On Quasi-weak Commutative Boolean-like Near-Rings

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Abstract

In this paper we establish a result that every quasi-weak commutative Boolean-like near-ring can be imbedded into a quasi-weak commutative Boolean-like commutative semi-ring with identity. Key words: Quasi-weak commutative near-ring, Boolean-like near-ring.

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1 Introduction

The concept of Boolean-like ring was coined by A.L.Foster\cite{1}. Foster proved that if $R$ is a Boolean ring with identity then $ab(1-a)(1-b) = 0$ for all $a,b \in R$. He generalized the concept of Boolean ring as Boolean-like ring as a ring $R$ with identity satisfying (i) $ab(1-a)(1-b) = 0$ and (ii) $2a = 0$ for all $a,b \in R$. He also observed that the equation $ab(1-a)(1-b) = 0$ can be re-written as $(ab)^2 - ab^2 a^2 b + ab = 0$. He re-defined a Boolean-like ring as a commutative ring with identity satisfying (i) $(ab)^2 - ab^2 a^2 b + ab = 0$ and (ii) $2a = 0$ for all $a,b \in R$. In 1962 Adil Yaqub \cite{8} proved that the condition ’commutativity’ is not necessary in the definition of Boolean-like rings. He proved that any ring $R$ with the conditions (i) $(ab)^2 - ab^2 a^2 b + ab = 0$ and (ii) $2a = 0$ for all $a,b \in R$ is necessarily commutative.

Ketsela Hailu and others \cite{4} have constructed the Boolean-like semi-ring of fractions of a weak commutative Boolean-like semi-ring. We have coined and studied the concept of quasi-weak commutative near-ring in \cite{2}. In this paper we define Boolean-like near ring (right) and prove that every quasi-weak commutative Boolean-like near ring can be imbedded into a quasi weak commutative semi ring with identity.

2 Preliminaries

In this section we recall some definitions and results which we use in the sequel.

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2.1. Definition

A non-empty set \( R \) together with two binary operations + and \( \cdot \) satisfying the following axioms is called a right near-ring

(i) \((R,+)\) is a group

(ii) \( \cdot \) is associative

(iii) \( \cdot \) is right distributive w.r.to +

(iv) \((a+b) \cdot c = a \cdot c + b \cdot c \quad \forall \ a,b,c \in R\)

2.2. Note

In a right near-ring \( R \), \( 0 \ a = 0 \quad \forall \ a \in R \).

If \((R,+)\) is an abelian group, then \((R, +, \cdot)\) is called a semi-ring.

2.3. Definition

A right near-ring \((R, +, \cdot)\) is called a Boolean-like near ring if

(i) \(2a = 0 \quad \forall \ a \in R\) and

(ii) \((a+b-ab)ab = ab \quad \forall \ a, b \in R\)

2.4. Remark

If \((R, +, \cdot)\) is a Boolean-like near ring, then \((R, +)\) is always an abelian group for \(2x = 0 \quad \forall \ x \in R\) implies \(x = -x \quad \forall \ x \in R\). We know, a group in which every element is its own inverse is always commutative.

2.5. Definition [5]

A right near ring \( R \) is said to be weak commutative if \(xyz = xzy \quad \forall \ x,y,z \in R\)

2.6. Definition [8]

A right near ring \( R \) is said to be pseudo commutative if \(xyz = zyx \quad \forall \ x,y,z \in R\)

2.7. Definition [2]

A right near ring \( R \) is said to be quasi-weak commutative if \(xyz = yxz \quad \forall \ x,y,z \in R\)

2.8. Definition

Let \( R \) be a right near ring. A subset \( B \) of \( R \) is said to be multiplicatively closed if \( a,b \in B \) implies \( ab \in B \).

3. Main Results

3.1. Lemma

In a Boolean-like near ring (right) \( R \) \( a \cdot 0 = 0 \quad \forall \ a \in R \)
Proof:

Since $R$ is Boolean-like near ring, $(a+b-ab)ab = ab \forall a,b \in R$
Taking $a=0$, we get
$(0 + b - 0b) 0b = 0b$
(i.e) $b \cdot 0 = 0$
Thus $a \cdot 0 = 0 \forall a \in R$.

3.2. Lemma

Let $R$ be a quasi-weak commutative right near ring $R$. Then $(ab)^n = a^n b^n \forall a,b \in R$ and for all $n \geq 1$.

Proof:

Let $a,b \in R$.
Then $(ab)^2 = (ab)(ab) = a(bab)$
   $= a(abb)$ ( quasi weak)
   $(ab)^2 = a^2 b^2$
Assume $(ab)^m = a^m b^m$
Now $(ab)^{m+1} = (ab)^m ab$
   $= a^m b^m ab$
   $= a^m (ab)^m b$
   $= a^{m+1} b^{m+1}$
Thus $(ab)^m = a^m b^m \forall a,b \in R$ and for all integer $m \geq 1$.

3.3 Lemma

Let $R$ be a quasi-weak commutative Boolean like near-ring. Then

$a^2 b + ab^2 = ab + (ab)^2 \forall a,b \in R$.

Proof:

$a^2 b + ab^2 = aab + abb$
   $= aab + bab$
   $= (a + b)ab$
   $= (a + b) (ab + ab)ab$
   $= (a + b)ab + (ab)^2$
$a^2 b + ab^2 = ab + (ab)^2$ ( $R$ is Boolean-like near-ring )

3.4 Lemma

In a quasi-weak commutative Boolean like near ring $(R,+, \cdot)$,

$(a + a^2)(b + b^2)c = 0 \forall a,b,c \in R$.

Proof:

$(a + a^2)(b + b^2)c = (a(b + b^2) + a^2(b + b^2))c$
   $= a(b + b^2)c + a^2(b + b^2)c$
Let \( r \in R \) be a quasi commutative Boolean like near-ring. Let \( S \) be a commutative subset of \( R \) which is multiplicatively closed. Define a relation \( N \) on \( R \) by \( (r,s) \sim (r,s) \) if and only if there exists an element \( s \in S \) such that \( rs-rs \in \mathbb{R} \) and \( (r,s) \sim (r,s) \) if and only if there exists an element \( s \in S \) such that \( rs-rs \in \mathbb{R} \). Then there exists \( p,q \in \mathbb{R} \) such that \( p+q = 0 \). Hence \( N \) is reflexive.

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\[ \Rightarrow (r_1 s_2 - r_2 s_1 s_3 + r_2 s_1 s_3 - r_3 s_1 s_2) pq = 0. \text{(S is commutative)} \]
\[ \Rightarrow (r_1 s_3 - r_3 s_1) s_2 pq = 0 \]
\[ \Rightarrow (r_1 s_3 - r_3 s_1) r = 0 \text{ where } r = s_2 pq \in S. \]

This implies \((r_1, s_1) \sim (r_3, s_3)\).

Hence \(\sim\) is transitive.

Hence the Lemma.

### 3.6 Remark

We denote the equivalence class containing \((r, s) \in R \times S\) by \(\frac{r}{s}\) and the set of all equivalence classes by \(S^{-1}R\).

### 3.8 Lemma

Let \(R\) be a quasi weak commutative Boolean like near-ring. Let \(S\) be a commutative subset of \(R\) which is also multiplicatively closed. If \(0 \notin S\) and \(R\) has no zero divisors, then \((r_1, s_1) \sim (r_2, s_2)\) if and only if \(r_1 s_2 = r_2 s_1\).

**Proof:**

Assume \((r_1, s_1) \sim (r_2, s_2)\). Then there exists an element \(s \in S\) such that \((r_1 s_2 - r_2 s_1) s = 0\).

Since \(0 \notin S\) and \(R\) has zero divisors, we get \((r_1 s_2 - r_2 s_1) = 0\).

(i.e) \(r_1 s_2 = r_2 s_1\)

Conversely assume \(r_1 s_2 = r_2 s_1\).

Then \(r_1 s_2 - r_2 s_1 = 0\) and so \((r_1 s_2 - r_2 s_1) s = 0\) for all \(s \in S\).

Hence \((r_1, s_1) \notin (r_2, s_2)\).

### 3.9 Lemma:

Let \(R\) be a quasi weak commutative Boolean like near-ring. Let \(S\) be a commutative subset of \(R\), which is also multiplicatively closed.

Then

(i) \(\frac{r}{s} = \frac{r'}{s'} = \frac{r''}{s''}\) for all \(r, r' \in R\) and for all \(s, s', s'' \in S\).

(ii) \(\frac{r}{s} = \frac{r}{s'} \) for all \(r \in R\) and for all \(s, s' \in S\).

(iii) \(\frac{s}{s} = \frac{s}{s'} \) for all \(s, s' \in S\).

(iv) If \(0 \in S\), then \(S^{-1} R\) contains exactly one element.

**Proof:**

The proof of (i), (ii) and (iii) are routine.

(iv) Since \(0 \in S\), \((r_1 s_2 - r_2 s_1) 0 = 0\) \(\forall r_1, r_2, s_1, s_2 \in S^{-1}R\).

and so \(\frac{r_1}{s_1} = \frac{r_2}{s_2}\).

Then \(S^{-1} R\) contains exactly one-element.

### 3.10 Theorem:

Let \(R\) be a quasi weak commutative Boolean like near ring. Let \(S\) be a commutative subset of \(R\) which is also multiplicatively closed. Define binary operation + and on \(S^{-1} R\) as follows:

\[ \frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2} \quad \text{and} \]

\[ \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2} \quad \text{for all } r_1, r_2 \in R, s_1, s_2 \in S. \]
Then $S^{-1}R$ is a commutative Boolean like semi-ring with identity.

**Proof:**

Let us first prove that $+$ and $\cdot$ are well defined. Let $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r'_1}{s'_1}$ and $\frac{r_2}{s_2} = \frac{r'_2}{s'_2}$ Then there exists $t_1$, $t_2 \in S$ such that

$(r_1s'_1 - r'_1s_1)t_1 = 0 \ldots \ldots (1)$

and $(r_2s'_2 - r'_2s_2)t_2 = 0 \ldots \ldots (2)$

Now $\left[(r_1s_2 + r_2s_1)s'_2 - (r'_1s'_2 + r'_2s'_2)s_1s_2\right]t_1t_2$

$= \left[r_1s_2s'_2 + r_2s_1s'_2 - r'_1s'_2s_1s_2 + r'_2s'_2s_2s'_1\right]t_1t_2$

$= \left[(r_1s'_1 - r'_1s_1)s_2s'_2 + (r_2s'_2 - r'_2s_2)s_1s'_1\right]t_1t_2$

$= \left[(r_1s'_1 - r'_1s_1)t_1s_2s'_2 + (r_2s'_2 - r'_2s_2)t_2s_1s'_1\right]t_1t_2$

$= 0 \cdot s_2s'_2t_2 + 0 \cdot s_1s'_1t_1$

$= 0$

**Hence** $\frac{r_1s_2 + r_2s_1}{s_1s_2} = \frac{r'_1s'_2 + r'_2s'_2}{s'_1s'_2}$

(i.e) $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r'_1}{s'_1} + \frac{r'_2}{s'_2}$

**Hence** $+$ is well defined.

From (1) we get

$(r_1s'_1 - r'_1s_1)t_1t_2r_2s'_2 = 0$

$t_1t_2(r_1s'_1 - r'_1s_1)r_2s'_2 = 0$ (quasi weak commutative)

$t_1t_2(r_1s'_1 - r'_1s_1)t_2s'_2 = 0$ (S is commutative subset)

$(r_1s'_1 - r'_1s_1)r_2s'_2 = 0$ (S is commutative subset)

From (2) we get

$(r_2s'_2 - r'_2s_2)t_2t_1r'_1s_1 = 0$

$(r_2s'_2 - r'_2s_2)t_2t_1r'_1s_1 = 0$ (S is commutative subset)

$t_1t_2(r_2s'_2 - r'_2s_2)r'_1s_1 = 0$ (quasi weak commutative)

$t_1t_2(r_2s'_2 - r'_2s_2)r'_1s_1 = 0$ (S is commutative subset)

$(r_2s'_2 - r'_2s_2)r'_1s_1t_1t_2 = 0$ (quasi weak commutative)

$(r_2s'_2 - r'_2s_2)r'_1s_1t_1t_2 = 0$ (S is commutative subset)

From (3) + (4) gives

$r_1r_2s'_2s_2t_2t_1t_2 - r'_1r'_2s'_2s_1s_2t_1t_2 = 0$

$(r_1r_2s'_2s_2t_2t_1t_2 - r'_1r'_2s'_2s_1s_2t_1t_2) = 0$

This means $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r'_1}{s'_1} \cdot \frac{r'_2}{s'_2}$

**Hence** is well-defined.

We note that $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1s_2 + r_2s_1}{s_1s_2} = \frac{(r_1 + r_2)s}{s^2}$

$= \frac{r_1 + r_2}{s}$ (by lemma 3.9) \ldots \ldots (5)

**Claim:** $(S^{-1}R,+)$ is an abelian group.

Let $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3} \in S^{-1}R$.

Then
\[
\frac{r_1 + (r_2 + r_3)}{s_1} = \frac{r_1}{s_1} + \left(\frac{r_2 + r_3}{s_1}\right) = \frac{r_1 s_2 s_3 + (r_2 + r_3) s_1}{s_1 s_2 s_3} = \frac{r_1 s_2 s_3 + r_2 s_3 s_1 + r_3 s_2 s_1}{s_1 s_2 s_3}
\]

Also \(\frac{r_1 + (r_2 + r_3)}{s_1} = \frac{r_1}{s_1} + \frac{r_2 + r_3}{s_1}\)

So \(\cdot\) is associative.

For any \(\frac{r}{s} \in R\), we have
\[
\frac{r}{s} + \frac{0}{s} = \frac{r + 0}{s} = \frac{r}{s}
\]

Also \(\frac{0}{s} + \frac{r}{s} = \frac{0 + r}{s} = \frac{r}{s}\)

Hence \(\frac{0}{s}\) is the additive identity of \(\frac{r}{s} \in S^{-1}R\) for all \(r \in R\)

Clearly \(\cdot\) is commutative.

Thus \((R, +)\) is an abelian group.

**Claim 2**: is associative.

Now \(\frac{r_1 + (r_2 + r_3)}{s_1} + \frac{r_3}{s_3} = \frac{r_1 + (r_2 + r_3)}{s_1} \cdot \frac{r_3}{s_3}\)

So \(\cdot\) is associative.

**Claim 3**: is right distributive with respect to +.

Let \(\frac{r_1, r_2, r_3}{s_1, s_2, s_3} \in S^{-1}R\).

Now \(\left(\frac{r_1 + r_2}{s_1} \cdot \frac{r_3}{s_3}\right) = \frac{r_1 s_2 s_3 + r_2 s_1 s_3}{s_1 s_2 s_3}\)

\(= \frac{r_1 s_2 s_3 + r_2 s_1 s_3}{s_1 s_2 s_3}\) (quasi weak commutative)

\(= \frac{r_1 s_2 s_3}{s_1 s_2 s_3} + \frac{r_2 s_1 s_3}{s_1 s_2 s_3}\) (using (5))

\(= \frac{r_1}{s_1} + \frac{r_2}{s_1} + \frac{r_3}{s_1}\)

This proves right - distributive law.

**Claim 4**: \(S^{-1}R\) is a Boolean-like ring.

It is already proved in claim 1 that
\(2(\frac{r}{s}) = 0\) for all \(\frac{r}{s} \in S^{-1}R\)

Let \(a = \frac{r_1}{s_1}\) and \(b = \frac{r_2}{s_2}\) be any two elements of \(S^{-1}R\). Let \(t \in S\) be any element.

Now by Lemma 3.5

\((a - a^2)(b - b^2)t = 0\)

\[(\frac{r_1}{s_1} - \frac{r_1^2}{s_1})(\frac{r_2}{s_2} - \frac{r_2^2}{s_2})t = 0\]

\[(\frac{r_1}{s_1})(\frac{r_2}{s_2})t - \frac{r_1^2}{s_1} - \frac{r_2^2}{s_2}t = 0\]

\[(\frac{r_2}{s_2})t(\frac{r_1}{s_1}) - (\frac{r_1}{s_1})^2t = 0\) (quasi weak commutative)

\([(\frac{r_2}{s_2})t(\frac{r_1}{s_1}) - (\frac{r_1}{s_1})^2t]t = 0\)

\([(\frac{r_2}{s_2} - \frac{r_1}{s_1})t(\frac{r_1}{s_1}) - (\frac{r_1}{s_1})^2t]t = 0\)

\([(\frac{r_2}{s_2} - \frac{r_1}{s_1})t(\frac{r_1}{s_1})]\frac{r_1}{s_1} - (\frac{r_1}{s_1})^2t\frac{r_1}{s_1}]t = 0\) (using Lemma 3.9)

\([(\frac{r_2}{s_2} - \frac{r_1}{s_1})t(\frac{r_1}{s_1})]r_1s_2 - (\frac{r_1}{s_1})^2t\frac{r_1}{s_1}]t = 0\)
Let $R$ be a quasi-weak commutative Boolean-like near ring. Let $S$ be a commutative subset of $R$.

3.12 Theorem

This proves $S^{-1}R$ is Boolean-like near ring.

Claim 5: Multiplication in $S^{-1}R$ is commutative

Let $\frac{r_1}{s_1}, \frac{r_2}{s_2}$ be any two elements of $S^{-1}R$.

Then $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$ (using Lemma 3.9)

Hence multiplication in $S^{-1}R$ is commutative.

Claim 6: Existence of multiplicative identity in $S^{-1}R$

Let $\frac{r}{s} \in S^{-1}R$ be any element.

Then $\frac{r}{s} \cdot \frac{s}{s} = \frac{rs}{ss} = \frac{r}{s}$

Also $\frac{s}{s} \cdot \frac{r}{s} = \frac{rs}{ss} = \frac{r}{s}$

Hence $\frac{s}{s}$ is the multiplicative identity of $S^{-1}R$.

Thus $S^{-1}R$ is a commutative Boolean-like near-ring with identity.

3.11 Theorem

$S^{-1}R$ is quasi-weak commutative near-ring.

Proof:

Let $a = \frac{r_1}{s_1}, b = \frac{r_2}{s_2}, c = \frac{r_3}{s_3}$ be any three elements of $S^{-1}R$.

Now $abc = \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} \cdot \frac{r_3}{s_3} = \frac{r_1 r_2 r_3}{s_1 s_2 s_3}$ (R is quasi-weak commutative)

$= \frac{r_1 r_2}{s_1 s_2}$ (S is commutative)

Then $abc = bac \quad \forall a,b,c \in S^{-1}R$.

This proves $S^{-1}R$ is quasi-weak commutative near-ring.

3.12 Theorem

Let $R$ be a quasi-weak commutative Boolean-like near ring. Let $S$ be a commutative subset of $R$ which is multiplicatively closed. Let $0 \neq s \in S$. Define a map $f_s: R \rightarrow S^{-1}R$ as $f_s(r) = \frac{r}{s} \quad \forall r \in R$. Then $f_s$ is a near-ring monomorphism.

Proof:

Let $r_1, r_2 \in R$.

Then $f_s \left( r_1 + r_2 \right) = \frac{(r_1 + r_2)s}{s} = \frac{r_1 s + r_2 s}{s}$ (By (5) of Theorem 3.11)

$= \frac{r_1 s + r_2 s}{s} = f_s(r_1) + f_s(r_2)$

Also $f_s \left( r_1 \cdot r_2 \right) = \frac{(r_1 r_2)s}{s} = \frac{r_1 r_2 s^2}{s^2}$
\[ r_1(s_{r_2}s) = \frac{r_1s}{s} \cdot \frac{r_2s}{s} \quad \text{(quasi weak commutative)} \]
\[ = f_s(r_1) - f_s(r_2) \]

Also \( f_s(r_1) = f_s(r_2) \Rightarrow \frac{r_1s}{s} = \frac{r_2s}{s} \)
\[ \Rightarrow \frac{r_1s - r_2s}{s} = 0 \]
\[ \Rightarrow \frac{(r_1 - r_2)s}{s} = 0 \]
\[ \Rightarrow (\frac{r_1 - r_2}{s})s = 0 \]
\[ \Rightarrow \frac{r_1}{s} = \frac{r_2}{s} \]

Hence \( f_s \) is a monomorphism

### 3.13 Theorem

Let \( R \) be a quasi-weak commutative Boolean-like near-ring. Then \( R \) be embedded into a quasi-weak commutative Boolean like commutative semi ring with identity.

**Proof:**

Follows from Theorem 3.11 and 3.12.

### References


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