Existence of solutions for nonlinear fuzzy impulsive integrodifferential equations

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Abstract

In this paper, we prove the existence and uniqueness of a nonlinear impulsive fuzzy integrodifferential equations with nonlocal condition. The results are obtained by using Banach fixed-point theorem approach. An example is provided to illustrate the theory.

Keywords: Existence, fuzzy set, fuzzy number, impulsive integrodifferential equation, fixed point theorem.

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1 Introduction

In various fields of engineering and physics, many problems that are related to linear viscoelasticity, nonlinear elasticity have mathematical models and are described by the problems of differential or integral equations or integrodifferential equations. Our work centers on the problems described by the integrodifferential models. It is important to note that when we describe the systems which are functions of space and time by partial differential equations, in some situations, such a formulation may not accurately model the physical system because, while describing the system as a function at a given time, it may fail to take into account the effect of past history. However, one may easily visualize that abrupt changes such as shock, harvesting and disasters may occur in nature. These phenomena are short time perturbations whose duration is negligible in comparison with the duration of the whole evolution process. Consequently, it is natural to assume, in modeling these problems, that these perturbations act instantaneously, that is in the form of impulses. The theory of impulsive differential equation \cite{12, 19} is much richer than the corresponding theory of differential equations without impulsive effects. The impulsive condition

$$\Delta u(t_i) = u(t_i^+) - u(t_i^-) = I_i(u(t_i^-)), \quad i = 1, 2, \ldots, m,$$

is a combination of traditional initial value problems and short-term perturbations whose duration is negligible in comparison with the duration of the process. Liu \cite{13} discussed the iterative methods for the solution of impulsive functional differential systems.

Generally, several systems are mostly related to uncertainty and inaccuracy. The problem of inaccuracy is considered in general an exact science and that of uncertainty is considered as vague or fuzzy and accidental. For fuzzy concepts, recently Diamand and Kloeden \cite{7} established the theory of metric space of fuzzy sets. In particular, Kaleva \cite{14} researched the fuzzy differential equations, Cauchy problem for continuous fuzzy differential equations was studied by Nieto \cite{16}, and Song et al. \cite{18} obtained the global solutions. Park and Han \cite{20} studied the existence and uniqueness theorem for a solution of fuzzy Volterra integral equations by

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using the method of successive approximation. Seikkala [17] proved the existence and uniqueness of the fuzzy solution for the following systems:

\[
\begin{align*}
    u'(t) &= f(t, u(t)), \quad t \in J = [0, a], \\
    u(0) &= u_0,
\end{align*}
\]

where \( f \) is a continuous mapping from \( \mathbb{R}^+ \times \mathbb{R} \) into \( \mathbb{R} \) and \( x_0 \) is a fuzzy number. Recently, the above concept has been extended to the integrodifferential equations by Balasubramaniam and Muralisankar [3]. Balachandran and Durairaj [4], Balachandran and Prakash [5] established the existence of perturbed fuzzy integral equations and fuzzy delay differential equations with nonlocal conditions respectively. Ding and Kandel [11] analyzed a way to combine differential equations with fuzzy sets to form a fuzzy logic systems called a fuzzy dynamical system, which can be regarded to form a fuzzy neutral functional differential equations. The study of abstract nonlocal initial value problems was initiated by Byczewski [4][5][6]. Because it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems.

Fuzzy differential equations are a very important topic both in theory and application, for example, in population models, in engineering, in chaotic systems and in modeling hydraulics. The study of fuzzy differential equations has been initiated as an independent subject in conjunction with fuzzy valued analysis and set valued differential equations. Hukuhara first introduced Hukuhara derivatives of a set-valued mapping in, and that paper was the starting point for the topic of set differential equations and, later, for fuzzy differential equations. The integral of fuzzy mapping was proposed by Dubois and Prade [8][9][10]. The H-differentiability of fuzzy mapping was introduced by Puri and Ralescu [21]. Especially, one always describes a model which possesses hereditary properties by integrodifferential equations in practice. The word "fuzzy" means "vagueness". Fuzziness occurs when the boundary of a piece of information is not clear-cut. Fuzzy set have been introduced by Zadeh [22] as an extension of the classical notion of set. Classical set theory allows the membership of the elements in the set in binary terms, a bivalent condition- an element either belongs or does not belong to the set. Fuzzy set theory permits the gradual assessment of the membership of elements in a set, described with the aid of a membership function valued in the real unit interval \([0, 1]\). Fuzzy sets have been able provide solutions to many real world problems. Fuzzy set theory is an extension of classical set theory where elements have degrees of membership.

Motivated by the above approach, the goal of this paper is to use the fixed point theorem to obtain the fuzzy solution of the nonlinear impulsive neutral integrodifferential equations with nonlocal conditions.

2 Preliminaries

Consider the nonlinear impulsive fuzzy integrodifferential equations with nonlocal condition of the form

\[
\begin{align*}
    \frac{du(t)}{dt} &= a(t)u(t) + \int_0^t k(t, s, u(s))ds + f(t, u(t)), \quad t \in [0, b], \quad t \neq t_k, \\
    u(0) + h(u) &= u_0, \\
    \Delta u(t_k) &= I_k(u(t_k^-)), \quad k = 1, 2, \ldots, m,
\end{align*}
\]

where \( a : J \to \mathcal{E}_N \) is a fuzzy coefficient, \( \mathcal{E}_N \) is the set of all upper semicontinuous convex normal fuzzy numbers with bounded \( \alpha \) - level intervals, \( f : J \times \mathcal{E}_N \to \mathcal{E}_N \), \( h : \mathcal{E}_N \to \mathcal{E}_N \) and \( k : J \times J \times \mathcal{E}_N \to \mathcal{E}_N \) are nonlinear continuous functions, \( \Delta u(t_k) = u(t_k^-) - u(t_k^+) \) and \( I_k \in \mathcal{C}(\mathcal{E}_N, \mathcal{E}_N) \) are continuous functions.

A fuzzy subset of \( \mathcal{R}^n \) is defined in terms of a membership function which assigns to each point \( x \in \mathcal{R}^n \) a grade of membership in the fuzzy set. Such a membership function is denoted by

\[
u : \mathcal{R}^n \to [0, 1].
\]

Throughout this paper, we assume that \( u \) maps \( \mathcal{R}^n \) onto \([0, 1]\), \( [u]^0 \) is a bounded subset of \( \mathcal{R}^n \), \( u \) is upper semicontinuous, and \( u \) is fuzzy convex. We denote by \( \mathcal{E}^n \) the space of all fuzzy subsets \( u \) of \( \mathcal{R}^n \) which are normal, fuzzy convex, and upper semicontinuous fuzzy sets with bounded supports. In particular, \( \mathcal{E}^1 \) denotes the space of all fuzzy subsets \( u \) of \( \mathcal{R} \).

A fuzzy number \( a \) in real line \( \mathcal{R} \) is a fuzzy set characterized by a membership function \( \chi_a \)

\[
\chi_a : \mathcal{R} \to [0, 1].
\]
A fuzzy number $a$ is expressed as

$$a = \int_{x \in \mathbb{R}} \frac{\chi_a}{x}$$

with the understanding that $\chi_a(x) \in [0, 1]$, represents the grade of membership of $x$ in $a$ and $\int$ denotes the union of $\chi_a$.

**Definition 2.1.** A fuzzy number $a \in \mathbb{R}$ is said to be convex if, for any real numbers $x$, $y$, $z$ in $\mathbb{R}$ with $x \leq y \leq z$,

$$\chi_a(y) \geq \min\{\chi_a(x), \chi_a(z)\}.$$  

**Definition 2.2.** The height of a fuzzy set is the largest membership value attained by any point.

**Definition 2.3.** If the height of a fuzzy set equals one, then the fuzzy set is called normal. Thus, a fuzzy number $a \in \mathbb{R}$ is called normal, if the followings holds:

$$\max_x \chi_a(x) = 1.$$  

Let $E_N$ be the set of all upper semicontinuous convex normal fuzzy numbers with bounded $\alpha$-level intervals (see [15]). This means that if $a \in E_N$, then $\alpha-$level set

$$[a]^{\alpha} = \{x \in \mathbb{R} : a(x) \geq \alpha, 0 \leq \alpha \leq 1\},$$

is a closed bounded interval, which we denote by

$$[a]^{\alpha} = [a_q^{\alpha}, a_r^{\alpha}]$$

and there exists a $t_0 \in \mathbb{R}$ such that $a(t_0) = 1$. Two fuzzy numbers $a$ and $b$ are called equal $a = b$, if $\chi_a(x) = \chi_b(x)$, for all $x \in \mathbb{R}$. It follows that

$$a = b \iff [a]^{\alpha} = [b]^{\alpha}, \text{ for all } \alpha \in (0, 1].$$

A fuzzy number $a$ may be decomposed into its level sets through the resolution identity

$$a = \int_0^1 \alpha [a]^{\alpha},$$

where $\alpha [a]^{\alpha}$ is the product of a scalar $\alpha$ with the set $[a]^{\alpha}$ and $\int$ is the union of $[a]^{\alpha}$ with $\alpha$ ranging from 0 to 1.

**Definition 2.4.** The support of a fuzzy set $A$ in the universal set $U$ is a crisp set that contains all the elements of $U$ that have nonzero membership values in $A$, that is,

$$\text{supp}(A) = \{x \in U : \chi_a(x) > 0\},$$

where $\text{supp}(A)$ denotes the support of fuzzy set $A$. Hence the support $\Gamma_a$ of a fuzzy number $a$ is defined, as a special case of level set, by the following:

$$\Gamma_a = \{x : \chi_a(x) > 0\}.$$  

**Definition 2.5.** A fuzzy number $a \in \mathbb{R}$ is said to be positive if $0 < a_1 < a_2$ holds for the support $\Gamma_a = [a_1, a_2]$ of $a$, that is, $\Gamma_a$ is in the positive real line. Similarly, $a$ is called negative if $a_1 \leq a_2 < 0$ and zero if $a_1 \leq 0 \leq a_2$.

**Lemma 2.1.** If $a, b \in E_N$, then for $\alpha \in (0, 1]$,

$$[a + b]^{\alpha} = [a_q^{\alpha} + b_q^{\alpha}, a_r^{\alpha} + b_r^{\alpha}],$$

$$[ab]^{\alpha} = \min\{a_i^{\alpha}b_i^{\alpha}, \max\{a_i^{\alpha}b_i^{\alpha}\}\}, \quad (i, j = q, r),$$

$$[a - b]^{\alpha} = [a_q^{\alpha} - b_q^{\alpha}, a_r^{\alpha} - b_r^{\alpha}].$$
Lemma 2.2. Let \([a^\alpha_q, a^\alpha_r], 0 < \alpha \leq 1\), be a given family of nonempty intervals. If
\[ [a^\alpha_1, a^\alpha_2] \subset [a^\alpha_q, a^\alpha_r] \text{ for } 0 < \alpha \leq \beta, \]
\[ \lim_{k \to \infty} a^\alpha_q, \lim_{k \to \infty} a^\alpha_r = [a^\alpha_q, a^\alpha_r], \]
whenever \((a_k)\) is nondecreasing sequence converting to \(\alpha \in (0, 1]\), then the family \([a^\alpha_q, a^\alpha_r], 0 < \alpha \leq 1\), are the \(\alpha\)-level sets of a fuzzy number \(a \in \mathcal{E}_N\).

Let \(x\) be a point in \(\mathcal{R}^n\) and \(A\) be a nonempty subsets of \(\mathcal{R}^n\). We define the Hausdroff separation of \(B\) from \(A\) by
\[ d(x, A) = \inf\{ \| x - a \| : a \in A \}. \]
Now let \(A\) and \(B\) be nonempty subsets of \(\mathcal{R}^n\). We define the Hausdroff separation of \(B\) from \(A\) by
\[ d^*_H(B, A) = \sup\{ d(b, A) : b \in B \}. \]
In general,
\[ d^*_H(A, B) \neq d^*_H(B, A). \]
We define the Hausdroff distance between nonempty subsets of \(A\) and \(B\) of \(\mathcal{R}^n\) by
\[ d_H(A, B) = \max\{ d^*_H(A, B), d^*_H(B, A) \}. \]
This is now symmetric in \(A\) and \(B\). Consequently,
(a) \(d_H(A, B) \geq 0\) with \(d_H(A, B) = 0\) if and only if \(\overline{A} = \overline{B}\);
(b) \(d_H(A, B) = d_H(B, A)\);
(c) \(d_H(A, B) \leq d_H(A, C) + d_H(C, B)\);
for any nonempty subsets of \(A, B\) and \(C\) of \(\mathcal{R}^n\). The Hausdroff distance is a metric, the Hausdroff metric.
The supremum metric \(d_\infty\) on \(\mathcal{E}^n\) is defined by
\[ d_\infty(u, v) = \sup\{ d_H([u]^\alpha, [v]^\alpha) : \alpha \in (0, 1]\}, \text{ for all } u, v \in \mathcal{E}^n, \]
and is obviously metric on \(\mathcal{E}^n\).

The supremum metric \(\mathcal{H}_1\) on \(C(J, \mathcal{E}^n)\) is defined by
\[ \mathcal{H}_1(x, y) = \sup\{ d_\infty(x(t), y(t)) : t \in J \}, \text{ for all } x, y \in C(J, \mathcal{E}^n)\}. \]

3 Existence Result
In this section, we establish the existence and uniqueness results for the impulsive nonlinear fuzzy integrodifferential equations (2.1) – (2.3). The results are obtained by using the Banach fixed point theorem and fuzzy technique. There exist a finite constants \(l_k, l_k, l_k, l_k\), and \(l_l\) and satisfy a global Lipschitz condition such that
\[
\begin{align*}
\left. \begin{array}{c}
\chi H([f(s, u_1(s))]|^{\alpha}, [f(s, u_2(s))]|^{\alpha}) \leq l_f d_H([u_1(s)]|^{\alpha}, [u_2(s)]|^{\alpha}), \\
\chi H([k(t, s, u_1(s))]|^{\alpha}, [k(t, s, u_2(s))]|^{\alpha}) \leq l_k d_H([u_1(s)]|^{\alpha}, [u_2(s)]|^{\alpha}), \\
\chi H([h(u_1(s))]|^{\alpha}, [h(u_2(s))]|^{\alpha}) \leq l_h d_H([u_1(s)]|^{\alpha}, [u_2(s)]|^{\alpha}), \\
\chi H([I_k(u_1(s_k))]|^{\alpha}, [I_k(u_2(s_k))]|^{\alpha}) \leq l_I d_H([u_1(s)]|^{\alpha}, [u_2(s)]|^{\alpha}), \\
\end{array} \right\} \quad (3.4)
\end{align*}
\]
for all \(u_1(t), u_2(t) \in \mathcal{E}_N\).
A mapping \( u : J \rightarrow \mathcal{E}_N \) of a fuzzy process \( u \), then

\[
[u'(t)]^\alpha = [(u^1_\alpha)'(u^2_\alpha)', \ldots], \quad 0 < \alpha \leq 1.
\]

The fuzzy integral \( \int_a^b u(t)dt \), \( a, b \in J \), is defined by \( \int_a^b [u(t)]^\alpha = [\int_a^b u^1_\alpha, \int_a^b u^2_\alpha] \) provided Lebesgue integrals on the right exist.

**Theorem 3.1.** Let \( b > 0, f, g \) and \( k \) satisfy a global Lipschitz condition, for every \( x_0 \in \mathcal{E}_N \), then the fuzzy neutral integrodifferential equation has a unique solution \( x \in \mathcal{C}(J, \mathcal{E}_N) \).

**Proof.** For each \( u(t) \in \mathcal{E}_N, t \in J \).

\[
(F_0 u(t)) = S(t)[u_0 - h(u)] + \int_0^t S(t-s)\left(\int_0^s k(s,r,u(r))dr\right)ds + \int_0^t S(t-s)f(s,u(s))ds + \sum_{0 < t_k < t} S(t-t_k)I_k(u(t_k^-)),
\]

where \( S(t) \) is a fuzzy number and

\[
[S(t)]^\alpha = [S^1_\alpha(t), S^\alpha_\alpha(t)] = \left[\exp\left\{\int_0^t \alpha_\alpha(s)\right\}, \exp\left\{\int_0^t \alpha_\alpha(s)\right\}\right],
\]

and \( S^i_\alpha(t)(i = q, r) \) is continuous. That is, there exist a constant \( l_s > 0 \) such that \( |S^i_\alpha(t)| \leq l_s \), for all \( t \in J \).

Thus \( F_0 : J \rightarrow \mathcal{E}_N \) is continuous, \( F_0 : \mathcal{C}(J, \mathcal{E}_N) \rightarrow \mathcal{C}(J, \mathcal{E}_N) \). For \( u_1, u_2 \in \mathcal{C}(J, \mathcal{E}_N) \), we have

\[
d_H \left( [(F_0(u_1)(t))]^\alpha, [(F_0(u_2)(t))]^\alpha \right) = d_H \left( [S(t)[u_0 - h(u_1)]]^\alpha + \left[ \int_0^t S(t-s)\left(\int_0^s k(s,r,u_1(r))dr\right)ds \right]^\alpha \right.
\]

\[
+ \left[ \int_0^t S(t-s)f(s,u_1(s))ds \right]^\alpha + \left[ \sum_{0 < t_k < t} S(t-t_k)I_k(u_1(t_k^-)) \right]^\alpha,
\]

\[
[S(t)[u_0 - h(u_2)]]^\alpha + \left[ \int_0^t S(t-s)\left(\int_0^s k(s,r,u_2(r))dr\right)ds \right]^\alpha
\]

\[
+ \left[ \int_0^t S(t-s)f(s,u_2(s))ds \right]^\alpha + \left[ \sum_{0 < t_k < t} S(t-t_k)I_k(u_2(t_k^-)) \right]^\alpha,
\]

\[
\leq d_H \left( [S(t)[u_0 - h(u_1)]]^\alpha, [S(t)[u_0 - h(u_2)]]^\alpha \right) + d_H \left( \left[ \int_0^t S(t-s)\left(\int_0^s k(s,r,u_1(r))dr\right)ds \right]^\alpha, \right.
\]

\[
\left. \left[ \int_0^t S(t-s)\left(\int_0^s k(s,r,u_2(r))dr\right)ds \right]^\alpha \right),
\]

\[
+ d_H \left( \left[ \int_0^t S(t-s)f(s,u_1(s))ds \right]^\alpha, \left[ \int_0^t S(t-s)f(s,u_2(s))ds \right]^\alpha \right),
\]

\[
+ d_H \left( \left[ \sum_{0 < t_k < t} S(t-t_k)I_k(u_1(t_k^-)) \right]^\alpha, \left[ \sum_{0 < t_k < t} S(t-t_k)I_k(u_2(t_k^-)) \right]^\alpha \right).
\]
\[
\begin{align*}
&= d_H \left( [S(t)[u_0] - [S(t)h(u_1)] - [S(t)[u_0] - [S(t)h(u_1)]] \right) \\
&\quad + d_H \left( \left[ \int_0^t S(t-s) \left( \int_0^s k(s, r, u_1(r))dr \right)ds \right]^\alpha, \left[ \int_0^t S(t-s) \left( \int_0^s k(s, r, u_2(r))dr \right)ds \right]^\alpha \right) \\
&\quad + d_H \left( \left[ \int_0^t S(t-s)f(s, u_1(s))ds \right]^\alpha, \left[ \int_0^t S(t-s)f(s, u_2(s))ds \right]^\alpha \right) \\
&\quad + d_H \left( \left[ \sum_{0<t_k<t} S(t-t_k)I_k(u_1(\tau_k)) \right]^\alpha, \left[ \sum_{0<t_k<t} S(t-t_k)I_k(u_2(\tau_k)) \right]^\alpha \right) \\
&= d_H \left( [S(t)h(u_1)]^\alpha, [S(t)h(u_2)]^\alpha \right) + d_H \left( \left[ \int_0^t S(t-s) \left( \int_0^s k(s, r, u_1(r))dr \right)ds \right]^\alpha, \right)
\end{align*}
\]
+ \int_0^t \max \left\{ \left| S_q^\alpha (t - s) \right| f_q^\alpha (s, u_1 (s)) - f_q^\alpha (s, u_2 (s)) \right\} ds + \max \left\{ \left[ \sum_{0 < t_k < t} \left| S_q^\alpha (t - t_k) \right| (I_k)^q_\alpha (u_1 (t_k^-)) - (I_k)^q_\alpha (u_2 (t_k^-)) \right] \right\}.

\leq l_s \max \left\{ \left| h_q^\alpha (u_1) - h_q^\alpha (u_2) \right|, \left| h_q^\alpha (u_1) - h_q^\alpha (u_2) \right| \right\}
+ l_s \int_0^t \max \left\{ \left| \int_0^s k_q^\alpha (s, r, u_1 (r)) dr - \int_0^s k_q^\alpha (s, r, u_2 (r)) dr \right| \right\} ds
+ l_s \int_0^t \max \left\{ \left| f_q^\alpha (s, u_1 (s)) - f_q^\alpha (s, u_2 (s)) \right|, \left| f_q^\alpha (s, u_1 (s)) - f_q^\alpha (s, u_2 (s)) \right| \right\} ds
+ l_s \max \left\{ \left[ \sum_{0 < t_k < t} \left| (I_k)^q_\alpha (u_1 (t_k^-)) - (I_k)^q_\alpha (u_2 (t_k^-)) \right| \right\},
\leq l_s l_d H ([u_1], [u_2]) + l_s l_d d_H ([u_1], [u_2]) ds + l_s l_d d_H ([u_1], [u_2]) ds
+ l_s l_d d_H ([u_1], [u_2]) = l_s (l_d + l_f) d_H ([u_1], [u_2]) + l_s (l_k b + l_f) \int_0^t d_H ([u_1], [u_2]) ds
= L d_H ([u_1], [u_2]) + L \int_0^t d_H ([u_1], [u_2]) ds,

where \( L_1 = l_s (l_d + l_f) \), \( L_2 = l_s (l_k b + l_f) \). Therefore

\[ d\infty (\mathcal{F}_0 u_1, \mathcal{F}_0 u_2) = \sup_{t \in J} d_H ([\mathcal{F}_0 u_1]^\alpha, ([\mathcal{F}_0 u_2]^\alpha) \]
\leq \sup_{t \in J} \left\{ L_1 d_H ([u_1], [u_2]) + L_2 \int_0^t d_H ([u_1], [u_2]) ds \right\}
= L_1 \sup_{t \in J} d_H ([u_1], [u_2]) + L_2 \sup_{t \in J} \int_0^t d_H ([u_1], [u_2]) ds
\leq L_1 d\infty ([u_1], [u_2]) + L_2 \int_0^t d\infty ([u_1], [u_2]) ds.

Hence

\[ H_1 (\mathcal{F}_0 u_1, \mathcal{F}_0 u_2) = \sup_{t \in J} d_H (\mathcal{F}_0 u_1, \mathcal{F}_0 u_2) \]
\leq \sup_{t \in J} \left\{ L_1 d\infty ([u_1], [u_2]) + L_2 \int_0^t d\infty ([u_1], [u_2]) ds \right\}
= L_1 \sup_{t \in J} d\infty ([u_1], [u_2]) + L_2 \sup_{t \in J} \int_0^t d\infty ([u_1], [u_2]) ds
\leq (L_1 + L_2 b) H_1 (u_1, u_2).

We take sufficiently small \( b \), \( (L_1 + L_2 b) < 1 \). Hence, \( \mathcal{F}_0 \) is a contraction mapping. By the Banach fixed point theorem, impulsively fuzzy integro-differential equation with nonlocal condition has a unique fixed point \( u \in \mathcal{C}(J, \mathcal{E}_N) \).
4 Example

Consider the fuzzy solution of the nonlinear fuzzy neutral integrodifferential equation of the form:

\[
\frac{du(t)}{dt} = 2u(t) + 2tu(t)^2 + 2tu(t)^2, \quad t \in J, \tag{4.1}
\]

\[
u(0) = \sum_{k=1}^{p} u(t_k), \tag{4.2}
\]

\[
\Delta u(t_k) = I_k(u(t_k)) = \frac{1}{1+u(t_k)}. \tag{4.3}
\]

The \( \alpha \)-level set of fuzzy number \( 2 \) is

\[
[2]^\alpha = [\alpha + 1, 3 - \alpha], \text{ for } \alpha \in [0, 1]
\]

Let \( \int_0^t k(t, s, u(s))ds = 2tu(t)^2, f(t, u(t)) = 2tu(t)^2, h(u(t)) = \sum_{k=1}^{p} u(t_k) \). Then \( \alpha \)-level set of \( \int_0^t k(t, s, u(s))ds = 2tu(t)^2 \) is

\[
\left[ \int_0^t k(t, s, u(s))ds \right]^\alpha = [2tu(t)^2]^\alpha
\]

\[
= t[2]^\alpha[u(t)^2]^\alpha
\]

\[
= t[\alpha + 1, 3 - \alpha][(u^q_0(t))^2, (u^0_0(t))^2]
\]

\[
= t[(\alpha + 1)(u^q_0(t))^2, (3 - \alpha)(u^0_0(t))^2],
\]

where \( [u(t)]^\alpha = [u^q_0(t), u^0_0(t)] \) and \( [2]^\alpha = [\alpha + 1, 3 - \alpha], \) for \( \alpha \in [0, 1] \) and the \( \alpha \)-level set of \( f(t, x(t)) \) is

\[
[f(t, u(t))]^\alpha = [2tu(t)^2]^\alpha
\]

\[
= t[2]^\alpha[u(t)^2]^\alpha
\]

\[
= t[\alpha + 1, 3 - \alpha][(u^q_0(t))^2, (u^0_0(t))^2]
\]

\[
= t[(\alpha + 1)(u^q_0(t))^2, (3 - \alpha)(u^0_0(t))^2],
\]

where \( [u(t)]^\alpha = [u^q_0(t), u^0_0(t)] \) and \( [2]^\alpha = [\alpha + 1, 3 - \alpha], \) for \( \alpha \in [0, 1] \) and the \( \alpha \)-level set of \( h(u(t)) \) is

\[
[h(u(t))]^\alpha = \sum_{k=1}^{p} c_k u(t_k)]^\alpha
\]

\[
= \sum_{k=1}^{p} c_k u^q_0(t_k), \sum_{k=1}^{p} c_k u^0_0(t_k)
\]

and \( \alpha \)-level set of \( I(u(t)) \) is

\[
[I(u(t))]^\alpha = \left[ \frac{1}{1+u(t_k)} \right]^\alpha
\]

\[
= \left[ \frac{1}{1+u^q_0(t_k) \cdot 1+u^0_0(t_k)} \right].
\]

Thus

\[
d_H\left([I(u(t))]^\alpha, [I(v(t))]^\alpha\right)
\]

\[
= d_H\left(\left[\frac{1}{1+u(t_k)}\right]^\alpha, \left[\frac{1}{1+v(t_k)}\right]^\alpha\right)
\]

\[
= d_H\left(\left[\frac{1}{1+u^q_0(t_k)} \cdot \frac{1}{1+u^0_0(t_k)}\right], \left[\frac{1}{1+v^q_0(t_k) \cdot 1+v^0_0(t_k)}\right]\right).
\]
Therefore

\[
\begin{align*}
0 & \leq \max_k \left\{ \left| \frac{1}{1 + u^p_0(t)} - \frac{1}{1 + v^p_0(t)} \right|, \left| \frac{1}{1 + u^p_0(t)} - \frac{1}{1 + v^p(t)} \right| \right\} \\
& \leq \max_k \left\{ \left| \frac{u^p_0(t) - v^p_0(t_k)}{(1 + v^p_0(t_k))(1 + u^p_0(t_k))} \right|, \left| \frac{u^p_0(t) - v^p(t_k)}{(1 + v^p(t_k))(1 + u^p(t_k))} \right| \right\} \\
& \leq \max_k \left\{ \left| \frac{u^p_0(t_k) - v^p_0(t_k)}{1 + |v^p_0(t_k)|} \right|, \left| \frac{u^p_0(t_k) - v^p(t_k)}{1 + |v^p(t_k)|} \right| \right\} \\
& \leq l_k \max_k \{|u^p_0 - v^p_0|, |u^p - v^p|\} \\
& = l_k d_H([u]^\alpha, [v]^\alpha).
\end{align*}
\]

\[
\begin{align*}
d_H([f(t, u(t))]^\alpha, [f(t, v(t))]^\alpha) &= d_H(t([\alpha + 1](u^p_0(t))^2, (3 - \alpha)(u^p_0(t))^2], \\
&= t \max\{|\alpha + 1|u^p_0(t) - (v^p_0(t))|^2, (3 - \alpha)|(u^p_0(t))^2 - (v^p(t))^2|\} \\
&= t \max\{|\alpha + 1|u^p_0(t) + v^p_0(t)|u^p_0(t) - v^p_0(t)|, \\
&= (3 - \alpha)|u^p_0(t)|u^p_0(t) - v^p_0(t)| \\
&\leq (3 - \alpha)t|u^p_0(t)| + (v^p_0(t)) \max\{|u^p_0(t)| - (v^p_0(t)), |u^p_0(t)| - (v^p(t))\} \\
&\leq (3 - \alpha)\beta|u^p_0(t)| + (v^p_0(t)) \max\{|u^p_0(t)| - (v^p_0(t)), |u^p_0(t)| - (v^p(t))\} \\
&\leq 3\beta|u^p_0(t)| + (v^p_0(t)) \max\{|u^p_0(t)| - (v^p_0(t)), |u^p_0(t)| - (v^p(t))\} \\
&= l_\beta d_H([u]^\beta, [v]^\beta).
\end{align*}
\]

Therefore

\[
\begin{align*}
\begin{align*}
d_H([h_1(u)]^\alpha, [h_1(v)]^\alpha) &= d_H(\left\{ \sum_{k=1}^{p} c_k u(t_k) \right\}^\alpha, \left\{ \sum_{k=1}^{p} c_k v(t_k) \right\}^\alpha) \\
&= d_H(\left\{ \sum_{k=1}^{p} c_k u(t_k) \right\}^\alpha, \left\{ \sum_{k=1}^{p} c_k v(t_k) \right\}^\alpha) \\
&\leq \left| \sum_{k=1}^{p} c_k \right| \max_k \{|u^p_0(t_k) - v^p_0(t_k)|, |u^p_0(t_k) - v^p(t_k)|\} \\
&\leq l_k \beta d_H([u]^\beta, [v]^\beta) \\
&\leq l_\beta d_H([u]^\beta, [v]^\beta),
\end{align*}
\end{align*}
\]

where $l_f$, $l_\beta$, $l_h$ and $l_l$ are constant and satisfies the equation (3.4). Since $f$, $k$ and $h$ are satisfy the global Lipschitz condition, and choose $b$ such that $b < (1 - l_\beta(l_f + l_k + l_h + l_l))/l_\alpha$. Then all conditions stated in Theorem 3.1 are satisfied, so problem (4.1) – (4.3) has a unique fuzzy solution.
References


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