On some boundary-value problems of functional integro-differential equations with nonlocal conditions

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Abstract

In this paper, we study the existence of solution for some boundary value problems of functional integro-differential equations with nonlocal boundary conditions.

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1 Introduction

Mathematical modelling of real-life problems usually results in functional equations, like ordinary or partial differential equations, integral and integro-differential equations, stochastic equations. Many mathematical formulation of physical phenomena contain integro-differential equations, these equations arises in many fields like fluid dynamics, biological models and chemical kinetics integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. Consider the following boundary value problems of functional integro-differential equations with the nonlocal boundary conditions.

\begin{align*}
&x'(t) = f(t, \int_0^1 k(t,s)x(s)ds), \quad t \in (0,1) \\
x(\tau) + \alpha x(\xi) = 0, \quad \tau, \xi \in [0,1], \alpha \neq -1. (1.2) \\
&x''(t) = f(t, \int_0^1 k(t,s)x'(s)ds), \quad t \in (0,1) \\
x(\tau) + \beta x(\xi) = 0, \quad \beta \neq -1, (1.4) \\
x'(\tau) + \alpha x'(\xi) = 0, \quad \tau, \xi \in [0,1], \alpha \neq -1. (1.5)
\end{align*}

Here we study the existence of at least one solution of each of the boundary value problems (1.1)-(1.2) and (1.3)-(1.5).

The existence of exactly one solution of them will be deduced.
2 Functional integral equation

Here we study the existence of at least one (and exactly one) continuous solution of the functional integral equation.

\[ y(t) = f(t, \int_0^1 k(t,s) \left[ \int_0^s y(\theta)d\theta \right] - \frac{1}{1+\alpha} \int_0^T y(\theta)d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y(\theta)d\theta]ds \]  \hspace{1cm} (2.6)

under the following assumptions

1. \( f : I = [0,1] \times R \to R \) is measurable in \( t \in [0,1] \) for all \( x \in R \) and continuous in \( x \in R \) for all \( t \in [0,1] \) and there exists integrable function \( a \in L^1[0,1] \) and positive constant \( b > 0 \) such that

\[ |f(t,x)| \leq a(t) + b|x| \quad t \in I. \]

2. \( a = \sup_t |a(t)|, \quad t \in [0,1] \)

3. \( k : I = [0,1] \times [0,1] \to R \) is continuous \( t \in [0,1] \) for every \( s \in [0,1] \) and measurable in \( s \in [0,1] \) for all \( t \in [0,1] \), such that

\[ \sup_t \int_0^1 k(t,s)dt \leq M \]

Now for the existence of at least one continuous solution of the functional integral equation (2.6), we have the following theorem.

**Theorem 2.1.** Let the assumptions (1)-(3) be satisfied. If \( 2Mb < 1 \), then the functional integral equation (2.6) has at least one solution \( y \in C[0,1] \).

**Proof.** Let \( C = C[0,1] \) and define the set \( Q_r \) by

\[ Q_r = \{ y \in C : |y| \leq r \} \subset C[0,1] \]

where \( r = \frac{a}{1-2Mb} \).

Define the operator \( F \) associated with the functional integral equation (2.6) by

\[ Fy(t) = f(t, \int_0^1 k(t,s) \left[ \int_0^s y(\theta)d\theta \right] - \frac{1}{1+\alpha} \int_0^T y(\theta)d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y(\theta)d\theta]ds \]

To show that \( F : Q_r \to Q_r \), let \( y \in Q_r \), then

\[ |Fy(t)| = |f(t, \int_0^1 k(t,s) \left[ \int_0^s y(\theta)d\theta \right] - \frac{1}{1+\alpha} \int_0^T y(\theta)d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y(\theta)d\theta]ds | \]

\[ \leq |a(t)| + b | \int_0^1 k(t,s) \left[ \int_0^s y(\theta)d\theta ] - \frac{1}{1+\alpha} \int_0^T y(\theta)d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y(\theta)d\theta]ds | \]

\[ \leq |a(t)| + b \left[ | \int_0^1 k(t,s) \left[ \int_0^s y(\theta)d\theta ] + | \int_0^1 k(t,s) \left[ \frac{1}{1+\alpha} \int_0^T y(\theta)d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y(\theta)d\theta]ds | \right] \]

\[ \leq |a(t)| + b \left[ | \int_0^1 k(t,s) |y(s)| ds + \int_0^1 |k(t,s)|[ \frac{1}{1+\alpha} + \frac{\alpha}{1+\alpha}] |y(s)| ds \right] \]

\[ \leq |a(t)| + 2bMr = r. \]

\[ a + 2bMr = r. \]
This proves that $F : Q_r \to Q_r$ and the class of functions $\{F(y)\}$ is uniformly bounded.

Let $t_1, t_2 \in [0, 1]$ and $|t_2 - t_1| \leq \delta$, then

$$
|Fy(t_2) - Fy(t_1)| \leq |f(t_2, \int_0^1 k(t_2, s)[\int_0^{\tau} y(\theta)d\theta - \frac{1}{1 + \alpha} \int_0^{\tau} y(\theta)d\theta - \frac{\alpha}{1 + \alpha} \int_0^{\tau} y(\theta)d\theta]ds| 
- f(t_1, \int_0^1 k(t_1, s)[\int_0^{\tau} y(\theta)d\theta - \frac{1}{1 + \alpha} \int_0^{\tau} y(\theta)d\theta - \frac{\alpha}{1 + \alpha} \int_0^{\tau} y(\theta)d\theta]ds| 
\quad \leq \frac{L}{\alpha} \int_0^1 k(t_2, s)[\int_0^{\tau} y(\theta)d\theta - \frac{1}{1 + \alpha} \int_0^{\tau} y(\theta)d\theta - \frac{\alpha}{1 + \alpha} \int_0^{\tau} y(\theta)d\theta]ds
- f(t_1, \int_0^1 k(t_2, s)[\int_0^{\tau} y(\theta)d\theta - \frac{1}{1 + \alpha} \int_0^{\tau} y(\theta)d\theta - \frac{\alpha}{1 + \alpha} \int_0^{\tau} y(\theta)d\theta]ds| 
\quad \leq 2L ||y|| \int_0^1 |k(t_2, s) - k(t_1, s)|ds,
$$

This means that the class of functions $F\{y\}$ is equi-continuous on $Q_r$.

Using Arzela-Ascoli Theorem (see [13]), we find that $F$ is compact.

Now we prove that $F : Q_r \to Q_r$ is continuous.
Let \( \{y_n\} \subset Q_r \), and \( y_n \to y \), then

\[
Fy_n(t) = f(t, \int_0^1 k(t,s) \int_0^s y_n(\theta)d\theta - \frac{1}{1+\alpha} \int_0^\tau y_n(\theta)d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y_n(\theta)d\theta|ds)
\]

\[
\lim_{n \to \infty} Fy_n(t) = \lim_{n \to \infty} f(t, \int_0^1 k(t,s) \int_0^s y_n(\theta)d\theta - \frac{1}{1+\alpha} \int_0^\tau y_n(\theta)d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y_n(\theta)d\theta|ds)
\]

Now

\[
\lim_{n \to \infty} f(t, \int_0^1 k(t,s)y_n(s)ds) = f(t, \lim_{n \to \infty} \int_0^1 k(t,s) \int_0^s y_n(\theta)d\theta - \frac{1}{1+\alpha} \int_0^\tau y_n(\theta)d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y_n(\theta)d\theta|ds)
\]

then using Lebesgue dominated convergence Theorem (see [13]), we have

\[
\lim_{n \to \infty} Fy_n = \lim_{n \to \infty} f(t, \int_0^1 k(t,s)f(t, \int_0^1 k(t,s) \int_0^s y_n(\theta)d\theta - \frac{1}{1+\alpha} \int_0^\tau y_n(\theta)d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y_n(\theta)d\theta|ds)
\]

\[
= f(t, \int_0^1 k(t,s) \int_0^s y(\theta)d\theta - \frac{1}{1+\alpha} \int_0^\tau y(\theta)d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y(\theta)d\theta|ds)
\]

Then \( Fy_n(t) \to Fy(t) \).

Which means that the operator \( F \) is continuous.

Since all conditions of Schauder fixed point theorem [12] are satisfied, then the operator \( F \) has at least one fixed point \( y \in C[0,1] \), which completes the proof.

Now for the uniqueness of the solution of the functional integral equation (2.6).

Consider following assumptions

\( (1^*) \) \( f : I = [0,1] \times R \to R \) is measurable in \( t \in [0,1] \) for all \( x \in R \) and satisfies the lipschitz such that

\[
|f(t,x) - f(t,y)| \leq b|x - y|, \quad b > 0 \tag{2.7}
\]

\( (2^*) \) \( f(t,0) \in L^1[0,1] \quad \sup_t|f(t,0)| \leq a. \)

Theorem 2.2. Let the assumptions \((1^*), (2^*) \) and \( (3) \) be satisfied. If \( 2Mb < 1 \), then the functional integral equation (2.6) has a unique solution \( y \in C[0,1] \).

Proof. From (2.7) we can obtain

\[
|f(t,x)| \leq |f(t,0)| + b|x|.
\]

This shows that the assumptions of Theorem (2.1) are satisfied.

Now let \( y_1, y_2 \) be two solution of functional integral equation (2.6)

\[
y_1(t) = f(t, \int_0^1 k(t,s) \int_0^s y_1(\theta)d\theta - \frac{1}{1+\alpha} \int_0^\tau y_1(\theta)d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y_1(\theta)d\theta|ds)
\]

\[
y_2(t) = f(t, \int_0^1 k(t,s) \int_0^s y_2(\theta)d\theta - \frac{1}{1+\alpha} \int_0^\tau y_2(\theta)d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y_2(\theta)d\theta|ds)
\]
\[ |y_1(t) - y_2(t)| = |f(t, \int_0^1 k(t, s) \int_0^s y_1(\theta) d\theta - \frac{1}{1 + \alpha} \int_0^\tau y_1(\theta) d\theta - \frac{\alpha}{1 + \alpha} \int_0^s y_1(\theta) d\theta| ds \\
- f(t, \int_0^1 k(t, s) \int_0^s y_2(\theta) d\theta - \frac{1}{1 + \alpha} \int_0^\tau y_2(\theta) d\theta - \frac{\alpha}{1 + \alpha} \int_0^s y_2(\theta) d\theta| ds) | \\
\leq b |\int_0^1 k(t, s) \int_0^s y_1(\theta) d\theta - \frac{1}{1 + \alpha} \int_0^\tau y_1(\theta) d\theta - \frac{\alpha}{1 + \alpha} \int_0^s y_1(\theta) d\theta| ds \\
- \int_0^1 k(t, s) \int_0^s y_2(\theta) d\theta - \frac{1}{1 + \alpha} \int_0^\tau y_2(\theta) d\theta - \frac{\alpha}{1 + \alpha} \int_0^s y_2(\theta) d\theta| ds) | \\
\leq b |\int_0^1 k(t, s) \int_0^s y_1(\theta) d\theta - \int_0^s y_2(\theta) d\theta| ds \\
+ b |\int_0^1 k(t, s) \int_0^s (y_2(\theta) - y_1(\theta)) d\theta + \frac{\alpha}{1 + \alpha} \int_0^s (y_2(\theta) - y_1(\theta)) d\theta| ds | \\
\leq b |\int_0^1 k(t, s) \int_0^s (y_1(\theta) - y_2(\theta)) d\theta| ds \\
+ b |\int_0^1 k(t, s) \int_0^s ||y_2 - y_1|| + \frac{\alpha}{1 + \alpha} ||y_2 - y_1||| ds | \\
\leq b ( ||y_1 - y_2|| \int_0^1 |k(t, s)| ds + ||y_1 - y_2|| \int_0^1 |k(t, s)| ds) \\
\leq 2bM ||y_1 - y_2||
\]

then

\[ ||y_1 - y_2|| \leq K||y_1 - y_2||
\]

where \( K = 2bM < 1 \), then

\[ ||y_1 - y_2||(1 - k) \leq 0
\]

and

\[ ||y_1 - y_2|| = 0
\]

which implies that \( y_1 = y_2 \) then the functional integral equation (2.6) has a unique continuous solution.

### 3 Nonlocal boundary value problems

Here we study the existence of at least one (and exactly one) solution of each of the functional integro-differential equations (1.1) - (1.3).

Consider the functional integro differential equation

\[ x'(t) = f(t, \int_0^1 k(t, s) x(s) ds) \quad t \in (0, 1).
\]

with the nonlocal boundary value condition

\[ x(\tau) + \alpha x(\xi) = 0 \quad \tau, \xi \in [0, 1], \alpha \neq -1
\]

**Theorem 3.3.** Let the assumptions of theorem (2.1) be satisfied, then the nonlocal boundary value problem (1.1) - (1.2) has at least one continuous solution \( x \in C[0, 1] \).
Proof. Let \( x'(t) = y(t) \). Integrating both sides we get

\[
    x(t) = x(0) + \int_0^t y(s) ds,
\]

\[
    x(\tau) = x(0) + \int_0^\tau y(s) ds
\]

and

\[
    x(\xi) = x(0) + \int_0^\xi y(s) ds
\]

Using the nonlocal boundary condition (1.2) we obtain

\[
    x(0) + \int_0^\tau y(s) ds = -\alpha x(0) - \frac{\alpha}{1 + \alpha} \int_0^\xi y(s) ds,
\]

and

\[
    x(0) = -\frac{1}{1 + \alpha} \int_0^\tau y(s) ds - \frac{\alpha}{1 + \alpha} \int_0^\xi y(s) ds
\]

then

\[
    x(t) = \int_0^t y(s) ds - \frac{1}{1 + \alpha} \int_0^\tau y(s) ds - \frac{\alpha}{1 + \alpha} \int_0^\xi y(s) ds
\]

(3.8)

where \( y \) satisfies the functional integral equation

\[
    y(t) = f\left(t, \int_0^1 k(t, s) [\int_0^s y(\theta) d\theta] - \frac{1}{1 + \alpha} \int_0^\tau y(\theta) d\theta - \frac{\alpha}{1 + \alpha} \int_0^\xi y(\theta) d\theta \right] ds.
\]

This complete the proof of equivalent between the nonlocal problem (1.1)-(1.2) and the functional integral equation (2.6). This implies that there exists at least one solution \( x \in C[0, 1] \) of the nonlocal problem (1.1)-(1.2).

**Corollary 3.1.** Let the assumptions (1'),(2') and (3) be satisfied, then the solution of nonlocal boundary value problem (1.1)-(1.2) has a unique continuous solution \( x \in C[0, 1] \).

Consider the functional integro-differential equation

\[
    x''(t) = f(t, \int_0^1 k(t, s) x'(s) ds) \quad t \in (0, 1)
\]

with the nonlocal boundary conditions

\[
    x(\tau) + \beta x(\xi) = 0,
\]

\[
    x'(\tau) + \alpha x'(\xi) = 0.
\]

**Theorem 3.4.** Let the assumptions of theorem (2.1) be satisfied then the boundary value problems (1.3)-(1.5) has at least one continuous solution \( x \in C[0, 1] \).

Proof. Let \( x''(t) = y(t) \) integrating both sides, we obtain

\[
    x'(t) = x'(0) + \int_0^t y(s) \, ds
\]

and

\[
    x(t) = x(0) + tx'(0) + \int_0^t (t - s) \, y(s) ds.
\]

then

\[
    x'(\tau) = x'(0) + \int_0^\tau y(s) \, ds,
\]

and

\[
    x'(\xi) = x'(0) + \int_0^\xi y(s) \, ds.
\]
Using the nonlocal condition (1.5) we obtain
\[
x'(0) = -\frac{1}{1+\alpha} \int_0^\tau y(s) ds - \frac{\alpha}{1+\alpha} \int_0^\xi y(s) ds
\]
and
\[
x(\tau) = x(0) + \tau x'(0) + \int_0^\tau (\tau - s) y(s) ds,
\]
\[
x(\xi) = x(0) + \xi x'(0) + \int_0^\xi (\xi - s) y(s) ds,
\]
\[
x'(0) = -\frac{1}{1+\alpha} \int_0^\tau y(s) ds - \frac{\alpha}{1+\alpha} \int_0^\xi y(s) ds.
\]
Using Boundary condition (1.4) we obtain
\[
x(0) = -\frac{\beta \xi - \tau}{1+\beta} x'(0) - \frac{1}{1+\alpha} \int_0^\tau (\tau - s)y(s) ds - \frac{1}{1+\beta} \int_0^\xi (\xi - s)y(s) ds,
\]
\[
x(t) = -\frac{\beta \xi - \tau}{1+\beta} \left[ -\frac{1}{1+\beta} \int_0^\tau y(s) ds - \frac{1}{1+\alpha} \int_0^\xi y(s) ds \right]
\]
\[
- \frac{1}{1+\beta} \int_0^\tau (\tau - s) y(s) ds - \frac{1}{1+\beta} \int_0^\xi (\xi - s) y(s) ds
\]
\[
+ t \left[ -\frac{1}{1+\alpha} \int_0^\tau y(s) ds - \frac{\alpha}{1+\alpha} \int_0^\xi y(s) ds \right] + \int_0^t (t - s) y(s) ds,
\]
\[
x'(t) = -\frac{1}{1+\alpha} \int_0^\tau y(s) ds - \frac{\alpha}{1+\alpha} \int_0^\xi y(s) ds + \int_0^t y(s) ds,
\]
and \( y \) satisfies the functional integral equation
\[
y(t) = f(t, \int_0^1 k(t,s) [\int_0^\tau y(\theta) d\theta - \frac{1}{1+\alpha} \int_0^\tau y(\theta) d\theta - \frac{\alpha}{1+\alpha} \int_0^\xi y(\theta) d\theta] ds).
\]
This complete the proof of equivalent between the nonlocal problem (1.3)-(1.5) and the functional integral equation (2.6). This implies that there exists at least one solution \( x \in C[0,1] \) of the nonlocal problem (1.3)-(1.5). \[\Box\]

**Corollary 3.2.** Let the assumptions \((1^*),(2^*)\) and (3) be satisfied, then the solution of nonlocal boundary value problem (1.3)-(1.5) has a unique continuous solution \( x \in C[0,1] \).

**References**


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