Sobolev type fractional stochastic integro-differential evolution

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Abstract

In this paper, we prove the existence of \( \alpha \)-mild solutions for a class of fractional stochastic integro-differential evolution equations of sobolev type with fractional sobolev stochastic nonlocal conditions in a real separable Hilbert space. To establish our main results, we use the Banach contraction mapping principle, fractional calculus, stochastic analysis and an analytic semigroup of linear operators. An example is given to illustrate the feasibility of our abstract result.

Keywords: Fractional stochastic evolution equations, Fixed point technique, fractional stochastic nonlocal condition.

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1 Introduction

Let \((\Omega, \Gamma, P)\) be a complete probability space equipped with a normal filtration \(\Gamma_t, t \in J\) satisfying the usual conditions (i.e., right continuous and \(\Gamma_0\) containing all \(P\)-null sets). We consider three real separable spaces \(X, Y\) and \(E\), and \(Q\)-Wiener process on \((\Omega, \Gamma, P)\) with the linear bounded covariance operator \(Q\) such that \(trQ < \infty\). We assume that there exist complete orthonormal systems \(\{e_{1,n}\}_{n \geq 1}, \{e_{2,n}\}_{n \geq 1}\) in \(E\), bounded sequences of non-negative real numbers \(\{\lambda_{1,n}\}, \{\lambda_{2,n}\}\) such that \(QE_{1,n} = \lambda_{1,n}e_{1,n}, QE_{2,n} = \lambda_{2,n}e_{2,n}, n = 1, 2, ...\), and sequences \(\{\beta_{1,n}\}_{n \geq 1}, \{\beta_{2,n}\}_{n \geq 1}\) of independent Brownian motions such that

\[
\langle w_1(t), e_1 \rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_{1,n}} \langle e_{1,n}, e_1 \rangle \beta_{1,n}(t), \quad e_1 \in E, t \in J,
\]

\[
\langle w_2(t), e_2 \rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_{2,n}} \langle e_{2,n}, e_2 \rangle \beta_{2,n}(t), \quad e_2 \in E, t \in J,
\]

and \(\Gamma_t = \Gamma_t^{\omega_1, \omega_2}\), where \(\Gamma_t^{\omega_1, \omega_2}\) is the sigma algebra generated by \(\{(w_1(s), w_2(s)) : 0 \leq s \leq t\}\). Let \(L_2^0 = L_2(Q^{1/2}E; X)\) be the space of all Hilbert–Schmidt operators from \(Q^{1/2}E\) to \(X\) with the inner product \(\langle \psi, \pi \rangle L_2^0 = tr[\psi^*Q^{1/2}]\).

In this paper we consider the following Sobolev type fractional stochastic integro-differential evolution equations with fractional sobolev stochastic nonlocal conditions of the form

\[
^cD_t^q [Lx(t)] = Ax(t) + \sigma_1(t, x(t), Hx(t)) \frac{dw_1(t)}{dt}, \quad t \in J, \quad (1.1)
\]

\[
^rD_t^{1-q} [Mx(0)] = \sigma_2(t, x(t)) \frac{dw_2(t)}{dt}, \quad (1.2)
\]

where \(^cD_t^q\) and \(^rD_t^{1-q}\) are the Caputo and Riemann–Liouville fractional derivatives with \(0 < q \leq 1\), the state \(x(\cdot)\) takes its values in the Hilbert space \(X\).

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The operators $A : D(A) \subset X \to X$, $L : D(L) \subset X \to X$ and $M : D(M) \subset X \to X$ are closed linear operators in $X$, $c_1$ and $c_2$ are given functions to be specified later, $J = [0, b]$, $b > 0$ is a constant. The term $Hx(t)$ is given by

$$Hx(t) = \int_0^t K(t, s)x(s)\,ds,$$

where $K \in C(\Delta, \mathbb{R}^+)$, $\Delta = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq b\}$.

During the past three decades, fractional differential equations and their applications have gained a lot of importance, mainly because this field has become a powerful tool in modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering (22, 3, 10, 11, 23, 24). Recently, there has been a significant development in the existence results for boundary value problems of nonlinear fractional differential equations (11, 27).

The problem with nonlocal condition, which is a generalization of the problem of classical condition, was motivated by physical problems. The pioneering work on nonlocal conditions is due to Byszewski see (4, 5, 6). Since it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems, differential equations with nonlocal conditions have been studied by many authors and some basic results on nonlocal problems have been obtained, see (12, 17, 22) and the references therein for more comments and citations. A. Debbouche, D. Baleanu and R. P. Agarwal [13] proved the existence of mild and strong solutions for fractional nonlocal nonlinear integro-differential equations of Sobolev type using Schauder fixed point theorem, Gelfand-Shilov principles combined with semigroup theory. A. Debbouche and J.J. Nieto [11] studied the existence and uniqueness of mild solutions for a class of Sobolev type fractional nonlocal abstract evolution equations with nonlocal conditions and optimal multi-controls in Banach spaces by using fractional calculus, semigroup theory, a singular version of Gronwall inequality and Leray–Schauder fixed point theorem.

Stochastic differential and integro-differential equations have attracted great interest due to its applications in various fields of science and engineering. There are many interesting results on the existence, uniqueness and asymptotic stability of solutions to stochastic differential equations, see (8, 14, 18, 19, 28, 29, 30, 35, 36) and the references therein. In particular, fractional stochastic differential equations have also been studied by several authors, see (9, 31, 32).

More recently, El-Borai [16] studied the existence of mild solutions for a class of semilinear stochastic fractional differential equations by using Leray-Schauder fixed point theorem. Cui and Yan [9] studied the existence of mild solutions for a class of fractional neutral stochastic integro-differential equations with infinite delay in Hilbert spaces by means of Sadovskii’s fixed point theorem. Sakthivel et al. [31] investigated the existence and asymptotic stability in $p$th moment of mild solutions to a class of nonlinear fractional neutral stochastic differential equations with infinite delays in Hilbert spaces by using semigroup theory and fixed point technique. The existence of mild solutions for impulsive fractional stochastic differential equations with infinite delay has also been established in [32].

For our best knowledge, there is no work reported on the existence of $\alpha$-mild solutions for Sobolev type fractional stochastic integro-differential evolution equations with fractional sobolev stochastic nonlocal conditions in fractional power space $X_\alpha$.

Motivated by the above works, we introduce here a new nonlocal fractional stochastic condition of Sobolev type, and we prove the existence of $\alpha$-mild solutions for the problem (1.1)-(1.2) by using a new strategy which relies on the compactness of the operator semigroup generated by $T = AL^{-1}$, Schauder fixed point theorem and approximating techniques. The rest of this paper is organized as follows. In Section 2 we present some essential facts in fractional calculus, semigroup theory, stochastic analysis that will be used to obtain our main results. In Section 3, we state and prove existence results on $\alpha$-mild solutions for Sobolev type fractional stochastic system (1.1)-(1.2). Finally, in Section 4, as an example, a fractional stochastic parabolic partial differential equation with a sobolev type fractional stochastic nonlocal condition is considered.

2 Preliminaries

Throughout this paper, $(X, ||||)$ is a separable Hilbert space.

The operators $A : D(A) \subset X \to X$, $L : D(L) \subset X \to X$ and $M : D(M) \subset X \to X$ satisfy the following conditions:
(A1) \( L, A \) and \( M \) are linear closed operators.

(A2) \( D(M) \subset D(L) \subset D(A) \) and \( L \) and \( M \) are bijective.

(A3) \( L^{-1} : X \to D(L) \subset X \) and \( M^{-1} : X \to D(M) \subset X \) are linear, bounded, and compact operators.

From (A3), we deduce that \( L^{-1} \) is bounded operators. Note (A3) also implies that \( L \) is closed since the fact: \( L^{-1} \) is closed and injective, then its inverse is also closed. It comes from (A1) – (A3) and the closed graph theorem, we obtain the boundedness of the linear operator \( AL^{-1} : X \to X \). Consequently, \( -AL^{-1} \) generates a semigroup \( \{ S(t) = e^{AL^{-1}t}, t \geq 0 \} \). We suppose that \( K_0 = \sup_{t \geq 0} ||S(t)|| < \infty \), and for short, we denote by \( C_1 = ||L^{-1}||, C_2 = ||M^{-1}|| \) and \( T = AL^{-1} \).

(A4) The resolvent \( R(\lambda, T) \) is compact for some \( \lambda \in \rho(T) \), the resolvent set of \( T \).

Without loss of generality, we assume that \( 0 \in \rho(T) \). Then it is possible to define the fractional power \( T^\alpha \) for \( 0 < \alpha \leq 1 \), as a closed linear operator on its domain \( D(T^\alpha) \) with inverse \( T^{-\alpha} \). Hereafter, we denote by \( X_\alpha \) the Banach space \( D(T^\alpha) \) normed with \( ||x||_{\alpha} \).

**Lemma 2.1** (See [26]). Let \( T \) be the infinitesimal generator of an analytic semigroup \( S(t) \). If \( 0 \in \rho(T) \), then

(a) \( D(T^\alpha) \) is a Hilbert space with the norm \( ||x||_{\alpha} = ||T^\alpha x|| \) for \( x \in D(T^\alpha) \).

(b) \( S(t) : X \to D(T^\alpha) \) for each \( t > 0 \) and \( \alpha \geq 0 \).

(c) \( S(t)T^\alpha x = T^\alpha S(t)x \) for each \( x \in D(T^\alpha) \) and \( t \geq 0 \).

(d) If \( 0 < \alpha \leq \beta \leq 1 \), then \( D(T^\beta) \hookrightarrow D(T^\alpha) \).

(e) For every \( t > 0 \), \( T^\alpha S(t) \) is bounded on \( X \) and there exist \( K_\alpha > 0 \) and \( \delta > 0 \) such that

\[
||T^\alpha S(t)|| \leq \frac{K_\alpha}{\delta^\alpha} e^{-\delta t} \leq \frac{K_\alpha}{\delta^\alpha}.
\]

(vi) \( T^{-\alpha} \) is a bounded linear operator in \( X \) with \( D(T^\alpha) = 1m(T^{-\alpha}) \).

**Definition 2.1** The fractional integral of order \( \alpha > 0 \) of a function \( f \in L^1([a, b], R^+) \) is given by

\[
I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,
\]

where \( \Gamma \) is the gamma function. If \( a = 0 \), we can write \( I^\alpha f(t) = (g_\alpha \ast f)(t) \), where

\[
g_\alpha(t) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\
0, & t \leq 0,
\end{cases}
\]

and as usual, \( \ast \) denotes the convolution of functions. Moreover, \( \lim_{\alpha \to 0} g_\alpha(t) = \delta(t) \), with \( \delta \) the delta Dirac function.

**Definition 2.2** The Riemann–Liouville derivative of order \( n-1 < \alpha < n, n \in N \), for a function \( f \in C([0, +\infty)) \) is given by

\[
_{0} \mathcal{D}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0.
\]

**Definition 2.3** The Caputo derivative of order \( n-1 < \alpha < n, n \in N \), for a function \( f \in C([0, +\infty)) \) is given by

\[
_{C} \mathcal{D}^\alpha f(t) = {_{0} \mathcal{D}^n} \left( f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0.
\]

**Remark 2.1** The following properties hold. Let \( n-1 < \alpha < n, n \in N \)

(i) If \( f(t) \in C^n([0, \infty)) \), then

\[
_{C} \mathcal{D}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(s), \quad t > 0.
\]
(ii) The Caputo derivative of a constant is equal to zero.

(iii) The Riemann–Liouville derivative of a constant function is given by

\[ L_D^a x = \frac{C}{\Gamma(1-a)} (x-a)^{-a}. \]

If \( f \) is an abstract function with values in \( X \), then integrals which appear in Definitions 2.1-2.3 are taken in Bochner’s sense.

According to previous definitions, it is suitable to rewrite problem (1.1)-(1.2) as the equivalent integral equation

\[
x(t) = x(0) + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q-1} A x(s) \, ds + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q-1} \sigma_1(s, x(s), Hx(s)) \, dw_1(s)
\]

(2.1)

**Remark 2.2** We note that:

(a) For the nonlocal condition, the function \( x(0) \) is dependent on \( t \).

(b) \( L_D^1 \frac{d}{dt} [Mx(t)] \) is well defined, i.e., if \( q = 1 \) and \( M \) is the identity, then (1.2) reduces to the usual nonlocal condition.

(c) The function \( x(0) \) takes the form

\[
M^{-1}x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q}M^{-1}\sigma_2(s, x(s)) \, dw_2(s),
\]

where \( Mx(0)|_{t=0} = x_0 \).

(d) The explicit and implicit integrals given in (2.1) exist (taken in Bochner’s sense).

Let \( L^2(\Gamma_b, X_a) \) be the Banach space of all \( \Gamma_b \)-measurable square integrable random variables with values in the Hilbert space \( X_a \). Let \( E(\cdot) \) denotes the expectation with respect to the measure \( P \). An important subspace of \( L^2(\Gamma_b, X_a) \) is given by \( L^2(\Gamma_b, X_a) = \{ x \in L^2(\Gamma_b, X_a) : x is \Gamma_b \text{ measurable} \} \).

Let \( C(J, L^2(\Gamma, X_a)) \) be the Banach space of continuous maps from \( J \) into \( L^2(\Gamma, X_a) \) satisfying \( \sup_{t \in J} E[|x(t)|^2] < \infty \). Let \( \mathcal{H} \) be the closed subspace of \( C(J, L^2(\Gamma, X_a)) \) consisting of measurable and \( \Gamma \)-adapted \( X_a \)-valued process \( x \in C(J, L^2(\Gamma, X_a)) \) endowed with the norm \( \|x\|_{\mathcal{H}} = (\sup_{t \in J} E[|x(t)|^2])^{1/2} \).

Then it easy to check that \( \langle \mathcal{H}, \| \cdot \|_{\mathcal{H}} \rangle \) is a Hilbert space. For any constant \( \tau > 0 \), let

\[ B_\tau = \{ x \in \mathcal{H} : \|x\|_{\mathcal{H}} \leq \tau \}, \]

clearly that \( B_\tau \) is a bounded closed convex set in \( \mathcal{H} \).

**Definition 2.4** By the \( \alpha \)-mild solution of the problem (1.1)-(1.2), we mean that the \( \Gamma \)-adapted stochastic process \( x \in \mathcal{H} \) which satisfies

1. \( x(0) \in L^2(\Gamma, X_a) \), where \( x(0) = M^{-1}x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q}M^{-1}\sigma_2(s, x(s)) \, dw_2(s) \) and \( Mx(0)|_{t=0} = x_0 \);

2. \( x(t) \in X_a \) has càdlàg paths on \( t \in J \) almost surely and for each \( t \in J \), \( x(t) \) satisfies the integral equation

\[
x(t) = S_q(t)x(0) + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q}S_q(t-s) \sigma_2(s, x(s)) \, dw_2(s)
\]

(2.1)

where \( S_q(t) = \int_0^{t+1} h_q(s) S(t+s) \, ds \) and \( T_q(t)x = q \int_0^{t+1} s h_q(s) S(t+s) \, ds \).

Here, \( S(t) \) is a \( C_0 \)-semigroup generated by a linear operator \( T = AL^{-1} : X \rightarrow X \), \( h_q \) is a probability density function defined on \( (0, \infty) \), that is \( h_q(s) \geq 0, s \in (0, \infty) \) and \( \int_0^{\infty} h_q(s) \, ds = 1 \).

The following lemma follows from the results in ([13, 16, 21, 33, 34]) and will be used throughout this paper.

**Lemma 2.2**. The operators \( S_q(t) \) and \( T_q(t) \) have the following properties:

1. For any fixed \( t \geq 0 \), \( S_q(t) \) and \( T_q(t) \) are linear and bounded operators in \( X_a \),

\[
i.e. \text{for any } x \in X_a. \quad \|S_q(t)x\| \leq K_0 \|x\|_a, \quad \|T_q(t)x\| \leq \frac{q K_0}{\Gamma(1+q)} \|x\|_a.
\]
(2) The operators \{S_q(t) : t \geq 0\} and \{T_q(t) : t \geq 0\} are strongly continuous.

(3) For every \( t > 0 \), \( S_q(t) \) and \( T_q(t) \) are compact operators in \( X \), and hence they are norm-continuous.

(4) For every \( t > 0 \), the restriction of \( S_q(t) \) to \( X_a \) and the restriction of \( T_q(t) \) to \( X_a \) are compact operators in \( X_a \).

(5) The restriction of \( S_q(t) \) to \( X_a \) and the restriction of \( T_q(t) \) to \( X_a \) are continuous in \( (0, +\infty) \) by the operator norm \( \| \cdot \|_a \).

(6) For any \( x \in X \) and \( t \in J \), \( \| T^a T_q(t)x \| \leq A_a t^{-aq} \| x \| \), where \( A_a = \frac{q K_a \Gamma(2-a)}{\Gamma(\{1+q(1-a)\})} \).

3 Main results

In this section, we give the existence of \( \alpha \)-mild solutions for the problem (1.1)-(1.2). We impose the following assumptions on the data of the problem (1.1)-(1.2).

(H1) The functions \( \sigma_1 : J \times X_a \times X_a \to L^0_\mathbb{1} \) satisfies the Carathéodory type conditions, i.e. \( \sigma_1(\cdot, x, Hx) \) is strongly measurable for all \( x \in X_a \), and \( \sigma_1(t, \cdot, \cdot) \) is continuous for a.e. \( t \in J \).

(H2) For some \( \tau > 0 \), there exist constants \( q_1 \in [\frac{1}{2}, q(1-\alpha)) \), \( \rho_1 > 0 \) and functions \( \varphi_\tau \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+) \) such that for a.e. \( t \in J \),

\[
\sup_{\| x \|_2^2 \leq \tau} E(\| \sigma_1(t, x, Hx) \|_2^2) \leq \varphi_\tau(t) \quad \text{and} \quad \liminf_{\tau \to +\infty} \frac{\| \varphi_\tau \|_{L^{\frac{1}{q_1}}(J, \mathbb{R}^+)}^{\frac{1}{q_1}}}{\tau} = \rho_1 < +\infty.
\]

(H3) The nonlocal function \( \sigma_2 : J \times X_a \to L^0_\mathbb{1} \) is continuous, and there exist a constant \( \rho_2 > 0 \) and a nondecreasing continuous function \( \Phi : \mathbb{R}^+ \to \mathbb{R}^+ \) such that for some \( \tau > 0 \) and all \( x \in B_\tau \),

\[
E(\| \sigma_2(t, x) \|_2^2) \leq \Phi(\tau) \quad \text{and} \quad \liminf_{\tau \to +\infty} \frac{\Phi(\tau)}{\tau} = \rho_2 < +\infty.
\]

(H4) There exists a constant \( \delta \in (0, b) \) such that \( \sigma_2(t, x) = \sigma_2(t, y) \) for any \( x, y \in H_2 \) with \( x(t) = y(t) \), \( t \in [\delta, b] \).

Theorem 3.1 If the assumptions (H1)-(H4) are satisfied, then the problem (1.1)-(1.2) has at least one \( \alpha \)-mild solution provided that

\[
\bigg\{ \left( \frac{C K_a}{\Gamma(1-q)} \right)^2 \text{Tr}(Q) b^{2q-1} \rho_2 \\
+ (C_1 A_a)^2 \text{Tr}(Q) b^{2(q-q_1-a)} \left( \frac{1-q_1}{q-q_1-a} \right)^{2-2q_1} \rho_1 \bigg\} < \frac{1}{2}.
\]

Proof Let \( \{ \delta_n : n \in \mathbb{N} \} \) be a decreasing sequence in \( (0, b) \) such that \( \lim_{n \to +\infty} \delta_n = 0 \). We first prove the following problem

\[
C D^\alpha_1 [Lx(t)] = Ax(t) + \sigma_1(t, x(t), Hx(t)) \frac{dw_1}{Dt}, \quad t \in J,
\]

(3.2)

\[
x(0) = S(\delta_n) \left( M^{-1} x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} M^{-1} \sigma_2(s, x(s)) dw_2(s) \right)
\]

(3.3)

has at least one \( \alpha \)-mild solution \( x_n \in H_2 \). To this end, for fixed \( n \in \mathbb{N} \), we define an operator \( \Psi_n : H_2 \to H_2 \) by

\[
(\Psi_n x)(t) = S_q(t) S(\delta_n) M^{-1} \left[ x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \sigma_2(s, x(s)) dw_2(s) \right] \\
+ \int_0^t (t-s)^{q-1} L^{-1} T_q(t-s) \sigma_1(s, x(s), Hx(s)) dw_1(s), \quad t \in J.
\]

(3.4)

By direct calculation, we know that \( \Psi_n \) is well defined. From Definition 2.4, it is easy to see that the \( \alpha \)-mild solution of the problem (3.2)-(3.3) is equivalent to the fixed point of the operator \( \Psi_n \).

In what follows, we prove that there exists a positive constant \( R \), such that \( \Psi_n(B_R) \subset B_R \).
If this is not true, then for each $\tau > 0$, there would exist $x_\tau \in B_\tau$ and $t_\tau \in J$ such that $E \| (\Psi_n x_\tau)(t_\tau) \|^2 > \tau$. It follows from Lemma 2.2 (1) and (6), the assumption (H2) and Hölder inequality that

$$
\tau < E \| (\Psi_n x_\tau)(t_\tau) \|_a^2 \\
\leq 2E \left\| S_q(t_\tau)S (\delta_n) M^{-1} \chi_0 + \frac{1}{\Gamma(1-q)} \int_0^{t_\tau} (t_\tau - s)^{-q} \sigma_2(s, x_\tau(s)) dw_2(s) \right\|_a^2 \\
+ 2E \left\| \int_0^{t_\tau} (t_\tau - s)^{q-1} L^{-1} T_\delta(t_\tau - s) \sigma_1(s, x_\tau(s), Hx_\tau(s)) dw_1(s) \right\|_a^2 \\
\leq 2 \left\| M^{-1} \right\|^2 \left\| S_q(t_\tau) \right\|^2 \| Tr(Q) \left( \frac{1}{\Gamma(1-q)} \right) \|_a^2 \\
\times \int_0^{t_\tau} (t_\tau - s)^{-2q} E \| \sigma_2(s, x_\tau(s)) \|_a^2 ds \\
+ 2 \left\| L^{-1} \right\|^2 \| Tr(Q) \|_a \int_0^{t_\tau} (t_\tau - s)^{2(q-1)} E \| T^\delta T_\delta(t_\tau - s) \sigma_1(s, x_\tau(s), Hx_\tau(s)) \|_a^2 ds \\
\leq 2C^2_2 K^2_0 \| Tr(Q) \left( \frac{1}{\Gamma(1-q)} \right) \|^2 \Phi(\tau) \left( \int_0^{t_\tau} (t_\tau - s)^{-2q} ds \right) \\
+ 2C^2_1 \| Tr(Q) \|_a^2 \left( \int_0^{t_\tau} (t_\tau - s)^{-2q} ds \right)^{2-2q_1} \times \left( \int_0^{t_\tau} \varphi_{q_1}^{-1} \left( \frac{1}{q_1} - 1 \right) (s) \right)^{2q_1-1} \\
\leq 2C^2_2 K^2_0 \| Tr(Q) \left( \frac{1}{\Gamma(1-q)} \right) \|^2 \Phi(\tau) \frac{1}{-2q+1} b^{-2q+1} + 2C^2_1 \| Tr(Q) \|_a^2 b^{2(q-1-aq)} \left( \frac{1-q}{q_1} \right)^{2-2q_1} \| \varphi_\eta \|_{L^{\eta^{-1}}[0,b]} + 1.
$$

Dividing both side of (3.5) by $\tau$, then taking the lower limit as $\tau \to +\infty$, we get

$$
\left\{ \begin{array}{l}
2C^2_2 K^2_0 \| Tr(Q) \left( \frac{1}{\Gamma(1-q)} \right) \|^2 \frac{1}{-2q+1} b^{-2q+1} \rho_2 \\
2C^2_1 \| Tr(Q) \|_a^2 b^{2(q-1-aq)} \left( \frac{1-q}{q_1} \right)^{2-2q_1} \rho_1 \end{array} \right\} \geq 1.
$$

which contradicts (3.1).

Next, we prove that $\Psi_n$ is continuous in $B_R$. To this end, let $\{x_m\}_{m=1}^{\infty} \subset B_R$ be a sequence such that $\lim_{m \to \infty} x_m = x \in B_R$. By the Carathéodory continuity of $\sigma_1$ and $\sigma_2$, we have

$$\lim_{m \to \infty} \sigma_1(s, x_m(s), Hx_m(s)) = \sigma_1(s, x(s), Hx(s)), \quad a.e. \ s \in I. \tag{3.6}$$

$$\lim_{m \to \infty} \sigma_2(s, x_m(s)) = \sigma_2(s, x(s)), \quad a.e. \ s \in I. \tag{3.7}$$

From the assumption (H2), we get that for each $t \in I$,

$$\begin{align*}
(t-s)^{2(q-1-aq)} E \| \sigma_1(s, x_m(s), Hx_m(s)) - \sigma_1(s, x(s), Hx(s)) \|^2 \\
&\leq (t-s)^{2(q-1-aq)} \left( 2E \| \sigma_1(s, x_m(s), Hx_m(s)) \|^2 \\
&+ 2E \| \sigma_1(s, x_m(s), Hx_m(s)) \|^2 \right) \leq 4 (t-s)^{2(q-1-aq)} q_R(s).
\end{align*}
$$

From the assumption (H3), we get that for each $t \in I$,

$$\begin{align*}
(t-s)^{-2q} E \| \sigma_2(s, x_m(s)) - \sigma_2(s, x(s)) \|^2 \\
&\leq 4 (t-s)^{-2q} q_R(s). \tag{3.9}
\end{align*}
$$

Using the fact that the functions $s \to 4 (t-s)^{2(q-1-aq)} q_R(s)$ and $s \to 4 (t-s)^{-2q} q_R(s)$ are Lebesgue integrable for $s \in [0, t]$, $t \in I$, by Lemma 2.2 (1) and (6), (3.6), (3.7), (3.8), (3.9) and the Lebesgue dominated
define the operator \\
\[ \Psi \] \\
that \\
the restriction of \\
[\s] is relatively compact in \\
[\x] \\
that the set \\
convergence theorem, we get that \\
\[
\begin{aligned}
E \| (\Psi_n x_m) (t) - (\Psi_n x) (t) \|_H^2 & \leq 2 \| M^{-1} \|_2 ^2 \| S_q (t) \| \operatorname{Tr} (Q) \left( \frac{1}{\Gamma (1 - q)} \right) ^2 \int_0 ^t (t - s)^{-2q} \\
& \times E \| \sigma_2 (s, x_m (s)) - \sigma_2 (s, x (s)) \| ^2 ds \\
& + 2 \| L^{-1} \|_2 \operatorname{Tr} (Q) \int_0 ^t (t - s)^{2(q - 1)} \\
& \times E \| T^n T_q (t - s) (\sigma_1 (s, x_m (s), Hx_m (s)) - \sigma_1 (s, x (s), Hx (s))) \| ^2 ds \\
& \leq 2 C_2^2 K_2^2 \operatorname{Tr} (Q) \left( \frac{1}{\Gamma (1 - q)} \right) ^2 \frac{1}{-2q + 1} t^{-2q + 1} \\
& \times E \| \sigma_2 (s, x_m (s)) - \sigma_2 (s, x (s)) \| ^2 \\
& + C_1^2 \operatorname{Tr} (Q) A_n^2 \int_0 ^t (t - s)^{2(q - 1 - \alpha q)} \\
& \times E \| \sigma_1 (s, x_m (s), Hx_m (s)) - \sigma_1 (s, x (s), Hx (s)) \| ^2 ds \\
& \to 0 \text{ as } m \to \infty.
\end{aligned}
\]

Therefore, by (3.10) we know that \\
\[
E \| (\Psi_n x_m) - (\Psi_n x) \|_H^2 = \sup_{t \in J} E \| (\Psi_n x_m) (t) - (\Psi_n x) (t) \|_H^2 \to 0 \text{ as } m \to \infty.
\]

which means that \( \Psi_n \) is continuous in \( B_R \).

Now, we demonstrate that \( \Psi_n : B_R \to B_R \) is a compact operator. We first prove that \{ \( (\Psi_n x) (t) : x \in B_R \) \} is relatively compact in \( X_n \) for all \( t \in J \). For \( t = 0 \), since the compactness of \( S(t) \) for every \( t > 0 \) implies that the restriction of \( S(t) \) to \( X_n \) is compact semigroup in \( X_n \), for \( \forall n \in \mathbb{N} \) we can deduce, by the assumption (H3), that \{ \( (\Psi_n x) (0) : x \in B_R \) \} is relatively compact in \( X_n \). For \( 0 < t \leq b, \ \varepsilon \in (0, t) \), arbitrary \( \delta > 0 \) and \( x \in B_R \), we define the operator \( \Psi_n^{\varepsilon, \delta} \) by \\
\[
\left( \Psi_n^{\varepsilon, \delta} x \right) (t) = S_q (t) S (\delta_n) M^{-1} \left[ x_0 + \frac{1}{\Gamma (1 - q)} \int_0 ^t (t - s)^{-q} \sigma_2 (s, x(s)) dw_2 (s) \right] 3.11
\]

Since the restriction of \( S (\varepsilon^{\delta}) \) \( (\varepsilon^{\delta} > 0) \) to \( X_n \) is compact semigroup in \( X_n \), by Lemma 2.2 (4) we know that the set \{ \( \Psi_n^{\varepsilon, \delta} x \) \( (t) : x \in B_R \) \} is relatively compact in \( X_n \) for \( \forall \varepsilon \in (0, t) \) and \( \forall \delta > 0 \). Moreover, for every
\(x \in B_R,\) by assumption (H2), Lemma 2.2 (6) and Hölder inequality we know that

\[
E \left\| (\Psi_n x) (t) - \left( \Psi_n^\delta x \right) (t) \right\|^2_a \\
\leq 2E \left\| \int_0^t \int_0^\delta q \tau (t-s)^{-\alpha} L^{-1} H_\tau (\tau) \left( S (t-s) \right)^{q} \tau \sigma_1 (s, x(s), Hx(s)) d\tau d\omega_1 (s) \right\|^2_a \\
+ 2E \left\| \int_{t-\delta}^t \int_0^{\infty} q \tau (t-s)^{-\alpha} L^{-1} H_\tau (\tau) \left( S (t-s) \right)^{q} \tau \sigma_1 (s, x(s), Hx(s)) d\tau d\omega_1 (s) \right\|^2_a \\
\leq 2 \left\| L^{-1} \right\|^2 Tr (Q) \int_0^t \left( (t-s)^{2(q-1)} \right) \left\| \int_0^\delta q \tau H_\tau (\tau) T^\alpha \left( S (t-s) \right)^{q} \tau \right\|^2_a \\
\times E \| \sigma_1 (s, x(s), Hx(s)) \|^2 ds \\
+ 2 \left\| L^{-1} \right\|^2 Tr (Q) \int_{t-\delta}^t \left( (t-s)^{2(q-1)} \right) \left\| \int_{\delta}^{\infty} q \tau H_\tau (\tau) T^\alpha \left( S (t-s) \right)^{q} \tau \right\|^2_a \\
\times E \| \sigma_1 (s, x(s), Hx(s)) \|^2 ds \\
\leq 2C^2 Tr (Q) M^2_n \| \varphi_R \|_{L^{\frac{1}{q-1}} [0,b]} \left( \frac{1 - q_1}{q - q_1 - aq} \right)^{2-2q_1} \\
\times 2^{(q-q_1-aq)} \left( \int_0^\delta q \tau^{1-a} H_\tau (\tau) \right)^2 \\
+ 2C^2 Tr (Q) A^2_n \| \varphi_R \|_{L^{\frac{1}{q-1}} [0,b]} \left( \frac{1 - q_1}{q - q_1 - aq} \right)^{2-2q_1} e^{2(q-q_1-aq)}.
\]

Therefore, letting \( \delta, \epsilon \to 0, \) we see that there are relatively compact sets arbitrarily close to the set \( \{ (\Psi_n x) (t) : x \in B_R \} \) in \( X_a \) for \( 0 < t \leq b. \) Hence, the set \( \{ (\Psi_n x) (t) : x \in B_R \} \) is also relatively compact in \( X_a \) for \( 0 < t \leq b. \) And since \( \{ (\Psi_n x) (t) : x \in B_R \} \) is relatively compact in \( X_a, \) we have the relatively compactness of \( \{ (\Psi_n x) (t) : x \in B_R \} \) in \( X_a \) for all \( t \in J. \)

Next, we prove that \( \Psi_n (B_R) \) is equicontinuous. For \( t = 0, \) since \( S(\delta_n) \) is a compact operator for \( \forall \alpha \in \mathbb{N}, \) we know that the functions \( \{ (\Psi_n x) (t) : x \in B_R \} \) are equicontinuous at \( t = 0. \) For any \( x \in B_R \) and \( 0 < t_1 < t_2 \leq b, \) we get that

\[
E \left\| (\Psi_n x) (t_2) - (\Psi_n x) (t_1) \right\|^2_a \\
\leq 4E \left\| S_q (t_2) S (\delta_n) M^{-1} \left[ x_0 + \frac{1}{\Gamma(1-q)} \int_0^{t_2} (t-s)^{-q} \sigma_2 (s, x(s)) d\omega_2 (s) \right] \\
- S_q (t_1) S (\delta_n) M^{-1} \left[ x_0 + \frac{1}{\Gamma(1-q)} \int_0^{t_1} (t-s)^{-q} \sigma_2 (s, x(s)) d\omega_2 (s) \right] \right\|^2_a \\
+ 4E \left\| \int_{t_1}^{t_2} (t_2-s)^{-q_1} L^{-1} T_q (t_2-s) \sigma_1 (s, x(s), Hx(s)) d\omega_1 (s) \right\|^2_a \\
+ 4E \left\| \int_{0}^{t_1} ((t_2-s)^{-q} - (t_1-s)^{-q_1}) L^{-1} T_q (t_2-s) \sigma_1 (s, x(s), Hx(s)) d\omega_1 (s) \right\|^2_a \\
+ 4E \left\| \int_{0}^{t_1} (t_1-s)^{-q_1} L^{-1} \left[ T_q (t_2-s) - T_q (t_1-s) \right] \sigma_1 (s, x(s), Hx(s)) d\omega_1 (s) \right\|^2_a \\
: = I_1 + I_2 + I_3 + I_4.
\]

where

\[
I_1 = 4E \left\| S_q (t_2) S (\delta_n) M^{-1} \left[ x_0 + \frac{1}{\Gamma(1-q)} \int_0^{t_2} (t-s)^{-q} \sigma_2 (s, x(s)) d\omega_2 (s) \right] \\
- S_q (t_1) S (\delta_n) M^{-1} \left[ x_0 + \frac{1}{\Gamma(1-q)} \int_0^{t_1} (t-s)^{-q} \sigma_2 (s, x(s)) d\omega_2 (s) \right] \right\|^2_a,
\]

\[
I_2 = 4E \left\| \int_{t_1}^{t_2} (t_2-s)^{-q_1} L^{-1} T_q (t_2-s) \sigma_1 (s, x(s), Hx(s)) d\omega_1 (s) \right\|^2_a,
\]

\[
I_3 = 4E \left\| \int_{0}^{t_1} ((t_2-s)^{-q} - (t_1-s)^{-q_1}) L^{-1} T_q (t_2-s) \sigma_1 (s, x(s), Hx(s)) d\omega_1 (s) \right\|^2_a,
\]

\[
I_4 = 4E \left\| \int_{0}^{t_1} (t_1-s)^{-q_1} L^{-1} \left[ T_q (t_2-s) - T_q (t_1-s) \right] \sigma_1 (s, x(s), Hx(s)) d\omega_1 (s) \right\|^2_a.
\]
\[ I_3 = 4E \left\| \int_0^{t_1} \left( (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) L^{-1} T_q^\alpha (t_2 - s) \sigma_1 (s, x(s), H_x(s)) dw_1(s) \right\|_\alpha^2, \]

\[ I_4 = 4E \left\| \int_0^{t_1} (t_1 - s)^{q-1} L^{-1} (T_q^\alpha (t_2 - s) - T_q^\alpha (t_1 - s)) \sigma_1 (s, x(s), H_x(s)) dw_1(s) \right\|_\alpha^2. \]

Therefore, we only need to check \( I_i \to 0 \) independently of \( x \in B_R \) when \( t_2 - t_1 \to 0, i = 1, 2, \ldots, 4. \)

For \( I_1 \), by Lemma 2.2 (1) and (5) and the assumption (H3), we know that

\[ I_1 \leq 4E \left\| (S_q^\alpha (t_2) - S_q^\alpha (t_1)) S (\delta_n) M^{-1} \left[ \frac{1}{\Gamma(1-q)} \int_0^{t_1} ((t_2 - s)^{-q} - (t_1 - s)^{-q}) \sigma_2 (s, x(s)) dw_2(s) \right] \right\|_\alpha^2 + 4E \left\| S_q^\alpha (t_2) S (\delta_n) M^{-1} \left[ \frac{1}{\Gamma(1-q)} \int_0^{t_1} ((t_2 - s)^{-q} - (t_1 - s)^{-q}) \sigma_2 (s, x(s)) dw_2(s) \right] \right\|_\alpha^2 + 4E \left\| S_q^\alpha (t_2) S (\delta_n) M^{-1} \left[ \frac{1}{\Gamma(1-q)} \int_0^{t_1} ((t_2 - s)^{-q} - (t_1 - s)^{-q}) \sigma_2 (s, x(s)) dw_2(s) \right] \right\|_\alpha^2 \leq 4 \left\| M^{-1} \right\|_\alpha^2 K_3^2 \left( t_1^{q-1} \int_0^{t_1} ((t_2 - s)^{-q} - (t_1 - s)^{-q}) \sigma_2 (s, x(s)) dw_2(s) \right) \leq 4 \left\| M^{-1} \right\|_\alpha^2 \left( \frac{1}{\Gamma(1-q)} \right) \left( t_1^{q-1} \int_0^{t_1} ((t_2 - s)^{-q} - (t_1 - s)^{-q}) \sigma_2 (s, x(s)) dw_2(s) \right) \]

\[ \to 0 \quad \text{as} \quad t_2 - t_1 \to 0. \]

For \( I_2 \), by the assumption (H2), Lemma 2.2 (6) and Hölder inequality, we have

\[ I_2 \leq 4 \left\| L^{-1} \right\|_\alpha^2 T (Q_1) \left( t_2 - s \right)^{q-1} \left\| \int_0^{t_1} ((t_2 - s)^{-q} - (t_1 - s)^{-q}) \sigma_2 (s, x(s), H_x(s)) \right\|_\alpha^2 ds \leq 4C_1^2 \left( Q_1 \right) A_\alpha^2 \left( t_2 - s \right)^{2q - 2q - 2aq} \varphi_R (s) ds \]

\[ \to 0 \quad \text{as} \quad t_2 - t_1 \to 0. \]

For \( I_3 \), by the assumption (H2), Lemma 2.2 (6) and Hölder inequality, we get that

\[ I_3 \leq 4 \left\| L^{-1} \right\|_\alpha^2 T (Q) \left( t_2 - s \right)^{q-1} \left\| \int_0^{t_1} ((t_2 - s)^{-q} - (t_1 - s)^{-q}) \sigma_2 (s, x(s), H_x(s)) \right\|_\alpha^2 ds \leq 4C_1^2 \left( Q \right) A_\alpha^2 \left( t_2 - s \right)^{2q - 2q - 2aq} \varphi_R (s) ds \]

\[ \leq 4C_1^2 \left( Q \right) A_\alpha^2 \left( t_2 - s \right)^{2q - 2q - 2aq} \varphi_R (s) ds \]

\[ \leq 4C_1^2 \left( Q \right) A_\alpha^2 \left( t_2 - s \right)^{2q - 2q - 2aq} \varphi_R (s) ds \]

\[ \to 0 \quad \text{as} \quad t_2 - t_1 \to 0. \]

For \( \epsilon > 0 \) small enough, by Lemma 2.3 (5) and (6), the assumption (H2) and Hölder inequality, we know
that
\[
I_4 \leq \sup_{s \in [0,t_1-\epsilon]} \left\| T_\eta(t_2-s) - T_\eta(t_1-s) \right\|_\alpha^2 \cdot 8 \left\| L^{-1} \right\|_\alpha^2 \left( \int_0^{t_1-\epsilon} (t_1-s)^2(2^{q-1}) \right) \\
\times E \left\| \sigma_1(s,x(s),Hx(s)) \right\|^2 ds \\
+ 8 \left\| L^{-1} \right\|_\alpha^2 \left( \int_0^{t_1} (t_1-s)^2(2^{q-1}) \right) \left( T_\eta(t_2-s) - T_\eta(t_1-s) \right) \left\| \frac{1}{1-\epsilon} \right\|_\alpha^2 \\
\times E \left\| \sigma_1(s,x(s),Hx(s)) \right\|^2 ds \\
\leq \sup_{s \in [0,t_1-\epsilon]} \left\| T_\eta(t_2-s) - T_\eta(t_1-s) \right\|_\alpha^2 \cdot 8 C_1^2 \left( \int_0^{t_1-\epsilon} (t_1-s)^2(2^{q-1}) \right) \left( \int_0^{t_1} (t_1-s)^2(2^{q-1}) \right) \varphi_R(s) ds \\
+ 16 C_1^2 \left( \int_0^{t_1} (t_1-s)^2(2^{q-1}) \right) \left( \int_0^{t_1} \left( (t_2-s)^{-2\alpha q} - (t_1-s)^{-2\alpha q} \right) \right) \varphi_R(s) ds \\
\leq \sup_{s \in [0,t_1-\epsilon]} \left\| T_\eta(t_2-s) - T_\eta(t_1-s) \right\|_\alpha^2 \cdot 8 C_1^2 \left( \int_0^{t_1-\epsilon} (t_1-s)^2(2^{q-1}) \right) \left( \int_0^{t_1} (t_1-s)^2(2^{q-1}) \right) \varphi_R(s) ds \\
\times \left( \frac{1-q_1}{q-q_1} \right)^{2-2q_1} \left( \frac{q-q_1}{1-\epsilon-q_1} \right)^{2-2q_1} \\
+ 32 C_1^2 \left( \int_0^{t_1} (t_1-s)^2(2^{q-1}) \right) \varphi_R(s) \left\| L^{-1} \right\|_{[0,b]} \\
\rightarrow 0 \quad \text{as} \quad t_2 - t_1 \to 0 \quad \text{and} \quad \epsilon \to 0.
\]

As a result, \( E \left\| (\Psi_n x)(t_2) - (\Psi_n x)(t_1) \right\|^2 \) tends to zero independently of \( x \in B_R \) as \( t_2 - t_1 \to 0 \), which means that \( \Psi_n : B_R \to B_R \) is equicontinuous. Hence by the Arzela-Ascoli theorem one has that \( \Psi_n : B_R \to B_R \) is a compact operator. Therefore, by Schauder fixed point theorem we obtain that for each \( n \in \mathbb{N}, \Psi_n \) has at least one fixed point \( x_n \in B_R \) which is in turn a \( \alpha \)-mild solution of the problem (3.2)-(3.3). Furthermore, for any \( t \in J \), \( x_n(t) \) is given by
\[
x_n(t) = S_\eta(t)S(\delta_\alpha) M^{-1} \left[ x_0 + \frac{1}{1-\eta} \int_0^t (t-s)^{-\eta} \sigma_2(s,x_n(s),Hx_n(s)) ds \right]
+ \int_0^t (t-s)^{-\eta} L^{-1} T_\eta(t-s) \sigma_1(s,x_n(s),Hx_n(s)) \, dw_1(s) \
\]  
(3.12)

Finally, we show that the set \( \{ x_n : n \in \mathbb{N} \} \subset B_R \) is precompact in \( H_2 \). Denote by
\[
x_n^1(t) = S_\eta(t)S(\delta_\alpha) M^{-1} \left[ x_0 + \frac{1}{1-\eta} \int_0^t (t-s)^{-\eta} \sigma_2(s,x_n(s),Hx_n(s)) ds \right], \quad t \in J
\]
and
\[
x_n^2(t) = \int_0^t (t-s)^{-\eta} L^{-1} T_\eta(t-s) \sigma_1(s,x_n(s),Hx_n(s)) \, dw_1(s), \quad t \in J.
\]

Therefore, it is sufficient to show that the sets \( \{ x_n^1 : n \in \mathbb{N} \} \) and \( \{ x_n^2 : n \in \mathbb{N} \} \) are precompact in \( H_2 \).

Let \( \zeta \in (0,\delta) \) be fixed, where \( \delta \) is the constant in (H4). By taking the method similar to the proof of the compactness of the operator \( \Psi_n \), we can see that the sets \( \{ x_n^1 : n \in \mathbb{N} \} |_{[\zeta,b]} \) and \( \{ x_n^2 : n \in \mathbb{N} \} \) are precompact in \( C([\zeta,b],L^2(\Gamma,X_\alpha)) \) and \( C([0,b],L^2(\Gamma,X_\alpha)) \), respectively. In particular, the set \( \{ x_n^2 : n \in \mathbb{N} \} |_{[\zeta,b]} \) is precompact in \( C([\zeta,b],L^2(\Gamma,X_\alpha)) \). Therefore, we have proved that the set \( \{ x_n : n \in \mathbb{N} \} |_{[\zeta,b]} \) is precompact in \( C([\zeta,b],L^2(\Gamma,X_\alpha)) \).

Without loss of generality, we let
\[
x_n \to x \quad \text{in} \quad C([\zeta,b],L^2(\Gamma,X_\alpha)) \quad \text{as} \quad n \to \infty.
\]

Denote by
\[
x_n^\delta(t) = \begin{cases} x_n(t), & t \in [0,\delta], \\
x_n(t), & t \in [\delta,b]. \end{cases}
\]
\[
x(t) = \begin{cases} x(t), & t \in [0,\delta], \\
x(t), & t \in [\delta,b]. \end{cases}
\]

Then we have
\[
x_n^\delta \to x^\delta \quad \text{in} \quad C([0,b],L^2(\Gamma,X_\alpha)) \quad \text{as} \quad n \to \infty.
\]
Therefore, to prove that the set \( \{ x_n : n \in \mathbb{N} \} \) is precompact in \( C([0, b], L^2(\Gamma, X_\alpha)) \), we only need to prove that the set \( \{ x_n^1 : n \in \mathbb{N} \} \) is precompact in \( C([0, \xi], L^2(\Gamma, X_\alpha)) \). By the strong continuity of the semigroup \( S(t)(t \geq 0) \) and the assumptions (H3) and (H4), we have

\[
E \left\| S(\delta_n) M^{-1} \left( x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \sigma_2(s, x_n(s)) dw_2(s) \right) \right\|^2_a \\
- M^{-1} \left( x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \sigma_2(s, x(s)) dw_2(s) \right)^2 \\
\leq 2 \left\| M^{-1} \right\|^2 \left( \frac{1}{\Gamma(1-q)} \right)^2 \\
\times E \left\| S(\delta_n) \int_0^t (t-s)^{-q} T^a \left( \sigma_2(s, x_n^1(s)) - \sigma_2(s, x^1(s)) \right) dw_2(s) \right\|^2 \\
+ 2 \left\| M^{-1} \right\|^2 \left( \frac{1}{\Gamma(1-q)} \right)^2 \times E \left\| S(\delta_n) \int_0^t (t-s)^{-q} T^a \sigma(s, x^1(s)) dw_2(s) \right\|^2 \\
- \int_0^t (t-s)^{-q} T^a \sigma_2(s, x^1(s)) dw_2(s) \right\|^2_a \\
\leq 2C_2^2 \left( \frac{1}{\Gamma(1-q)} \right)^2 K_2^2 E \left\| \int_0^t (t-s)^{-q} \sigma_2(s, x_n^1(s)) dw_2(s) \right\|^2 \\
- \int_0^t (t-s)^{-q} \sigma_2(s, x^1(s)), H_2 x^1(s) dw_2(s) \right\|^2_a \\
+ 2C_2^2 \left( \frac{1}{\Gamma(1-q)} \right)^2 \times E \left\| S(\delta_n) - I \right\| \int_0^t (t-s)^{-q} T^a \sigma_2(s, x^1(s)) dw_2(s) \right\|^2 \\
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

which means that the set \( \left\{ S(\delta_n) M^{-1} \left( x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \sigma_2(s, x_n(s)) dw_2(s) \right) : n \in \mathbb{N} \right\} \) is precompact in \( X_\alpha \). By the continuity of the operator \( S(\delta_n) M^{-1} \left( x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \sigma_2(s, x_n(s)) dw_2(s) \right) : n \in \mathbb{N} \), we know that the set \( \left\{ S(\delta_n) M^{-1} \left( x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \sigma_2(s, x_n(s)) dw_2(s) \right) : n \in \mathbb{N} \right\} \) is precompact in \( X_\alpha \) for \( t \in [0, \xi] \). By Lemma 2.2 (2) and the assumption (H3), we know that for every \( n \in \mathbb{N} \) and \( t_1, t_2 \in [0, \xi] \) with \( t_1 < t_2 \),

\[
E \left\| S(t_2) S(\delta_n) M^{-1} \left( x_0 + \frac{1}{\Gamma(1-q)} \int_0^{t_2} (t_2-s)^{-q} \sigma_2(s, x_n(s)) dw_2(s) \right) \right\|^2_a \\
- S(t_1) S(\delta_n) M^{-1} \left( x_0 + \frac{1}{\Gamma(1-q)} \int_0^{t_1} (t_1-s)^{-q} \sigma_2(s, x_n(s)) dw_2(s) \right) \right\|^2_a \\
\leq 3E \left\| S(t_2) - S(t_1) \right\| \left\| S(\delta_n) M^{-1} \left[ \frac{1}{\Gamma(1-q)} \int_0^{t_2} (t_2-s)^{-q} T^a \sigma_2(s, x_n(s)) dw_2(s) \right) \right\|^2_a \\
+ 3E \left\| S(t_2) S(\delta_n) M^{-1} \left[ \frac{1}{\Gamma(1-q)} \int_0^{t_1} (t_2-s)^{-q} - (t_1-s)^{-q} \right) T^a \sigma_2(s, x_n(s)) dw_2(s) \right\|^2_a \\
+ 3E \left\| S(t_2) S(\delta_n) M^{-1} \left[ \frac{1}{\Gamma(1-q)} \int_{t_1}^{t_2} (t_2-s)^{-q} T^a \sigma_2(s, x_n(s)) dw_2(s) \right) \right\|^2_a \\
\rightarrow 0 \quad \text{as} \quad t_2 - t_1 \rightarrow 0.
\]

This means that the set \( \left\{ x_1^1(t) : n \in \mathbb{N} \right\} \) is equicontinuous for \( t \in [0, \xi] \). Therefore, applying Arzela-Ascoli theorem again one obtains that the set \( \left\{ x_n^1 : n \in \mathbb{N} \right\} \) is precompact in \( C([0, \xi], L^2(\Gamma, X_\alpha)) \).

Therefore, we have proved that the set \( \left\{ x_n : n \in \mathbb{N} \right\} \) is precompact in \( C([0, b], L^2(\Gamma, X_\alpha)) \).

Hence, without losing the generality, we may suppose that

\[
x_n \rightarrow x^* \quad \text{in} \quad C([0, b], L^2(\Gamma, X_\alpha)) \quad \text{as} \quad n \rightarrow \infty.
\]

Taking limits in (3.12) one has

\[
x^*(t) = S(t) S(\delta_n) M^{-1} \left[ x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \sigma_2(s, x(s)) dw_2(s) \right) \\
+ \int_0^t (t-s)^{q-1} L^{-1} T_0(t-s) \sigma_1(s, x^*(s), Hx^*(s)) dw_1(s).
\]
for all \( t \in J \), which means that \( x^+ \in C([0,b], L^2(\Gamma, X_\delta)) \) is a \( \alpha \)-mild solution of the problem (1.1)-(1.2) and the proof of Theorem 3.1 is completed.

4 An example

In this section, we present an example, which do not aim at generality but indicate how our abstract result can be applied to concrete problem. Let \( N \geq 1 \) be an integer, \( U \subset \mathbb{R}^N \) be a bounded domain, whose boundary \( \partial \Omega \) is an \( (N-1) \)-dimensional \( C^{2+\mu} \)-manifold for some \( 0 < \mu < 1 \). We consider the nonlocal problem of Sobolev type fractional stochastic parabolic partial differential equation of the form

\[
C D_t^\frac{3}{2} [x(z,t) - x_{zz}(z,t)] - \frac{\partial^2}{\partial z^2} x(z,t) = \frac{\sin \left( z, t, x(z, t) \right) - \int_0^t K(t,s) x(z,s) \, ds}{e^t \cdot dt}, \quad z \in U, \quad t \in J, \quad (4.1)
\]

\[
x(z,0) = \frac{\partial^2}{\partial z^2} \left[ x_0(z) + \frac{1}{\Gamma \left( \frac{3}{2} \right)} \sum_{k=1}^m c_k \int_0^t (t-s)^{-\frac{3}{2}} x(z, t_k) \, \omega_2(s) \right], \quad z \in U \quad (4.2)
\]

where \( C D_t^\frac{3}{2} \) is the Caputo fractional derivative of order \( q \in (0,1) \), \( 0 < t_1 < ... < t_m < b \) and \( c_k \) are positive constants, \( k = 1,...,m \); the functions \( x(t)(z) = x(z,t), \sigma_1(t,x(t),Hx(t))(z) = \frac{\sin \left( z,t,x(z,t) \right) - \int_0^t K(t,s) x(z,s) \, ds}{e^t \cdot dt} \) and \( \sigma_2(t,x(t))(z) = \sum_{k=1}^m c_k x(z,t_k) \); \( \omega_1(t) \) and \( \omega_2(t) \) are two sided and standard one dimensional Brownian motions defined on the filtered probability space \( (\Omega, \Gamma, P) \), \( J = [0,b], K \in C(\Delta, \mathbb{R}^+), \Delta = \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq b \} \).

Let \( X = L^2(U) \), define the operators \( L : D(L) \subset X \rightarrow X, A : D(A) = H^2(U) \cap H_0^1(U) \subset X \rightarrow X \) and \( M : D(M) \subset X \rightarrow X \) by \( Lx = x - x'' \), \( Ax = -x'' \) and \( M^{-1}x = x'' \) where the domains \( D(L), D(A) \) and \( D(M) \) are given by

\[ \{ x \in X : x, x' \text{ are absolutely continuous}, x'' \in X, x|_{\partial U} = 0 \} \]

It is easy to see that \( L^{-1} \) is compact, bounded with \( ||L^{-1}|| \leq 1 \) and \( T = AL^{-1} \) generates the above strongly continuous semigroup \( S(t) \) on \( L^2(U) \) with \( ||S(t)|| \leq e^t \leq 1 \). Therefore, with the above choices, the system (4.1)-(4.2) can be written as an abstract formulation of (1.1)-(1.2).

From the definitions of \( \sigma_1 \) and \( \sigma_2 \), it is easy to verify that \( \sigma_1 : X_0 \times X_0 \rightarrow L_0^2 \) and \( \sigma_2 : X_0 \rightarrow L_0^2 \) whenever \( x \in C(J, L^2(\Gamma, X_0)) \). Moreover, we see that the assumptions (H1)-(H4) and the condition (3.1) hold with

\[ q_1 = \frac{1}{2}, \quad \varphi_\tau(t) = \frac{|U|}{e^t}, \quad \Phi(\tau) = m|U| \sum_{k=1}^m c_k^2 \tau^2, \quad \rho_1 = \rho_2 = 0, \quad \delta = t_1. \]

Therefore, by Theorem 3.1, we have the following result.

**Theorem 4.1** The nonlocal problem of Sobolev type fractional stochastic parabolic partial differential equation has at least one \( 0 \)-mild solution.

References


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