, Y]
\]
\[\phi[\bar{\phi} X Y] = -[X, Y] + \eta([X, Y]) \xi + \phi(A_{\phi X} Y - A_{\phi Y} X) + \phi(\nabla^{\perp}_{\nabla X} \phi Y - \nabla^{\perp}_{\nabla Y} \phi X) \]  
(5.47)
\[
(\bar{\phi} X Y) Y - (\bar{\phi} Y X) X = [X, Y] - \phi(A_{\phi X} Y - A_{\phi Y} X) - \phi(\nabla^{\perp}_{\nabla X} \phi Y - \nabla^{\perp}_{\nabla Y} \phi X) \]  
(5.48)
Using (5.13), (5.14) and (5.11) we get for \( Y, X \in D^0 \)
\[
N^1(Y, X) = -2[X, Y] + 2/3 \phi P[X, Y] + 2\phi(\nabla^{\perp}_{\nabla X} \phi Y - \nabla^{\perp}_{\nabla Y} \phi X) \]  
(5.49)
\section{Theorem 5.7.}
Let \( M \) be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. Then the distribution \( D^0 \oplus \{ \xi \} \) is integrable if and only if
\[
A_{FX} Y = A_{FY} X \quad \text{for all } X, Y \in D^0 \oplus \{ \xi \}
\]
Integrability of the distribution \( D^0 \): We calculate the torsion tensor \( N^1(Y, X) \) for \( Y, X \in D^0 \). It can be verified that
\[
\phi((\nabla_X \phi) Y - (\nabla_Y \phi) X) = -[X, Y] + \eta([X, Y]) \xi + \phi(A_{\phi X} Y - A_{\phi Y} X) + \phi(\nabla^{\perp}_{\nabla X} \phi Y - \nabla^{\perp}_{\nabla Y} \phi X)
\]
(5.47)
\[
(\nabla_{\phi X} Y - (\nabla_{\phi Y} X) X = [X, Y] - \phi(A_{\phi X} Y - A_{\phi Y} X) - \phi(\nabla^{\perp}_{\nabla X} \phi Y - \nabla^{\perp}_{\nabla Y} \phi X)
\]
Using (5.13), (5.14) and (5.11) we get for \( Y, X \in D^0 \)
\[
N^1(Y, X) = -2[X, Y] + 2/3 \phi P[X, Y] + 2\phi(\nabla^{\perp}_{\nabla X} \phi Y - \nabla^{\perp}_{\nabla Y} \phi X) \]  
(5.49)
\section{Theorem 5.8.}
The distribution \( D^0 \) is integrable on a semi-invariant submanifold \( M \) of a nearly trans-hyperbolic Sasakian manifold if and only if
\[
N^{11}(Y, X) \in D^0 \oplus D^1 \quad X, Y \in D^0
\]
\[
A_{FX} Y = A_{FY} X 
\quad X, Y \in D^0 
\]
(5.50)
(5.51)
\section{Proof.}
If \( D^0 \) is integrable, then in view of (5.48) and (5.49), the relation (5.50) and (5.51) follow easily. Conversely, let \( X, Y \in D^0 \) and let the relation (5.50) and (5.51) be true. Then in view (5.48), we get \( P[X, Y] = 0 \) and in view of (5.49), we get
\[
0 = g(\xi, N^1(Y, X)) = g(\xi, 2[Y, X]).
\]
Thus \( [X, Y] \in D^0 \).
Non-integrability of the distribution \( D^1 \):
\section{Theorem 5.9.}
Let \( M \) be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold with \( \alpha \neq 0 \). Then the non-zero invariant distribution \( D^1 \) is not integrable.
\section{Proof.}
If \( D^1 \) is integrable then for \( X, Y \in D^1 \) it follows that \( d\eta(X, Y) = 0 \) and \( [\phi, \phi](X, Y) \in D^1 \). Therefore, for \( X \in D^1 \) in view of (3.24), we get
\[
0 = \eta([\phi, \phi](X, PX) + 2d\eta(X, PX) \xi)
\]
\[
= \eta(N^1(X, PX) = 4\alpha g(\phi X, PX) = 4\alpha g(PX, PX),
\]
which is a contradiction.
References


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