\section*{Abstract}

In this paper, we give the characterizations of \( n \)-power Quasi-isometry and \( n \)-power normal composition operators. Further, we also discuss the characterization of the \( n \)-power Quasi-isometry composite multiplication operator.

\textbf{Keywords:} \( n \)-power quasi-isometry operator, \( n \)-power normal operator, composite multiplication operator.

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\section{Introduction}

Let \( B(H) \) be the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space \( H \). An operator \( A \) is an \( n \)-power quasi-isometry if \( A^{n-1}A^*A^2 = A^*AA^{n-1} \) for all \( n \in \mathbb{Z}^+ \) \cite{5}. The operator \( A \) is normal if \( A^*A = AA^* \) and \( n \)-power normal if \( A^nA^* = A^*A^n \) for all \( n \geq 2 \) \cite{5}. We denote the class of \( n \)-power normal operators and \( n \)-power quasi-isometry operators by \([nN]\) and \([nQI]\) respectively. The class of normal operators \( \subset \text{class}[nN] \). Also \( A \) is an \( n \)-power normal operator if and only if \( A^n \) is normal\cite{5}.

Let \((X, \Sigma, \lambda)\) be a sigma-finite measure space and let \( T \) be a measurable transformation from \( X \) to itself. If \( T \) is a measurable transformation then \( T^n \) is also a measurable transformation. Further, if \( T \) is non-singular then \( \lambda T^{-1} \) is absolutely continuous with respect to \( \lambda \) and it follows that \( \lambda(T^{-1})^n \) is absolutely continuous with respect to \( \lambda \). The Radon-Nikodym derivative of \( \lambda(T^{-1})^n \) with respect to \( \lambda \) is denoted by \( h_n \).

Associated with each transformation \( T \) is a conditional expectation operator \( E(f|T^{-1}(\Sigma)) = E(f) \) is defined for each non-negative function \( f \in L^p \) \((1 \leq p < \infty)\) and is uniquely determined by the following conditions:

(i) \( E(f) \) is \( T^{-1}(\Sigma) \) measurable.

(ii) If \( B \) is any \( T^{-1}(\Sigma) \) measurable set for which \( \int_B f d\lambda \) converges then we have \( \int_B f d\lambda = \int_B E(f) d\lambda \).

The conditional expectation operator \( E \) has the following properties:

(i) \( E(f.(g \circ T)) = E(f).E(g) \)

(ii) \( E \) is monotonically increasing. (i.e) if \( f \leq g \) a.e then \( E(f) \leq E(g) \) a.e.

(iii) \( E(1) = 1 \).

When \( E \) is defined on a possible infinite \( \sigma \)-finite measure space it behaves similarly to expectations on standard probability spaces. As an operator on \( L^2(\lambda) \), \( E \) is the projection operator onto the closure of the range of \( C_T \).

Let \( \pi \) be an essentially bounded function. The multiplication operator \( M_\pi \) on \( L^2(\lambda) \) induced by \( \pi \), is given by \( M_\pi f = \pi.f \) for \( f \in L^2(\lambda) \).

A composition operator \( C_T \) on \( L^2(\lambda) \) is a bounded linear operator given by composition with a map \( T : X \rightarrow X \) as, \( C_T \equiv f \circ T \) for all \( f \in L^2(\lambda) \) and \( C_T^* \) is given by \( C_T^*f = hE(f) \circ T^{-1} \) for all \( f \in L^2(\lambda) \). A weighted
composition operator $W$ is a linear transformation acting on a set of complex valued $\sum$-measurable functions $f$ of the form, $Wf = \omega C_T f$ where $\omega$ is a complex valued $\sum$-measurable function. When $\omega = 1$ we say $W$ is a composition operator.

The adjoint $W^*$ is defined as, $W^* f = h E(\omega f) \circ T^{-1}$ for $f \in L^2(\lambda)$. Also, $w_u = w.(w \circ T).(w \circ T^2)....(w \circ T^{n-1})$. For $f \in L^2(\lambda)$, $W^* f = w_u f \circ T^n$, $W^*_u f = h_u E(w_u,f) \circ T^{-n}$.

A composite multiplication operator is a linear transformation acting on a set of complex valued $\sum$ measurable functions $f$ of the form $M_u T(f) = C_T M_u (f) = (uf) \circ T = u \circ T . f \circ T$ where $u$ is a complex valued, $\sum$ measurable function. In case, $u = 1$ almost everywhere, $M_u T$ becomes a composition operator. The adjoint of $M_u T$ is given by $M^*_u T f = uh.E(f) \circ T^{-1}$.

Various properties of composition operators and weighted composition operators on $L^2$ spaces have been analyzed by many authors. In particular, spectra of composition operators and their generalized Aluthge transformations as weighted composition operators are characterized in [4]. In this paper we study the characterisations of the $[nQI]$ and $[nN]$ class of composition operators. The characterisations of class $[nN]$ operators $A$ are evaluated mainly by the aid of the normality of $A^n$. In [7], the characterisations of $n$-power normal and $n$-power quasinormal composite multiplication operators are studied. We study the characterisations of quasi-isometry and $n$-power quasi-isometry composite multiplication operators.

## 2 Characterization of the class $[nQI]$ composition operators

The following Lemmas of [2] and [8] play an important role in the following Theorems:

**Lemma 2.1.** [2, 8] Let $P$ be the projection of $L^2(\lambda)$ onto $\overline{R(C_T)}$, where $\overline{R(C_T)}$ denotes the closure of the range of $C_T$. Then,

(i) $C_T^* C_T f = hf$ and $C_T C_T^* f = h \circ T P f \ \forall f \in L^2(\lambda)$.

(ii) $\overline{R(C_T)} = \{ f \in L^2(\lambda) : f \text{ is } T^{-1}(\sum) \text{ measurable} \}$.

(iii) If $f$ is $T^{-1}(\sum)$ measurable, and $g$ and $fg$ belong to $L^2(\lambda)$, then $P(fg) = P(fP(g))$ ($f$ need not be in $L^2(\lambda)$).

Also, for $k \in \mathbb{N}$

(iv) $(C_T^k C_T)^* f = h^k f$.

(v) $(C_T C_T^k)^* f = (h \circ T)^k P f$.

(vi) $E$ is the identity operator on $L^2(\lambda)$ if and only if $T^{-1}(\sum) = \sum$.

The following Theorem gives the characterization of $n$-power quasi-isometry operators.

**Theorem 2.1.** Let $C_T \in B(L^2(\lambda))$. Then $C_T$ is in the class $[nQI]$ if and only if $h \circ T^{n-1} E(h) \circ T^{n-2} = h$.

Proof.

$$C_T \in [nQI] \iff C_T^{-1} C_T^2 C_T^3 f = C_T^4 C_T f, \text{ where } C_T f = h.E(f) \circ T^{-1}.$$

$$\iff C_T^{-1} C_T^2 (f \circ T^2) = C_T^4 (f \circ T^n)$$

$$\iff C_T^{-1} C_T^2 [h.E(f \circ T^2) \circ T^{-1}] = h.E(f \circ T^n) \circ T^{-1}$$

$$\iff C_T^{-1} C_T^2 [h, f \circ T] = h, f \circ T^{n-1}$$

$$\iff C_T^{-1} [h, E(h, f \circ T) \circ T^{-1}] = h, f \circ T^{n-1}$$

$$\iff C_T^{-1} [h, E(h) \circ T^{-1}, f] = h, f \circ T^{n-1}$$

$$\iff C_T^{-2} [h, E(h) \circ T^{-1}, f] \circ T = h, f \circ T^{n-1}$$

$$\iff C_T^{-2} [h, T, E(h), f \circ T] = h, f \circ T^{n-1}$$

$$\iff C_T^{-3} [h \circ T^2, E(h) \circ T, f \circ T^2] = h, f \circ T^{n-1}$$

$$\iff C_T^{-4} [h \circ T^3, E(h) \circ T, f \circ T^3] = h, f \circ T^{n-1}$$

$$\iff h \circ T^{n-1} E(h) \circ T^{n-2} = h$$

$$\iff h \circ T^{n-1} E(h) \circ T^{n-2} = h$$


Theorem 2.2. Let $C_T \in B(L^2(\lambda))$. Then $C_T^*$ is in the class $[nQI]$ if and only if $h.E(h) \circ T^{-1} E(h) \circ T^{-2} ... E(h) \circ T^{-(n-2)} E(h) \circ T^{-(n-3)} E(h) \circ T^{-(n-2)} E(f) \circ T^{-(n-1)} = h \circ T.E[h].E[h] \circ T^{-1} ... E[h] \circ T^{-(n-2)} E(f) \circ T^{-(n-1)}.$

Proof.

$C_T^* \in [nQI] \iff C_T^{n-1} C_T^2 C_T^2 f = C_T C_T^{n} f.$ \hspace{1cm} (2.1)

Now,

$$C_T^{n-1} C_T^2 C_T^2 f = C_T^{n-1} C_T^2 h.E(h) \circ T^{-1} E(f) \circ T^{-2} = C_T^{n-1} h \circ T^2 E(h) \circ T.E(f) = C_T^{n-2} h.E[h \circ T^2 E(h) \circ T.E(f)] \circ T^{-1} = C_T^{n-2} h \circ T.E[h].E[f) \circ T^{-1} = C_T^{n-3} h.E[h] \circ T^{-1} h.E[h] \circ T^{-1}.E[f) \circ T^{-2} = C_T^{n-4} h.E[h] \circ T^{-1} E[h] \circ T^{-2}.E[h] \circ T^{-1}.E[f] \circ T^{-3} = C_T^{n-5} h.E[h] \circ T^{-1} E[h] \circ T^{-2}.E[h] \circ T^{-3}.E[h] \circ T^{-2}.E[f] \circ T^{-4} = h.E[h] \circ T^{-1} E[h] \circ T^{-2} ... E[h] \circ T^{-(n-2)}.E[h] \circ T^{-(n-3)}.E[h] \circ T^{-(n-2)}.E[f) \circ T^{-(n-1)}.$

And

$$C_TC_T^{n} f = C_TC_T^{n-1} h.E(f) \circ T^{-1} = C_TC_T^{n-2} h.E[h.E(f) \circ T^{-1}] \circ T^{-1} = C_TC_T^{n-2} h.E[h] \circ T^{-1}.E(f) \circ T^{-2} = C_TC_T^{n-3} h.E[h] \circ T^{-1}.E[h] \circ T^{-2}.E(f) \circ T^{-3} = C_TC_T^{n-4} h.E[h] \circ T^{-1}.E[h] \circ T^{-2} ... E[h] \circ T^{-(n-1)}.E(f) \circ T^{-n} = h \circ T.E[h].E[h] \circ T^{-1} ... E[h] \circ T^{-(n-2)}.E[f) \circ T^{-(n-1)}.$$

Now (2.1) becomes $C_T^* \in [nQI] \iff h.E[h] \circ T^{-1} E[h] \circ T^{-2} ... E[h] \circ T^{-(n-2)}.E[h] \circ T^{-(n-3)}.E[h] \circ T^{-(n-2)}.E(f) \circ T^{-(n-1)} = h \circ T.E[h].E[h] \circ T^{-1} ... E[h] \circ T^{-(n-2)}.E(f) \circ T^{-(n-1)}.$ \hspace{1cm} \(\Box\)

Example 2.1. Let $X = \mathbb{N}$, the set of all natural numbers and $\lambda$ be a counting measure on it. $T : \mathbb{N} \rightarrow \mathbb{N}$ is defined by $T(k) = k + 1, \forall k \in \mathbb{N}$. Since $T^{n-1}(k) = k_1$, where $k_1 \in \mathbb{N}$, $h_2 \circ T^{n-1}(k) = 1$ and $h(k) = 1$, $C_T$ is of class $[nQI]$. Here $C_T$ is the unilateral shift operator on $l^2$.

3 Weighted composition operators of class $[nQI]$

Theorem 3.3. Let $W$ be a weighted composition operator then $W \in [nQI]$ if and only if $w_{n-1} h \circ T^{n-1}.E(w) \circ T^{n-2}.E(h) \circ T^{n-2}.E(w.w_2) \circ T^{n-3}.f \circ T^{n-1} = h.E[w.w_n] \circ T^{-1}.f \circ T^{n-1}.$

Proof.

$W \in [nQI] \iff W^{n-1} W^2 W^2 f = W^* W^n f.$ \hspace{1cm} (3.2)
Consider

\[ W^{n-1}W^2 f = W^{n-1}W^2 W[w \circ T] \]
\[ = W^{n-1}W^2 w[w.f \circ T] \circ T \]
\[ = W^{n-1}W^2 w.f \circ T^2 \]
\[ = W^{n-1}W h.E[w.w_2.f \circ T^2] \circ T^{-1} \]
\[ = W^{n-1}W h.E[w.w_2] \circ T^{-1}.f \circ T \]
\[ = W^{n-1}h.E(w) \circ T^{-1}.E(h) \circ T^{-1}.E(w.w_2) \circ T^{-2}.f \]
\[ = W^{n-2}h.E(w).E(h).E(w.w_2) \circ T^{-1}.f \circ T \]
\[ = W^{n-3}w.w \circ T.h \circ T^2.E(w) \circ T.E(h) \circ T.E(w.w_2).f \circ T^2 \]
\[ = W^{n-4}w.w \circ T.w \circ T^2.h \circ T^3.E(w) \circ T^2.E(h) \circ T^2 \]
\[ E(w.w_2) \circ T.f \circ T^3 \]
\[ = w.w \circ T.w \circ T^2...w \circ T^{n-2}.h \circ T^{n-1}.E(w) \circ T^{n-2}.E(h) \circ T^{n-2}. \]
\[ E(w.w_2) \circ T^{n-3}.f \circ T^{n-1} \]
\[ = w_{n-1}h \circ T^{n-1}.E(w) \circ T^{n-2}.E(h) \circ T^{n-2}. \]
\[ E(w.w_2) \circ T^{n-3}.f \circ T^{n-1} \]

And

\[ W^* W^n f = W^* W^{n-1}_w.f \circ T \]
\[ = W^* W^{n-2}_w.[w.f \circ T] \circ T \]
\[ = W^* W^{n-2}_w w \circ T.f \circ T^2 \]
\[ = W^{n}w.w \circ T...w \circ T^{n-1}.f \circ T^n \]
\[ = h.E[w.w.w \circ T...w \circ T^{n-1}.f \circ T^n] \circ T^{-1} \]
\[ = h.E[w.w_n] \circ T^{-1}.f \circ T^{n-1}. \]

Now \(3.2\) becomes

\[ W \in [nQI] \Leftrightarrow w_{n-1}h \circ T^{n-1}.E(w) \circ T^{n-2}.E(h) \circ T^{n-2}.E(w.w_2) \circ T^{n-3}.f \circ T^{n-1} = h.E[w.w_n] \circ T^{-1}.f \circ T^{n-1}. \]

**Theorem 3.4.** Let \( W \) be a weighted composition operator then \( W^* \in [nQI] \) if and only if \( h.E[w] \circ T^{-1}.E[h] \circ T^{-1}.E[w] \circ T^{-2}.E[h] \circ T^{-2}.E[w] \circ T^{-3}.E[h] \circ T^{-3}...E[w] \circ T^{n-2}.E[h] \circ T^{n-2}.E[w.w_2] \circ T^{-(n-1)}.h \circ T^{-(n-3)}.E[h] \circ T^{-(n-2)}.E[w] \circ T^{-(n-2)}.E(w.f) \circ T^{-(n-1)} = w.h \circ T.E[w].E[h].E[w] \circ T^{-1}.E[h] \circ T^{-1}...E[w] \circ T^{-(n-2)}.E[h] \circ T^{-(n-2)}.E[w] \circ T^{-(n-1)}. \)

**Proof.**

\[ W^* \in [nQI] \Leftrightarrow W^{n-1}W^2 w^2 f = WW^* f. \] (3.3)
\[ W^{n-1}W^2 f = W^{n-1}W^2.E(w) \circ T^{-1}.E(h) \circ T^{-1}.E(wf) \circ T^{-2} \]
\[ = W^{n-1}W.w.h \circ T.E[w].E[h].E(wf) \circ T^{-1} \]
\[ = W^{n-1} w_2.h \circ T^2.E[w] \circ T.E[h] \circ T.E(wf) \]
\[ = W^{n-2}.E[w].w_2.h \circ T^2.E[w] \circ T.E[h] \circ T.E(wf) \circ T^{-1} \]
\[ = W^{n-2}.E[w].w_2.h \circ T^{-1}.h \circ T^2.E[w].E[h].E(wf) \circ T^{-1} \]
\[ = W^{n-3}.E[w].w_2 \circ T^{-1}.E[h] \circ T^{-1}.E[w].w_2 \circ T^{-2}.h. \]
\[ E[w] \circ T^{-1}.E[h] \circ T^{-1}.E(wf) \circ T^{-2} \]
\[ = W^{n-4}.E[w] \circ T^{-1}.E[h] \circ T^{-1}.E[w] \circ T^{-2}.E[h] \circ T^{-2}.E[w].w_2 \circ T^{-3}. \]
\[ h \circ T^{-1}.E[w] \circ T^{-2}.E[h] \circ T^{-2}.E(wf) \circ T^{-3}. \]
\[ = E[w] \circ T^4.E[h] \circ T^{n-2}.E[w].w_2 \circ T^{-(n-1)}.h \circ T^{-(n-3)}.E[w] \circ T^{-(n-2)}.E[h] \circ T^{-(n-1)}.E(wf) \circ T^{-(n-1)}. \]

And,
\[ W^{n^2} f = WW^{n-1}h.E(wf) \circ T^{-1} \]
\[ = WW^{n-2}h.E[w].h.E(wf) \circ T^{-1} \]
\[ = WW^{n-2}h.E[w] \circ T^{-1}.E[h] \circ T^{-1}.E(wf) \circ T^{-2} \]
\[ = WW^{n-3}h.E[w] \circ T^{-1}.E[h] \circ T^{-1}.E[w] \circ T^{-2}.E[h] \circ T^{-2}.E(wf) \circ T^{-3}. \]
\[ Wh.E[w] \circ T^{-1}.E[h] \circ T^{-1}.E[w] \circ T^{-2}.E[h] \circ T^{-2} \]
\[ = E[w] \circ T^{-(n-1)}.E[h] \circ T^{-(n-1)}.E[w] \circ T^{-(n-2)}.E[h] \circ T^{-(n-2)}.E(wf) \circ T^{-(n-1)}. \]

Now (3.3) becomes \( W^* \in [nQ] \Leftrightarrow h.E[w] \circ T^{-1}.E[h] \circ T^{-1}.E[w] \circ T^{-2}.E[h] \circ T^{-2}.E[w] \circ T^{-3}.E[h] \circ T^{-3}.E[w] \circ T^{n-2}.E[h] \circ T^{n-2}.E[w].w_2 \circ T^{-(n-1)}.h \circ T^{-(n-3)}.E[w] \circ T^{-(n-2)}.E[h] \circ T^{-(n-2)}.E(wf) \circ T^{-(n-1)} = w.h \circ E[w].E[h].E[w] \circ T^{-1}.E[h] \circ T^{-1}.E[w] \circ T^{-1}.E[w].w_2 \circ T^{-(n-1)}.E[h] \circ T^{-(n-2)}.E[w] \circ T^{-(n-2)}.E(hf) \circ T^{-(n-1)}. \)

\[ \]

\section{Characterisations of class \([nN]\) composition operators}

In this section we discuss the characterization of \( n \)-power normal composition operators on \( L^2 \) spaces.

\begin{lemma}
\[ \] Let \( \alpha \) and \( \beta \) be non-negative functions with \( S = \text{support } \alpha \). Then the following are equivalent:

(i) For every \( f \in L^2(X, \Sigma, \lambda) \), \( \int_X \alpha |f|^2 \, d\lambda \geq \int_X |E(\beta f)|^2 \, d\lambda \), where \( F \) is a sub-\( \sigma \)-algebra of \( \Sigma \).

(ii) Support \( \beta \subset S \) and \( E(\frac{\beta^2}{\pi} \chi_S |F|) \leq 1 a.e. \)

\end{lemma}

\begin{theorem}
\[ C_T \in B(L^2(\lambda)) \] is of class \([nN]\) if and only if \( h_n > 0 \) and \( E(\chi_{\frac{1}{h_n} T^n}) = \frac{1}{h_n} \chi_{\frac{1}{h_n} T^n} \).

Proof.
\[ \langle C_T^n C_T^n f, f \rangle = \langle h_n f, f \rangle = \int_X h_n |f|^2 \, d\lambda \]
\[ \langle C_T^n C_T^n f, f \rangle = \langle h_n \circ T^n E(f), f \rangle = \int_X h_n \circ T^n E(f) \, d\lambda = \int_X |E(h_n \circ T^n f)|^2 \, d\lambda. \]

Let \( S = \text{support } h_n \). By Lemma 4.2 \( C_T \) is of class \([nN]\) if and only if support \( h_n \circ T^n \subset \text{support } h_n \) and \( E(\chi_{\frac{1}{h_n} T^n}) \leq 1 \).

As \( h_n \circ T^n \geq 0 \), the condition involving supports is true if and only if \( h_n > 0 \) (so that \( \chi_S = 1 \)). The inequality is then equivalent to \( E(\chi_{\frac{1}{h_n} T^n}) = \frac{1}{h_n} \chi_{\frac{1}{h_n} T^n} \) since \( h_n \circ T \) is \( T^{-1}(\Sigma) \) measurable.

\end{theorem}
Theorem 4.6. \( C_T \) is of class \([nN] \Leftrightarrow \left\| h^\frac{1}{2} h_{n-1}^1 f \right\|^2 = \left\| h_{n-1}^1 h^\frac{1}{2} \circ TPf \right\|^2 \)
\( \Leftrightarrow \left\| h^\frac{1}{2} f \right\|^2 = \left\| (h_n \circ T^n)^\frac{1}{2} Pf \right\|^2 \).

(ii) follows directly from (i)

\[ \left\| h^\frac{1}{2} h_{n-1}^2 Pf \right\|^2 = \left\| h_{n-1}^2 h^\frac{1}{2} \circ TPf \right\|^2. \]

Proof. \( 0 = \left( C_T^n C_T^m f, f \circ T^n \right) \Rightarrow \left( C_T^n C_T^m f, f \circ T^n - 1 \right) \]
\[ = \left( h C_T^n f, f \circ T^n \right) - \left( C_T^n - 1 h \circ TPf, f \circ T^n \right) \]
\[ = \left( h_f \circ T^n - 1 f, f \circ T^n - 1 \right) - \left( h \circ TPf \circ T^n - 1 f, f \circ T^n - 1 \right) \]
\[ = \left( h_f \circ T^n - 1 f, f \circ T^n - 1 \right) - \left( Ph \circ T f \circ T^n - 1 f, f \circ T^n - 1 \right) \]
\[ = \int h |f|^2 \circ T^n - 1 d\lambda - \int Ph \circ T |f|^2 \circ T^n - 1 d\lambda \]
\[ = \langle hh_{n-1} - 1 f, f \rangle - \langle Ph \circ Th_{n-1} f, f \rangle \]
\[ = \langle hh_{n-1} - 1 f, f \rangle - \langle h \circ Th_{n-1} f, f \rangle \]
\[ = \langle hh_{n-1} - 1 f, f \rangle - \langle h \circ Th_{n-1} f, f \rangle \]

Since \( C_T \) and \( C_T^n \) commutes \( \forall n \in \mathbb{N}, h, h_n \) also commutes \( \forall n \in \mathbb{N} \).

\[ \left\| h^\frac{1}{2} h_{n-1} f \right\|^2 = \left\| h_{n-1}^2 h^\frac{1}{2} \circ TPf \right\|^2 \]

Also,

\[ \left( C_T^n C_T^m f, f \right) = \left( (h_n - h_T \circ T^n) f, f \right) \]

And hence it follows that \( \left\| h^\frac{1}{2} f \right\|^2 = \left\| (h_n \circ T^n)^\frac{1}{2} Pf \right\|^2 \).

(ii) follows directly from (i)

\[ \square \]

Theorem 4.7. Let \( C_T \) be an \( n \)-power normal composition operator on \( L^2(\lambda) \) then for all \( m > n, f \in L^2 \) and \( i = m - n \) we have

\[ \left( C_T^n f, f \right) = \left( Ph_i \circ Th_{n} f, f \right). \]

(4.4)

Proof. For \( m = n + 1 \), we have

\[ \left( C_T^{n+1} f, f \right) = \left( C_T f, f \right) \]
\[ = \left( h \circ T f, f \right) \]
\[ = \left( Ph \circ Th_{n} f, f \right) \]

Suppose (4.4) holds for \( m = n + 1, n + 2, \ldots n + k, i = 1, 2, \ldots k \) and all \( f \in L^2 \). Then

\[ \left( C_T^{n+k+1} f, f \right) = \left( C_T^{n+k} f, f \right) \]
\[ = \left( h_{k+1} \circ T^{k+1} f, f \right) \]
\[ = \left( Ph_{k+1} \circ T^{k+1} f, f \right) \]

And hence (4.4) follows by induction.

\[ \square \]
Theorem 4.8. Let $C_T$ be an $n$-power normal composition operator on $L^2(\lambda)$ then for all $m > n, f \in L^2$ and $i = m - n$ we have

$$\langle C_T^nC_T^m f, f \rangle = \langle h_nh_n f, f \rangle.$$ \hfill (4.5)

Proof. For $m = n + 1$, we have

$$\langle C_T^{n+1}C_T^{m+1} f, f \rangle = \langle C_T^nC_T^m C_T^n f, f \rangle = \langle h_nh_n f, f \rangle = \langle h_nh_n f, f \rangle$$

Suppose (4.5) holds for $m = n + 1, n + 2, \ldots n + k, i = 1, 2, \ldots k$ and all $f \in L^2$. Then

$$\langle C_T^{n+k+1}C_T^{m+k+1} f, f \rangle = \langle C_T^{k+1}C_T^nC_T^m f, f \rangle = \langle h_{k+1}h_n f, f \rangle = \langle h_{k+1}h_n f, f \rangle$$

And hence (4.5) follows by induction. \hfill \square

Theorem 4.9. Let $C_{T_1}$ and $C_{T_2}$ be $n$-power normal composition operators on $L^2(\lambda)$ then for all $m > n, p > n$, such that $m$ and $p$ are multiples of $n, C_{T_1}^n C_{T_2}^p$ is normal.

Proof. On applying Theorem 4.8 in the subsequent equations the assertion is proved. Denote $C_{T_1}^k C_{T_2}^k$ by $M_{h(1)k}$ and $C_{T_1}^k C_{T_2}^k$ by $M_{h(2)k}$ respectively.

$$\langle (C_{T_1}^n C_{T_2}^p)^* (C_{T_1}^m C_{T_2}^p) f, f \rangle = \langle C_{T_1}^m C_{T_1}^n C_{T_2}^p C_{T_2}^p f, f \rangle = \langle h_{(1)m-n} h_{(1)n} C_{T_2}^p f, C_{T_2}^p f \rangle = \langle h_{(1)m-n} h_{(1)n} C_{T_2}^p f, f \rangle = \langle h_{(1)m-n} h_{(1)n} h(2)_p-n h(2)_n f, f \rangle = \langle h_{(1)m-n} h_{(1)n} h(2)_p-n h(2)_n f, f \rangle$$

$$\langle (C_{T_1}^n C_{T_2}^p)(C_{T_1}^m C_{T_2}^p)^* f, f \rangle = \langle C_{T_1}^m C_{T_2}^p C_{T_1}^n C_{T_2}^p f, C_{T_1}^n f \rangle = \langle h_{(2)p-n} h_{(2)n} C_{T_1}^n C_{T_2}^p f, f \rangle = \langle h_{(1)m-n} h_{(1)n} h(2)_p-n h(2)_n f, f \rangle = \langle h_{(1)m-n} h_{(1)n} h(2)_p-n h(2)_n f, f \rangle$$

From the above equalities it follows that $C_{T_1}^m C_{T_2}^p$ is normal. \hfill \square

Corollary 4.1. Let $C_{T_1}$ and $C_{T_2}$ be $n$-power normal composition operators on $L^2(\lambda)$ then for all $m > n, p > n$, such that $m$ and $p$ are multiples of $n, (C_{T_1}^n C_{T_2}^p)^q, \forall q \in \mathbb{N}$ is normal.

In particular, for $q = n, C_{T_1}^m C_{T_2}^p$ is of class $[nN]$.

Proof. From Theorem 4.9 it is obvious that $(C_{T_1}^m C_{T_2}^p)^q, \forall q \in \mathbb{N}$ is normal. And for $q = n$, it follows from the normality of class $[nN]$ operators that $C_{T_1}^m C_{T_2}^p$ is of class $[nN]$. \hfill \square

Now we establish a close relationship between $\sigma(C_T)$ and $E(h)$ where $E(h)$ denotes the essential range of the Radon-Nikodym derivative $h$.

Theorem 4.10. Let $C_T$ be an $n$-power normal composition operator on $L^2(\lambda)$ then

$$\sigma(C_T) \subset \left\{ \lambda \frac{1}{n} : \lambda^\frac{1}{n} \in \mathbb{C} \text{ and } |\lambda|^{\frac{1}{n}} \in E(h_n)^{\frac{1}{n}} \right\}$$

Proof. $C_T$ is in class $[nN]$ implies $C_T^n$ is normal and hence by Spectral mapping Theorem, for normal operators, $\sigma(C_T^n) = \left\{ |\lambda|^\frac{1}{n} : \lambda \in \sigma(C_T^n) \right\}$. But $C_T^n C_T = M_{h_n}$. 


Therefore $\sigma(M_{h_n}) = \left\{ |\alpha|^\frac{2}{n} : \alpha \in \sigma(C^2_n) \right\}$. Because $\sigma(M_{h_n}) = E(h_n)$. We have,

$$E(h_n) = \left\{ |\alpha|^\frac{2}{n} : \alpha \in \sigma(C^2_n) \right\} = \left\{ |\alpha|^\frac{2}{n} : \alpha \in \sigma(C_T)^n \right\}.$$ 

Thus $\sigma(C_T)^n \subset \left\{ \alpha : \alpha \in C \text{ and } |\alpha|^\frac{2}{n} \in E(h_n) \right\}$, which implies $\sigma(C_T) \subset \left\{ \alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in C \text{ and } |\alpha|^\frac{2}{n} \in E(h_n)^\frac{1}{n} \right\}$.

\[\square\]

**Theorem 4.11.** Let $C_T$ be an $n$-power normal composition operator on $L^2(\lambda)$ such that 1 does not belong to $E(h_n)^\frac{1}{n}$. Then $\sigma(C_T) = \left\{ \alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in C \text{ and } |\alpha|^\frac{2}{n} \in E(h_n)^\frac{1}{n} \right\}$ and hence $\sigma(C_T)$ has cyclic symmetry.

\begin{proof}
From the proof of Theorem 4.10 we have $E(h_n)^\frac{1}{n} = \left\{ |\alpha|^\frac{2}{n} : \alpha \in \sigma(C_T) \right\}$. Thus for every $m \in E(h_n)^\frac{1}{n}$ there is an $\alpha^{\frac{1}{n}} \in \sigma(C_T)$ such that $|\alpha|^\frac{2}{n} = m$. If $\alpha \in \sigma(C_T)$ and $|\alpha| \neq 1$, then every $\beta$ such that $|\alpha| = |\beta|$ is in $\sigma(C_T)^n$. Since by assumption, $1 \notin E(h_n)^\frac{1}{n}$ there is no $\alpha^{\frac{1}{n}} \in \sigma(C_T)$ such that $|\alpha|^\frac{2}{n} = 1$. Hence $\left\{ \alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in C \text{ and } |\alpha|^\frac{2}{n} \in E(h_n)^\frac{1}{n} \right\} \subset \sigma(C_T)$. The opposite inclusion follows from Theorem 4.10 and hence $\sigma(C_T) = \left\{ \alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in C \text{ and } |\alpha|^\frac{2}{n} \in E(h_n)^\frac{1}{n} \right\}$.
\[\square\]

## 5 Quasi-isometry and $n$-power Quasi-isometry Composite multiplication operators

In this section we give a characterization of quasi-isometry and $n$-power quasi-isometry composite multiplication operators.

**Theorem 5.12.** Let $M_{u,T}$ on $L^2(\lambda)$ be a composite multiplication operator, then for $\lambda \geq 0$, $M_{u,T}$ is a quasi-isometry if and only if $u.h.E[uh] \circ T^{-1}.E[u \circ Tu \circ T^2] \circ T^{-2}.f = u^2.h.f$.

\begin{proof}
\begin{align*}
M_{u,T} \text{ is a quasi-isometry} \iff & \quad M_{u,T}^2 M_{u,T}^2 f = M_{u,T}^* M_{u,T} f \\
& \iff M_{u,T}^2 M_{u,T} u \circ T.f \circ T = M_{u,T}^* u \circ T.f \circ T \\
& \iff M_{u,T}^2 [u \circ T(u \circ T.f \circ T) \circ T] = u.h.E[u \circ T.f \circ T] \circ T^{-1} \\
& \iff M_{u,T}^2 [u \circ T.u \circ T^2.f \circ T^2] = u.h.u \circ T^{-1}.f \circ T \circ T^{-1} \\
& \iff M_{u,T}^* u.h.E[u \circ T.u \circ T^2.f \circ T^2] \circ T^{-1} = u.h.u.f \\
& \iff M_{u,T}^* u.h.E[u \circ T.u \circ T^2] \circ T^{-1}.f \circ T = u^2.h.f \\
& \iff u.h.E[u.h.E[u \circ T.u \circ T^2] \circ T^{-1}.f \circ T] \circ T^{-1} = u^2.h.f \\
& \iff u.h.E[uh] \circ T^{-1}.E[u \circ T.u \circ T^2] \circ T^{-2}.f = u^2.h.f.
\end{align*}
\[\square\]

**Corollary 5.2.** $C_T \in B(L^2(\lambda))$ is quasi-isometry if and only if $h.E[h] \circ T^{-1}.f = h.f$.

\begin{proof}
The proof is obtained by putting $u = 1$ in Theorem 5.12
\[\square\]

**Theorem 5.13.** Let $M_{u,T}$ on $L^2(\lambda)$ be a composite multiplication operator, then for $\lambda \geq 0$, $M_{u,T}^*$ is a quasi-isometry if and only if $u \circ T.u \circ T^2.h \circ T^2.E[uh] \circ T.E[f] = u \circ T.u \circ T.h \circ T.E[f]$.
Proof.

\[ M_{u,T}^* \text{ is a quasi-isometry} \iff M_{u,T}^2 M_{u,T}^* f = M_{u,T} M_{u,T}^* f \]
\[ \iff M_{u,T}^2 M_{u,T}^* u.h.E[f] \circ T^{-1} = M_{u,T} u.h.E[f] \circ T^{-1} \]
\[ \iff M_{u,T}^2 u.h.E[u.h.E[f] \circ T^{-1}] \circ T^{-1} = u \circ T.[u.h.E[f] \circ T^{-1}] \circ T \]
\[ \iff M_{u,T}^2 u.h.E[u.h] \circ T^{-2} = u \circ T.h \circ T.E[f] \]
\[ \iff M_{u,T} u \circ T.[u.h.E[u.h] \circ T^{-1}] \circ T.E[f] \circ T^{-2} = u \circ T.u \circ T.h \circ T.E[f] \]
\[ = u \circ T.u \circ T.h \circ T.E[f] \]
\[ \iff M_{u,T} u \circ T.h \circ T.E[u.h].E[f] \circ T^{-1} \]
\[ = u \circ T.u \circ T.h \circ T.E[f] \]
\[ \iff u \circ T[u \circ T.u \circ T.h \circ T.E[u.h].E[f] \circ T^{-1}] \circ T \]
\[ = u \circ T.u \circ T.h \circ T.E[f] \]
\[ \iff u \circ T.u \circ T^2.h \circ T^2.E[u.h] \circ T.E[f] \]
\[ = u \circ T.u \circ T.h \circ T.E[f]. \]

\[ \square \]

**Theorem 5.14.** Let \( M_{u,T} \) on \( L^2(\lambda) \) be a composite multiplication operator, then for \( \lambda \geq 0 \), \( M_{u,T} \) is an \( n \)-power quasi-isometry operator if and only if \( u \circ T.u \circ T^2.u \circ T^3...u \circ T^{n-1}.h \circ T^{n-1}.E[u.h] \circ T^{n-2}E[u \circ T.u \circ T^2] \circ T^{n-3} \circ T^{n-1} = u.h.E[u \circ T.u \circ T^2.u \circ T^3...u \circ T^n] \circ T^{-1}.f \circ T^{n-1} \).

**Proof.**

\( M_{u,T} \) is \( n \)-power quasi-isometry \( \iff M_{u,T}^{n-1} M_{u,T}^2 M_{u,T}^* f = M_{u,T}^* M_{u,T} M_{u,T}^{n-1} f \) (5.6)

Now,

\[ M_{u,T}^{n-1} M_{u,T}^2 M_{u,T}^* f = M_{u,T}^{n-1} u.h.E[u.h] \circ T^{-1}.E[u \circ T.u \circ T^2] \circ T^{-2}.f \]
\[ = M_{u,T}^{n-1} u \circ T[u.h.E[u.h] \circ T^{-1}.E[u \circ T.u \circ T^2] \circ T^{-2}.f] \circ T \]
\[ = M_{u,T}^{n-2} u \circ T.u \circ T.h \circ T.E[u.h].E[u \circ T.u \circ T^2] \circ T^{-1}.f \circ T \]
\[ = M_{u,T}^{n-3} u \circ T.u \circ T.h \circ T.E[u.h].E[u \circ T.u \circ T^2] \circ T^{-1}.f \circ T \]
\[ = M_{u,T}^{n-4} u \circ T.u \circ T^2.h \circ T^2.E[u.h] \circ T.E[u \circ T.u \circ T^2].f \circ T^2. \]

Continuing in a similar manner, we arrive at the following expression,

\[ M_{u,T}^{n-1} M_{u,T}^2 M_{u,T}^* f = u \circ T.u \circ T^2.u \circ T^3...u \circ T^{n-1}.h \circ T^{n-1}.E[u.h] \circ T^{n-2}E[u \circ T.u \circ T^2] \circ T^{n-3}.f \circ T^{n-1}. \]

\[ M_{u,T}^* M_{u,T}^n f = M_{u,T}^* M_{u,T}^{n-1} u.T.f \circ T \]
\[ = M_{u,T}^* M_{u,T}^{n-2} u.T[ u.T.f \circ T] \circ T \]
\[ = M_{u,T}^* M_{u,T}^{n-3} u.T.u \circ T^2.f \circ T^2 \]
\[ = . \]
\[ = . \]
\[ = . \]
\[ = M_{u,T}^* u \circ T.u \circ T^2.u \circ T^3...u \circ T^n.f \circ T^n \]
\[ = u.h.E[u \circ T.u \circ T^2.u \circ T^3...u \circ T^n].f \circ T^{-1} \]
\[ = u.h.E[u \circ T.u \circ T^2.u \circ T^3...u \circ T^n] \circ T^{-1}.f \circ T^{n-1}. \]
Hence equation \([5.6]\) becomes,

\[
M_{u,T} \text{ is } n\text{-power quasi-isometry } \iff u \circ T.u \circ T^2.u \circ T^3...u \circ T^{n-1}.u \circ T^n.
\]

\[
h \circ T^{n-1}.E[u.h] \circ T^{n-2}.E[u \circ T.u \circ T^2] \circ T^{n-3}.f \circ T^{n-1} = u.h.E[u \circ T.u \circ T^2.u \circ T^3...u \circ T^n] \circ T^{-1}.f \circ T^{-n-1}.
\]

**Corollary 5.3.** \(C_T \in B(L^2(\lambda))\) is \(n\text{-power quasi-isometry if and only if } h \circ T^{n-1}.E[h] \circ T^{n-2}.f \circ T^{n-1} = h.f \circ T^{n-1}.\)

**Proof.** The proof is obtained by putting \(u = 1\) in Theorem \([5.14]\). \(\square\)

**Theorem 5.15.** Let \(M_{u,T}\) on \(L^2(\lambda)\) be a composite multiplication operator, then for \(\lambda \geq 0\), \(M^*_{u,T}\) is an \(n\text{-power quasi-isometry operator if and only if } u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}...E[u.h] \circ T^{-(n-2)}.E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-(n-1)}.E[h] \circ T^{-(n-3)}.E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)} = u \circ T.u \circ T.h \circ T.E[u.h],E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}...E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}.E[u.h] \circ T^{-1}.E[f] \circ T^{-2}.
\]

**Proof.**

\[
M^*_{u,T} \text{ is } n\text{-power quasi-isometry } \iff M^{n-1}_{u,T}M^2_{u,T}M^{n-2}_{u,T}f = M_{u,T}M^{n}_{u,T}M^{n-1}_{u,T}f
\]  \(\tag{5.7}\)

Now,

\[
M^{n-1}_{u,T}M^2_{u,T}M^{n-2}_{u,T}f = M^{n-1}_{u,T}[u \circ T.u \circ T^2.u \circ T^2.h \circ T^2.E[u.h] \circ T.E[f]]
\]

\[
= M^{n-2}_{u,T}u.h.E[u \circ T.u \circ T^2.u \circ T^2.h \circ T^2.E[u.h] \circ T.E[f]] \circ T^{-1}
\]

\[
= M^{n-2}_{u,T}u.h.E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-1}.h \circ T.E[u.h].E[f] \circ T^{-1}
\]

\[
= M^{n-3}_{u,T}u.h.E[u.h.E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-1}.
\]

\[
h.E[u.h] \circ T^{-1}.E[f] \circ T^{-2}.
\]

And,

\[
M_{u,T}M^{n}_{u,T} = M_{u,T}M^{n-1}_{u,T}u.h.E[f] \circ T^{-1}
\]

\[
= M_{u,T}M^{n-2}_{u,T}u.h.E[u.h.E[f] \circ T^{-1}] \circ T^{-1}
\]

\[
= M_{u,T}M^{n-2}_{u,T}u.h.E[u.h] \circ T^{-1}.E[f] \circ T^{-2}
\]

\[
= M_{u,T}u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}...E[u.h] \circ T^{-3}...E[u.h] \circ T^{-(n-1)}...E[u.h] \circ T^{-n-1}.E[f] \circ T^{-n-1}
\]

\[
= u \circ T.u \circ T.h \circ T.E[u.h],E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}...E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}
\]

Hence equation \([5.7]\) becomes,

\[
M^*_{u,T} \text{ is } n\text{-power quasi-isometry } \iff u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}...E[u.h] \circ T^{-(n-2)}.E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-1}.f \circ T^{-n-1}
\]
\[ T^{-(n-1)}.E[h] \circ T^{-(n-3)}.E[h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)} = u \circ T. u \circ T. h \circ T.E[u.h].E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2} \]
\[ \ldots E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}. \]

**Corollary 5.4.** \( C^*_\alpha \in B(L^2(\lambda)) \) is \( n \)-power quasi-isometry if and only if
\[ h \circ T^{-1}.E[h] \circ T^{-2} \ldots E[h] \circ T^{-(n-2)}.E[h] \circ T^{-(n-1)} = h \circ T.E[h].E[h] \circ T^{-1}.E[h] \circ T^{-2} \ldots E[h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}. \]

**Proof.** The proof is obtained by putting \( u = 1 \) in Theorem 5.15.

**References**


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