Existence of solutions for fractional impulsive integro-differential systems

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Abstract

In this manuscript, we establish the existence results for abstract Cauchy problem for a class of mixed type impulsive fractional functional integro-differential equations involving Caputo fractional derivative in the order $\alpha \in (0, 1]$. The results are obtained by using fractional calculus theory and fixed point techniques.

Keywords: Impulsive differential equations, Caputo fractional derivatives, existence, fixed point theorem.

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1 Introduction

The theory of impulsive differential equations has become and important area of investigation in recent years stimulated by their various real-world problems in physical and engineering sciences subject to abrupt changes at certain instants during the evolution process. There has been significant improvement in impulsive theory especially in the area of impulsive differential equations with fixed moments (see [4, 21]) and applications (see [2, 5, 12, 15, 19, 29]) and references therein.

During the last few decades, differential equations in fractional order with impulsive conditions as an interesting and improvement field of research. Fractional differential and integro-differential equations occur from different phenomena in many engineering and applied sciences such as control theory, aerodynamics, the fluid-dynamic traffic model, polymer rheology and fitting experimental data etc. For more details about fractional calculus theory and applications we refer the monographs [20, 22, 24, 25, 30] and the papers of [6, 7, 13, 14, 15, 16, 17, 23, 28, 31, 32, 33, 37].

In the past few years, the existence, uniqueness and some other properties of mild solutions of differential and fractional evolution equations (1.1)-(1.3) with (or without impulsive) $f(t, u(t), Tu(t), Su(t)) = f(t, u(t))$ have been extensively studied by various authors by using different types of fixed point theorems, when $T(t), t \geq 0$ is a compact semigroup (see [1, 9, 10, 11, 26, 27, 31, 33]) and recently, Feckan et al. [8] give a counter example to show that the formula of solutions for impulsive fractional differential equations. Motivated by above discussion and inspired by above mentioned works [1, 8, 26, 27], in this paper, we are concern the existence results for the abstract mixed type impulsive fractional semilinear evolution equation of the form:

\[ C^{\alpha}D_{0+}^{\alpha}u(t) = f(t, u(t), Tu(t), Su(t)), \quad t \in J' := J \setminus \{t_1, \cdots, t_m\}, J := [0, T], \]
\[ u(t_k^+) = u(t_k^-) + y_k, \quad k = 1, 2, \cdots, m \]
\[ u(0) = u_0 \]

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where $^{C}D_{0}^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$ with the lower limit zero, $u_{0} \in R$, $f : J \times R \times R \rightarrow R$ is jointly continuous, $T$ and $S$ are integral operators is given by

$$Tu(t) = \int_{0}^{t} K(t,s)u(s)ds, \quad K \in C[D,R^{+}],$$

$$Su(t) = \int_{0}^{t} H(t,s)u(s)ds, \quad H \in C[D_{0}, R^{+}]$$

$D = \{(t,s) \in R^{2} : 0 \leq s \leq t \leq T\}$, $D_{0} = \{(t,s) \in R^{2} : 0 \leq t, s \leq T\}$, $I_{k} : R \rightarrow R$ and $t_{k}$ satisfy $0 = t_{0} < t_{1} < \cdots < t_{m} < t_{m+1} = T$, $u(t_{k}^{+}) = \lim_{\epsilon \rightarrow 0+} u(t_{k} + \epsilon)$ and $u(t_{k}^{-}) = \lim_{\epsilon \rightarrow 0-} u(t_{k} - \epsilon)$ represent the right and left limits of $u(t)$ at $t = t_{k}$.

In Section 2, we give some notations, recall some concepts and preparation results, and concept of a piecewise continuous solution for our problem. In Section 3, the main results are obtained by using Banach contraction principle and the Schaefer’s fixed point theorem. Conclusions are given to explain our work in this paper.

## 2 Preliminaries

To start with this section, we need some basic definitions and preliminary facts and properties from the fractional calculus theory. Throughout this paper, let $C(J,R)$ be the Banach space of all continuous functions from $J$ into $R$ with the norm $||u||_{c} := sup \{ ||u(t)|| : t \in J \}$ for $u \in C(J,R)$. We also introduce the Banach space $PC(J,R) = \{ u : J \rightarrow R : u \in C((t_{k}, t_{k+1}), R), k = 0, \ldots, m and there exists $u(t_{k}^{+}), u(t_{k}^{-}), k = 1, \ldots, m$ with $u(t_{k}^{+}) = u(t_{k}) \}$ with the norm $||u||_{PC} := sup \{ ||u(t)|| : t \in J \}$. Denote $PC^{1}(J,R) \equiv \{ u \in PC(J,R) \mid u \in PC(J,R) \}$. Set $||u||_{PC^{1}} = ||u||_{PC} + ||\dot{u}||_{PC}$. It can be seen that endowed with the norm $||.||_{PC^{1}}$, $PC^{1}(J,R)$ is also Banach space.

Let us recall the following known definitions. For more details [20, 24, 25].

**Definition 2.1** The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in N$, is defined as

$$(R-L)D_{0}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^{n} \int_{0}^{t} (t-s)^{n-\alpha-1} f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

**Definition 2.2** The Caputo derivative of order $\alpha$ for a function $f : [0, \infty) \rightarrow R$ can be written as

$$D^{\alpha}f(t) = D^{\alpha} \left( f(t) - \sum_{k=0}^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor} \frac{t^{k}}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n - 1 < \alpha < n.$$

**Remark 2.1** (i) If $f(t) \in C^{n}[0, \infty)$, then

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} \frac{f^{(n)}(s)}{(t-s)^{n-\alpha+1}}ds = I^{n-\alpha}f^{(n)}(t), \quad t > 0, \quad n - 1 < \alpha < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If $f$ is an abstract function with values in $X$, then integrals which appear in Definition 2.1 and 2.2 are taken in Bochner’s sense.

**Definition 2.3** ([5]) A function $u \in PC^{1}(J,R)$ is said to be a solution of the problem (1.1)-(1.3) if it satisfies the following integral equation

$$u(t) = \begin{cases} 
    u_{0} + \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} f(s, u(s), Tu(s), Su(s))ds, & \text{for } t \in [0, t_{1}), \\
    u_{0} + y_{1} + \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} f(s, u(s), Tu(s), Su(s))ds, & \text{for } t \in (t_{1}, t_{2}), \\
    u_{0} + y_{1} + y_{2} + \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} f(s, u(s), Tu(s), Su(s))ds, & \text{for } t \in (t_{2}, t_{3}), \\
    \vdots & \vdots \\vdots \\
    u_{0} + \sum_{i=1}^{m} y_{i} + \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} f(s, u(s), Tu(s), Su(s))ds, & \text{for } t \in (t_{m}, T). 
\end{cases}$$
**Remark 2.2** For measurable functions $m : J \to R$, define the norm

$$
\|m\|_{L^p(J)} = \left\{ \begin{array}{ll}
(f_J |m(t)|^p dt)^{\frac{1}{p}}, & 1 \leq p < \infty, \\
\inf_{t_0(J)} \{ \sup_{t \in J} |m(t)| \}, & p = \infty
\end{array} \right.
$$

where $\mu(J)$ is the Lebesgue measure on $J$. Let $L^p(J, R)$ be the Banach space of all Lebesgue measurable functions $m : J \to R$ with $\|m\|_{L^p(J)} < \infty$.

**Theorem 2.1** (Hölder inequality): Assume that $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $l \in L^p(J, R)$, $m \in L^q(J, R)$, then for $1 \leq q < \infty$, $lm \in L^1(J, R)$ and

$$
\|lm\|_{L^1(J)} \leq \|l\|_{L^p(J)} \|m\|_{L^q(J)}
$$

The following generalized Gronwall inequality with Caputo singular type kernel which is introduce by Ye et al. [36] and can be used in fractional differential equations with initial value conditions.

**Theorem 2.2** [36, Theorem 1]. Suppose $\beta > 0$, $\bar{a}(t)$ is a nonnegative function locally integrable on $[a, b)$ and $\bar{g}$ is a nonnegative, nondecreasing continuous function defined on $\bar{g} \leq M, t \in [a, b)$, and suppose $y(t)$ is nonnegative and locally integrable on $[a, b)$ with

$$
y(t) \leq \bar{a}(t) + \bar{g}(t) \int_0^t (t - s)^{\beta - 1} y(s) ds, \quad t \in [a, b).
$$

Then

$$
y(t) \leq \bar{a}(t) + \int_0^t \left[ \sum_{n=1}^{\infty} \frac{(\bar{g}(t)\Gamma(\beta))^{n}}{\Gamma(n\beta)}(t - s)^{n\beta - 1} \bar{a}(s) \right] ds, \quad t \in [a, b).
$$

**Remark 2.3** Under the hypothesis of Theorem 2.2, let $\bar{a}(t)$ be a nondecreasing function on $[a, b)$. Then we have

$$
y(t) \leq \bar{a}(t)E_\beta(\bar{g}(t)\Gamma(\beta) t^{\beta})
$$

where $E_\beta$ is the Mittag-Leffler function [20] defined by

$$
E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + 1)}, \quad z \in \mathbb{C}, \quad \Re(\beta) > 0
$$

To end this section, we collect some useful theorems including PC-type Ascoli - Arzela theorem, Schaefer’s fixed point theorem and nonlinear alternative of Leray - Schauder type which are used in the sequel.

**Theorem 2.3** (PC-type Ascoli - Arzela theorem, Theorem 2.1 of [34]). Let $X$ be a Banach space and $W \subset PC(J, X)$. If the following conditions are satisfied:

(i) $W$ is uniformly bounded subset of $PC(J, X)$;

(ii) $W$ is equicontinuous in $(t_k, t_{k+1}), k = 0, 1, 2, \ldots, m$, where $t_0 = 0, t_{m+1} = T$;

(iii) $W(t) \equiv \{ u(t) | u \in W; t \in J \backslash \{t_1, \ldots, t_m\} \}, W(t^+_k) = \{ u(t^+_k) | u \in W \}$ and $W(t^-_k) = \{ u(t^-_k) | u \in W \}$ is a relatively compact subsets of $X$.

Then $W$ is a relatively compact subset of $PC(J, X)$.

**Theorem 2.4** (Schaefer’s fixed point theorem). Let $X$ be a Banach space and $F : X \to X$ be a completely continuous operator. If the set $E(F) = \{ y \in X : y = \lambda Fy \text{ for some } \lambda \in [0, 1] \}$ is bounded, then $F$ has at least a fixed point.
3 Existence results

This section deals with the existence and uniqueness of solutions for the problem (1.1)-(1.3). Before starting and proving the main results, we introduce the following hypotheses.

(H1) \( f : J \times R \times R \times R \rightarrow R \) is jointly continuous.

(H2) There exists \( q_1 \in (0, q) \) and a real function \( m(.) \in L^\frac{1}{q_1} (J, R) \) such that \( |f(t, x, y, z)| \leq m(t) \), for all \( x, y, z \in R \).

(H3) There exists \( q_2 \in (0, q) \) and a real function \( h(\cdot, \cdot, \cdot) \in L^\frac{1}{q_2} (J, R) \) such that \( |f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq h(t) \|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\| \), for all \( x_1, y_1, z_1 \in R, i = 1, 2 \).

For brevity, let
\[
\gamma = \frac{T^q}{\Gamma(q+1)}, \quad \beta = \frac{q-1}{1-q_1}, \quad \alpha = \frac{q-1}{1-q_2}, \quad K^* = \sup_{t \in [0, T]} |K(t, s)| < \infty,
\]
\[
H^* = \sup_{t \in [0, T]} |H(t, s)| < \infty, \quad T^* = 1 + K^* T + H^* T.
\]

Theorem 3.5 Assume that [H1]-[H3] hold. If
\[
\frac{T^{\alpha+1} |h|}{\Gamma(q)(1+\beta)^{1-q_2}} < 1,
\]
then the problem (1.1)-(1.3) has an unique solution on the interval \( J \).

Proof 1 Transform the problem (1.1)-(1.3) into a fixed point problem. Consider the operator \( F : PC(J, R) \rightarrow PC(J, R) \) defined by
\[
(Fu)(t) = u_0 + \sum_{i=1}^{k} y_i \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} f(s, u(s), Tu(s), Su(s)) ds.
\]

It is obvious that \( F \) is well defined due to [H1].

Step 1. \( Fu \in PC(J, R) \) for every \( u \in PC(J, R) \). If \( t \in [0, t_1] \), then for every \( u \in C([0, t_1], R) \) and any \( \delta > 0 \), by using Hölder inequality, we get
\[
|(Fu)(t+\delta) - (Fu)(t)| \leq \frac{1}{\Gamma(q)} \left| \int_{0}^{t} ((t+\delta-s)^{q-1} - (t-s)^{q-1}) f(s, u(s), Tu(s), Su(s)) ds \right|
\]
\[
+ \frac{1}{\Gamma(q)} \left| \int_{t}^{t+\delta} (t+\delta-s)^{q-1} f(s, u(s), Tu(s), Su(s)) ds \right|
\]
\[
\leq \frac{1}{\Gamma(q)} \left( \int_{0}^{t} ((t+\delta-s)^{q-1} - (t-s)^{q-1})^{\frac{1}{1-q_1}} ds \right)^{1-q_1} \left( \int_{0}^{t} (m(s))^{\frac{1}{q_1}} ds \right)^{q_1}
\]
\[
+ \frac{1}{\Gamma(q)} \left( \int_{t}^{t+\delta} (t+\delta-s)^{q-1}^{\frac{1}{1-q_1}} ds \right)^{1-q_1} \left( \int_{t}^{t+\delta} m(s)^{\frac{1}{q_1}} ds \right)^{q_1}
\]

which implies that
\[
|(Fu)(t+\delta) - (Fu)(t)| \leq \frac{2\delta^{(1+\beta)(1-q_1)}}{\Gamma(q)(1+\beta)^{1-q_1} \|m\|_{L^\frac{1}{q_1}([0, t_1])}}
\]

It is easy to see that the right-hand side of the above inequality tends to zero as \( \delta \to 0 \). Thus, \( Fu \in C([0, t_1], R) \). If \( t \in (t_1, t_2] \), then for every \( u \in C((t_1, t_2], R) \) and any \( \delta > 0 \), repeating the same process, one can obtain
\[
|(Fu)(t+\delta) - (Fu)(t)| \leq \frac{2\delta^{(1+\beta)(1-q_1)}}{\Gamma(q)(1+\beta)^{1-q_1} \|m\|_{L^\frac{1}{q_1}([t_1, t_2])}}
\]
which implies that $Fu \in C((t_1, t_2], R)$. If $t \in (t_k, t_{k+1}], k = 1, 2, \ldots, m$, then for every $u \in C((t_k, t_{k+1}], R)$ and any $\delta > 0$, repeating the same process again, one can obtain

$$|(Fu)(t+\delta) - (Fu)(t)| \leq \frac{2\delta^{(1+\beta)(1-q_1)}||m||_{L^{\frac{1}{q_1}}((t_k, t_{k+1})}}{\Gamma(q)(1+\beta)^{1-q_1}}$$

which implies that $Fu \in C((t_k, t_{k+1}], R)$. From the above discussion, we must have $Fu \in PC(J, R)$ for every $u \in PC(J, R)$.

**Step 2.** $F$ is a contraction operator on $PC(J, R)$.
In fact, for arbitrary $u_1, u_2 \in PC(J, R)$, we get

$$|(Fu_1)(t) - (Fu_2)(t)| \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}|f(s, u_1(s), Tu_1(s), Su_1(s)) - f(s, u_2(s), Tu_2(s), Su_2(s))|ds$$

$$\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}h(s) \left[ ||u_1(s) - u_2(s)|| + ||Tu_1(s) - Tu_2(s)|| + ||Su_1(s) - Su_2(s)|| \right] ds.$$

Now

$$h(t)||Tu_1(s) - Tu_2(s)|| \leq h(t)K^* ||u_1 - u_2||_{PC}$$

Similarly

$$h(t)||Su_1(s) - Su_2(s)|| \leq h(t) \int_0^t |H(t, s) (u_1(s) - u_2(s))| ds$$

$$\leq h(t)H^* ||u_1 - u_2||_{PC}.$$

Therefore

$$|(Fu_1)(t) - (Fu_2)(t)| \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}h(s)||u_1(s) - u_2(s)||_{PC} [1 + K^* T + H^* T] ds$$

$$\leq \frac{[1 + K^* T + H^* T]}{\Gamma(q)} ||u_1 - u_2||_{PC} \int_0^t (t-s)^{q-1}h(s)ds$$

$$\leq \frac{T^* ||u_1 - u_2||_{PC}}{\Gamma(q)}$$

$$\times \left\{ \left( \int_0^t (t-s)^{q-1} \frac{1}{1-q_2} \right)^{1-q_2} \left( \int_0^t (h(s)) \frac{1}{q_2} \right)^{q_2} \right\}$$

$$\leq \frac{T^* ||u_1 - u_2||_{PC}}{\Gamma(q)} \left\{ \int_0^t ((t-s)^{\alpha})^{1-q_2} ||h||_{L^{\frac{1}{q_2}}(J)} \right\}$$

$$\leq \frac{T^* ||u_1 - u_2||_{PC} T^{(\alpha+1)(1-q_2)}}{(\alpha + 1)^{1-q_2}} ||h||_{L^{\frac{1}{q_2}}(J)}$$

$$\leq \frac{T^* T^{(\alpha+1)(1-q_2)} ||h||_{L^{\frac{1}{q_2}}(J)} ||u_1 - u_2||_{PC} \Gamma(q)(\alpha + 1)^{1-q_2}}{\Gamma(q)(1+\beta)^{1-q_1}}$$

Thus, $F$ is a contraction mapping on $PC(J, R)$ due to the condition (3.1). By applying the well-known Banach contraction mapping principle we know that the operator $F$ has a unique fixed point on $PC(J, R)$. Therefore, the problem (1.1)-(1.3) has a unique solution.
This completes the proof.

[H2'] There exists a constant $L > 0$ such that

$$|f(t, x, y, z)| \leq L(1 + |x| + |y| + |z|)$$

for each $t \in J$ and all $x, y, z \in R$. 
Theorem 3.6 Assume that [H1] and [H2'] hold. Then the problem (1.1)-(1.3) has at least one solution.

Proof 2 Transform the problem (1.1)-(1.3) into a fixed point problem.
Consider the operator

\[ F : PC(J, R) \to PC(J, R) \]

defined as (3.2). For the sake of convenience, the proof are in four steps.

Step 1. F is continuous. Let \( \{u_n\} \) be a sequence such that \( u_n \to u \) in \( PC(J, R) \). Then for each \( t \in J \), we have

\[
| (Fu_n)(t) - (Fu)(t) | 
\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \| f(s, u_n(s), Tu_n(s), Su_n(s)) - f(s, u(s), Tu(s), Su(s)) \| ds 
\leq \frac{T^q}{\Gamma(q+1)} \| f(\cdot, u_n(\cdot), Tu_n(\cdot), Su_n(\cdot)) - f(\cdot, u(\cdot), Tu(\cdot), Su(\cdot)) \|_{PC}.
\]

Due to [H1], \( f \) is jointly continuous, then we have \( ||Fu_n - Fu||_{PC} \to 0 \) as \( n \to \infty \).

Step 2. F maps bounded sets into bounded sets in \( PC(J, R) \).

Indeed, it is enough to show that for any \( r > 0 \), there exists a \( l > 0 \) such that for each \( u \in B_r = \{ u \in PC(J, R) : ||u|| \leq r \} \), we have \( ||Fu||_{PC} \leq l \).

For each \( t \in J \), we get

\[
\| (Fu(t)) \| 
\leq ||u_0|| + \sum_{i=1}^m \| y_i \| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \| f(s, u(s), Tu(s), Su(s)) \| ds 
\leq ||u_0|| + \sum_{i=1}^m \| y_i \| + \frac{L(1 + r + K^*Tr + H^*Tr) T^q}{\Gamma(q+1)} := l.
\]

Step 3. F maps bounded sets into equicontinuous sets of \( PC(J, R) \).

For interval \([0, t_1]\), \( 0 < s_1 < s_2 < t_1 \); \( u \in B_r \). Using [H2'] we have

\[
| (Fu(s_2)) - (Fu(s_1)) | 
\leq \frac{1}{\Gamma(q)} \int_0^{s_1} (s_1 - s)^{q-1} - (s_2 - s)^{q-1} \| f(s, u(s), Tu(s), Su(s)) \| ds 
+ \frac{1}{\Gamma(q)} \int_{s_1}^{s_2} (s_2 - s)^{q-1} \| f(s, u(s), Tu(s), Su(s)) \| ds 
\leq \frac{L}{\Gamma(q)} \int_0^{s_1} (s_1 - s)^{q-1} - (s_2 - s)^{q-1} (1 + ||u|| + ||Tu|| + ||Su||) ds 
+ \frac{L}{\Gamma(q)} \int_{s_1}^{s_2} (s_2 - s)^{q-1} (1 + ||u|| + ||Tu|| + ||Su||) ds 
\leq \frac{L(1 + r + K^*Tr + H^*Tr)}{\Gamma(q)} \int_0^{s_1} (s_1 - s)^{q-1} - (s_2 - s)^{q-1} ds 
+ \frac{L(1 + r + K^*Tr + H^*Tr)}{\Gamma(q)} \int_{s_1}^{s_2} (s_2 - s)^{q-1} ds 
\leq \frac{L(1 + r + K^*Tr + H^*Tr)}{\Gamma(q)} (|s_2^q - s_1^q| + 2(s_2 - s_1)^q) 
\leq \frac{3L(1 + r + K^*Tr + H^*Tr)}{\Gamma(q+1)} (s_2 - s_1)^q.
\]

As \( s_2 \to s_1 \), the right-hand side of the above inequality tends to zero, therefore \( F \) is equicontinuous on interval \([0, t_1]\). In general, for the time interval \([t_k, t_{k+1}]\), we similarly obtain the following inequality

\[
| (Fu(s_2)) - (Fu(s_1)) | 
\leq \frac{3L(1 + r + K^*Tr + H^*Tr)}{\Gamma(q+1)} (s_2 - s_1)^q \to 0, as \ s_2 \to s_1.
\]

This yields that \( F \) is equicontinuous on interval \((t_k, t_{k+1})\).

As a consequence of Step 1-3 together with the PC-type Arzela-Ascoli theorem (Lemma 2.11 in the case of \( X = R \)), we can conclude that \( F : B_r \to B_r \) is continuous and completely continuous.

Step 4. A priori bounds.
Now it remains to show that the set 

\[ E(F) = \{ u \in PC(J, R) : u = \lambda Fu, \text{ for some } \lambda \in (0, 1) \} \]

is bounded.

Let \( u \in E(F) \), then \( u = \lambda Fu \) for some \( \lambda \in (0, 1) \).

Without loss of generality, for the time interval \( t \in (t_k, t_{k+1}] \),

\[ |u(t)| \leq |u_0| + \sum_{i=1}^{k} |y_i| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, u(s), Tu(s), Su(s))| \, ds \]

\[ \leq |u_0| + \sum_{i=1}^{k} |y_i| + \frac{LT^n}{\Gamma(q+1)} + \frac{L}{\Gamma(q)} \int_0^t (t-s)^{q-1} |u(s)| \, ds \]

\[ + \frac{L}{\Gamma(q)} \int_0^t (t-s)^{q-1} |Tu(s)| \, ds + \frac{L}{\Gamma(q)} \int_0^t (t-s)^{q-1} |Su(s)| \, ds \]

By Theorem 2.2, there exists a \( M_k^* > 0 \) such that

\[ |u(t)| \leq M_k^*, \ t \in (t_k, t_{k+1}]. \]

Set \( M^* = \max_{1 \leq k \leq m} M_k^* \). Thus for every \( t \in J \), we have \( \|u\|_{PC} \leq M^* \).

This shows that the set \( E(F) \) is bounded.

As a consequence of Schaefer’s fixed point theorem, we deduce that \( F \) has a fixed point which is a solution of the problem (1.1)-(1.3). The proof is completed.

4 Example

Consider the following fractional integro-differential equations with impulsive conditions of the form

\[ C^\gamma D_t^\alpha = \frac{e^{-\gamma t}|u(t)|}{(9 + e^t)(1 + u(t))} + \frac{1}{10} \int_0^t e^{-(1/2)u(s)} \, ds + \frac{1}{10} \int_0^t e^{-(1/49)u(s)} \, ds, \ t \in [0, 1]/\{t_1\}, \]

\[ u(0) = u_0, \]

\[ u(t_{1+}) = u(t_{1-}) + 1, \]

where \( \gamma > 0 \) is a constant.

Set

\[ f(t, u) = \frac{e^{-\gamma t}|u|}{(9 + e^t)(1 + u)}; \ Tu(t) = \int_0^t e^{-(1/2)u(s)} \, ds \]

and \( Su(t) = \int_0^t e^{-(1/49)u(s)} \, ds \),

\[ f(t, u, Tu, Su) = \frac{e^{-\gamma t}u}{(9 + e^t)(1 + u)} + Tu + Su. \]

Then we have

\[ \|Tu - Tv\| = \left| \int_0^t e^{-(1/2)u(s)} \, ds - \int_0^t e^{-(1/2)v(s)} \, ds \right| \]

\[ \leq \frac{1}{2} |u - v| \]

\[ \|Su - Sv\| = \left| \int_0^t e^{-(1/49)u(s)} \, ds - \int_0^t e^{-(1/49)v(s)} \, ds \right| \]

\[ \leq \frac{1}{49} |u - v| \]
\[
\left\| f(t, u, Tu, Su) - f(t, v, Tv, Sv) \right\| = \frac{e^{-t}u}{(9 + e^t)(1 + u)} - \frac{e^{-t}v}{(9 + e^t)(1 + v)} + \frac{1}{10} \left( (Tu - Tv) + (Su - Sv) \right) \\
\leq \frac{e^{-t}u}{(9 + e^t)}|u - v| + \frac{1}{10} \left( \|Tu - Tv\| + \|Su - Sv\| \right) \\
\leq \frac{1}{10} \left( |u - v| + \|Tu - Tv\| + \|Su - Sv\| \right). 
\]

Then, all the assumptions in Theorem 3.1 are satisfied.

5 Conclusions

In the current paper, we are focused to study on some sufficient conditions for the existence results for abstract Cauchy problem for a class of mixed type impulsive fractional differential equations involving Caputo fractional derivative in Banach spaces. The proof of the main theorem is based on the fractional calculus theory and fixed point concepts.

References


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