Abstract

The aim of present paper is to define a pair of $k$-Saigo fractional integral and derivative operators involving generalized $k$-hypergeometric function. The Saigo-$k$ generalized fractional operators involving $k$-hypergeometric function in the kernel are applied to the generalized $k$-Mittag-Leffler function and evaluate the formula

$$2F_{1,k}\left[ \begin{array}{c} (\alpha,k), (\beta,k) \\ (\gamma,k) \end{array} ; \frac{1}{k} \right] = \frac{\Gamma_k(\gamma)\Gamma_k(\gamma - \alpha - \beta)}{\Gamma_k(\gamma - \alpha)\Gamma_k(\gamma - \beta)}$$

using the integral representation for $k$-hypergeometric function.

Keywords: $k$-functions and $k$-fractional calculus.

1 Introduction

The fractional $k$-calculus is the $k$-extension of the classical fractional calculus. The theory of $k$-calculus operators in recent past have been applied in different and numerous investigations.

Several authors that were dedicated to study such operators and since Diaz et al. defined the $k$-gamma function and the $k$-symbol. Very recently, Rehman et al. studied the properties of $k$-beta function. Musbeen and Rehman discuss extension of $k$-gamma and Pochhammer $k$-symbol. Musbeen and Habibullah defined $k$-fractional integration and gave an its application. Musbeen and Habibullah also introduced an integral representation of some generalized confluent $k$-hypergeometric functions $mF_{m,k}$ and $k$-hypergeometric functions $m+1F_{m,k}$ by using the properties of Pochhammer $k$-symbols, $k$-gamma and $k$-beta functions.

In this paper we evaluate the Saigo $k$-fractional integral operators and derivatives involving generalized $k$-hypergeometric function on the $k$-new generalized Mittag-Leffler function introduced by us.

2 Definitions and Preliminaries

In this section, we state some known results and some important definitions which will be used in the sequel.
Definition 2.1. Generalized k-Gamma function $\Gamma_k(x)$ defined as \[ \Gamma_k(x) = \lim_{n \to \infty} n! k^n (nk)^{x-1}, \quad k > 0, x \in \mathbb{C} \] (2.1)

where $(x)_{n,k}$ is the k-Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k),$$

(2.2)

$x \in \mathbb{C}, k \in \mathbb{R}, n \in \mathbb{N}^+$.

For $\text{Re}(x) > 0$ and $k > 0$, then $\Gamma_k(x)$ defined as the integral

$$\Gamma_k(x) = k^{x-1} \Gamma\left(\frac{x}{k}\right) = \int_0^\infty t^{x-1} e^{-\frac{t}{k}} dt,$$

(2.3)

and $\Gamma_k(x+k) = x \Gamma_k(x)$.

(2.4)

This give rise to $k$-beta function defined by

$$B_k(x, y) = \frac{1}{k} \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x > 0, y > 0.$$ (2.5)

They have also provided some useful and applicable relations

$$B_k(x, y) = \frac{1}{k} B\left(\frac{x}{k}, \frac{y}{k}\right) \quad \text{and} \quad B_k(x, y) = \frac{\Gamma_k(x) \Gamma_k(y)}{\Gamma_k(x+y)},$$ (2.6)

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)},$$ (2.7)

$$(1-kx)^{\frac{x}{k}} = \sum_{n=0}^{\infty} (\alpha)_{nk} \frac{x^n}{n!},$$ (2.8)

$$(1-x)^{-\frac{x}{k}} = \sum_{n=0}^{\infty} \frac{1}{k^n} (\alpha)_{nk} \frac{x^n}{n!}.$$ (2.9)

Definition 2.2. $k$-hypergeometric function $F_k$ define by the series as \[ F_k((\beta, k); (\gamma, k); x) = \sum_{n=0}^{\infty} \frac{\beta_{n,k} x^n}{(\gamma)_{n,k} n!}, \quad k \in \mathbb{R}^+, \beta, \gamma \in \mathbb{C}, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0. \] (2.10)

Its integral representation can be determined as follows

$$1_{F_1}((\beta, k); (\gamma, k); x) = \frac{\Gamma_k(\gamma)}{k \Gamma(\beta) \Gamma_k(\gamma - \beta)} \int_0^1 t^{\frac{\beta}{k} - 1} (1-t)^{\frac{\gamma}{k} - 1} e^{xt} dt.$$ (2.11)

And if $\text{Re}(\gamma) > \text{Re}(\beta) > 0, k > 0, m \geq 1, m \in \mathbb{Z}^+$ and $|x| < 1$, then

$$m+1_{F_m,k} \left[ \begin{array}{c} (\alpha, k), (\beta_{m,k}, k), (\beta_{m,k} + k), \cdots, (\frac{\gamma + (m-1)k}{m}, k) \\ (\gamma_{m,k}, k), (\gamma_{m,k} + k), \cdots, (\gamma_{m,k} + (m-1)k) \end{array} \right]; x \right] = \frac{\Gamma_k(\gamma)}{k \Gamma(\beta) \Gamma_k(\gamma - \beta)} \int_0^1 t^{\frac{\beta}{k} - 1} (1-t)^{\frac{\gamma}{k} - 1} (1-kxt)^{\frac{\alpha}{k}} dt.$$ (2.12)

And if $\text{Re}(\gamma) > \text{Re}(\beta) > 0$ and $|x| < 1$, then

$$2_{F_1}((\alpha, k), (\beta, k); (\gamma, k); x) = \frac{\Gamma_k(\gamma)}{k \Gamma(\beta) \Gamma_k(\gamma - \beta)} \int_0^1 t^{\frac{\beta}{k} - 1} (1-t)^{\frac{\gamma}{k} - 1} (1-kxt)^{\frac{\alpha}{k}} dt.$$ (2.13)
The corresponding fractional differential operators have their respective forms as

\[ p^k \Psi_q^k(x) = p^k \Psi_q^k \left[ \begin{array}{c} (a_i, a_i)_1, p \\ (b_j, b_j)_1, q \end{array} \right] \sum_{n=0}^{\infty} \frac{\Gamma_k(a_i + a_i, n)}{\Gamma_k(b_j + b_j, n)} \frac{(x)^n}{n!}, \quad (2.14) \]

where \( \Gamma_k(\cdot) \) denote the \( k \)-gamma function and satisfies the condition

\[ \sum_{j=1}^{q} \frac{\beta_j}{k} - \sum_{i=1}^{p} \frac{\alpha_i}{k} > -1. \quad (2.15) \]

Definition 2.4. The \( k \)-new generalized Mittag-Leffler function \( E_{\alpha, \beta, \gamma}^\delta(x) \) defined by \( [8] \) for \( k \in \mathbb{R}, \alpha, \beta, \gamma, \delta \in \mathbb{C}, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0, \text{Re}(\delta) > 0, p, q > 0 \) and \( q \leq \text{Re}(\alpha) + p \)

\[ E_{\alpha, \beta, \gamma}^\delta(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_q x^n}{\Gamma_k(an + \beta, \gamma)} \quad (2.16) \]

3 Generalized \( k \)-Saigo Fractional Calculus Operators

In this section we define the left and right-sided Saigo \( k \)-fractional calculus operators. Let \( \alpha, \beta, \gamma \in \mathbb{C}, K > 0, k \in \mathbb{R}^+ \), then the generalized \( k \)-fractional integration and differentiation operators associated with the \( k \)-Gauss hypergeometric function are defined as follows:

\[ \left( t_{0+}^{\alpha, \beta, \gamma} f \right)(x) = \frac{x^{-\alpha - \beta}}{k \Gamma_k(\alpha)} \int_{0}^{\infty} \frac{(x - t)^{\frac{\delta}{k} - 1}}{\Gamma_k(\alpha, n + \beta + \gamma, n)} (\alpha + \beta, k; (-\gamma, k); (\alpha, k); (1 - \frac{1}{x}) f(t) dt; \quad (\text{Re}(\alpha) > 0, k > 0), (3.1) \]

\[ \left( t_{-k}^{\alpha, \beta, \gamma} f \right)(x) = \frac{1}{k \Gamma_k(\alpha)} \int_{x}^{\infty} (t - x)^{\frac{\delta}{k} - 1 - \frac{\alpha - \beta}{k}} \Gamma_k(\alpha, n + \beta + \gamma, n) (\alpha, k; (-\gamma, k); (\alpha, k); (1 - \frac{x}{t}) f(t) dt; \quad (\text{Re}(\alpha) > 0, k > 0), (3.2) \]

Here \( \frac{\alpha}{k} \) is the \( k \)-Gauss hypergeometric function defined by \( [16] \) for \( x \in \mathbb{C}, |x| < 1, \text{Re}(\gamma) > \text{Re}(\beta) > 0 \)

\[ \left( \frac{\alpha}{k} \right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_{n,k} x^n}{(\gamma)_{n,k} n!}; \quad (3.3) \]

The corresponding fractional differential operators have their respective forms as

\[ \left( D_{0+}^{\alpha, \beta, \gamma} f \right)(x) = \left( \frac{d}{dx} \right) \left( t_{0+}^{\alpha, \beta, \gamma} f \right)(x); \quad \text{Re}(\alpha) > 0, k > 0; n = [\text{Re}(\alpha) + 1] \quad (3.4) \]

\[ \left( D_{0+}^{k} f \right)(x) = \left( \frac{d}{dx} \right) \left( t_{0+}^{\alpha, \beta, \gamma} f \right)(x); \quad \text{Re}(\alpha) > 0, k > 0; n = [\text{Re}(\alpha) + 1] \quad (3.5) \]

\[ \left( D_{-k}^{\alpha, \beta, \gamma} f \right)(x) = \left( -\frac{d}{dx} \right) \left( t_{-k}^{\alpha, \beta, \gamma} f \right)(x); \quad \text{Re}(\alpha) > 0, k > 0; n = [\text{Re}(\alpha) + 1] \quad (3.6) \]

\[ \left( D_{-k}^{\alpha, \beta, \gamma} f \right)(x) = \left( -\frac{d}{dx} \right) \left( t_{-k}^{\alpha, \beta, \gamma} f \right)(x); \quad \text{Re}(\alpha) > 0, k > 0; n = [\text{Re}(\alpha) + 1] \quad (3.7) \]
where $x > 0, \alpha \in C, \text{Re}(\alpha) > 0, k > 0$ and $[\text{Re}(\alpha)]$ is the integer part of $\text{Re}(\alpha)$.

For $K \to 1$, the operators (3.1) to (3.5) reduce to Saigo’s [19] fractional integer and differentiation operators. If we set $\beta = -\alpha$, operators (3.1) to (3.5) reduce to $k$-Riemann-Liouville operators as follows:

\[
\begin{align*}
(i_{0+}^{\alpha,-\gamma}f)_{k}(x) &= (i_{0+}^k f)(x), \\
(i_{k}^{\alpha,-\gamma}f)(x) &= (I_{-k}^\alpha f)(x), \\
(D_{0+}^{\alpha,-\gamma}f)(x) &= (D_{0+}^\alpha f)(x), \\
(D_{-k}^{\alpha,-\gamma}f)(x) &= (D_{-k}^\alpha f)(x).
\end{align*}
\]

\section{Main Result}

In this section, we find out the main result.

\begin{theorem}
    \[2F_{1,k} \left[ \begin{array}{c} (a, k), (\beta, k) \\ \gamma, k \end{array} ; \frac{1}{k} \right] = \frac{\Gamma_k(\gamma)(\gamma - \alpha - \beta)}{\Gamma_k(\gamma - \alpha) \Gamma_k(\gamma - \beta)}.\]
\end{theorem}

\begin{proof}
    From equation (2.13), we have the following result
    \[2F_{1,k} \left[ \begin{array}{c} (a, k), (\beta, k) \\ \gamma, k \end{array} ; \frac{1}{k} \right] = \frac{\Gamma_k(\gamma)}{k \Gamma_k(\beta) \Gamma_k(\gamma - \beta)} \int_0^1 t^{\frac{\beta}{k} - 1} (1 - t)^{\frac{\gamma - \beta}{k} - 1} (1 - xt)^{\frac{\alpha}{k}} dt.\]

    Put $x = 1$ in equation (4.1), we obtain the following
    \[2F_{1,k} \left[ \begin{array}{c} (a, k), (\beta, k) \\ \gamma, k \end{array} ; \frac{1}{k} \right] = \frac{\Gamma_k(\gamma)}{k \Gamma_k(\beta) \Gamma_k(\gamma - \beta)} \int_0^1 t^{\frac{\beta}{k} - 1} (1 - t)^{\frac{\gamma - \beta}{k} - 1} dt.\]

    On applying the definition of $k$-beta function, we get the required result
    \[2F_{1,k} \left[ \begin{array}{c} (a, k), (\beta, k) \\ \gamma, k \end{array} ; \frac{1}{k} \right] = \frac{\Gamma_k(\gamma) \Gamma_k(\gamma - \alpha - \beta)}{k \Gamma_k(\beta) \Gamma_k(\gamma - \beta)} B_k(\beta, \gamma - \alpha - \beta).\]
\end{proof}

\begin{lemma}
    Let $a, \beta, \gamma, \rho \in C, \text{Re}(\alpha) > 0, k \in R^+(0, \infty)$

    (a) If $\text{Re}(\rho) > \max[0, \text{Re}(\beta - \gamma)]$, then
    \[\left(i_{0+}^{\alpha, \beta, \gamma, \frac{\rho}{k} - 1} f\right)(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho) \Gamma_k(\rho - \beta + \gamma)}{\Gamma_k(\rho - \beta) \Gamma_k(\rho + \alpha + \gamma)} x^{\frac{\rho - \beta}{k} - 1}.\]

    (b) If $\text{Re}(\rho) > \max[\text{Re}(\beta), \text{Re}(\gamma)]$, then
    \[\left(i_{-k}^{\alpha, \beta, \gamma} t^{\frac{\rho}{k} - \frac{\beta}{k} - 1} f\right)(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho + \beta) \Gamma_k(\rho + \gamma)}{\Gamma_k(\rho) \Gamma_k(\rho + \alpha + \beta + \gamma)} x^{\frac{\rho - \beta}{k} \frac{k}{k} - \frac{\beta}{k} - 1}.\]
\end{lemma}
Proof. (a): Taking \( f(x) = t^{\frac{p}{k}} \) in (3.1), we get
\[
(I_{0+}^{a,b,\gamma} t^{\frac{p}{k}-1})(x) = \frac{x^{-a-b}}{k \Gamma_k(a)} \int_0^x (x - t)^{\frac{p}{k} - 1} F_{1,k} \left( (\alpha + \beta, k), (-\gamma, k); (\alpha, k), (1 - \frac{t}{x}) \right) t^{\frac{p}{k} - 1} dt.
\]
We invoke \( k \)-Gauss hypergeometric series \([16]\) and on changing the order of integration and summation, we have
\[
(I_{0+}^{a,b,\gamma} t^{\frac{p}{k}-1})(x) = \frac{x^{-a-b}}{k \Gamma_k(a)} \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k}n!} \int_0^x (x - t)^{\frac{p}{k} - 1} (1 - \frac{t}{x}) t^{\frac{p}{k} - 1} dt.
\]
On setting \( t = xu \), we get
\[
(I_{0+}^{a,b,\gamma} t^{\frac{p}{k}-1})(x) = \frac{x^{-a-b}}{k \Gamma_k(a)} \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k}n!} \int_0^1 (1 - u)^{\frac{p}{k} + n - 1} u^{\frac{p}{k} - 1} du.
\]
On evaluating the inner integral by \( k \)-beta function and using relation (2.3) and (2.7), we have
\[
= x^{\frac{p}{k} - 1} \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k}n!} \frac{\Gamma_k(\rho)}{\Gamma_k(\alpha + \rho)}
= x^{\frac{p}{k} - 1} \frac{\Gamma_k(\rho)}{\Gamma_k(\alpha + \rho)} \sum_{n=0}^{\infty} k^n_n F_{2,k} \left( (\alpha + \beta, k), (-\gamma, k); (\alpha + \rho, k); 1, \frac{1}{k} \right).
\]
Finally use theorem (4.1) and rearrange terms, expression (4.4) yields
\[
(I_{0+}^{a,b,\gamma} t^{\frac{p}{k}-1})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho)(\rho - k + \gamma)}{\Gamma_k(\rho - \beta) \Gamma_k(\rho + \alpha + \gamma)} x^{\frac{p}{k} - 1}.
\]
proof (b): Taking \( f(x) = t^{\frac{p}{k}} \) in (3.2), we get
\[
(I_{-k}^{a,b,\gamma} t^{\frac{p}{k}})(x) = \frac{1}{k \Gamma_k(a)} \int_x^\infty (t - x)^{\frac{p}{k} - 1} t^{\frac{p}{k} - 1} F_{1,k} \left( (\alpha + \beta, k), (-\gamma, k); (\alpha, k), (1 - \frac{x}{t}) \right) t^{\frac{p}{k} - 1} dt.
\]
On applying \( k \)-Gauss hypergeometric series \([16]\) and on changing the order of integration and summation, and
\[
(I_{-k}^{a,b,\gamma} t^{\frac{p}{k}})(x) = \frac{1}{k \Gamma_k(a)} \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k} n!} \int_x^\infty (t - x)^{\frac{p}{k} - 1} t^{\frac{p}{k} - 1} (1 - \frac{x}{t})^n t^{\frac{p}{k} - 1} dt.
\]
Put \( t = \frac{x}{u} \), we have
\[
(I_{-k}^{a,b,\gamma} t^{\frac{p}{k}})(x) = \frac{x^{-a-b}}{k \Gamma_k(a)} \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k} n!} \int_0^1 (1 - u)^{\frac{p}{k} + n - 1} u^{\frac{p}{k} - 1} du.
\]
On evaluating the inner integral by \( k \)-beta function and using relation (2.3) and (2.7), we have
\[
= x^{\frac{p}{k} - 1} \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_{n,k}(-\gamma)_{n,k}}{(\alpha)_{n,k} n!} \frac{\Gamma_k(\rho + \beta)}{\Gamma_k(\alpha + \beta + \rho)}
= x^{\frac{p}{k} - 1} \frac{\Gamma_k(\rho + \beta)}{\Gamma_k(\alpha + \beta + \rho)} \sum_{n=0}^{\infty} k^n n F_{2,k} \left( (\alpha + \beta, k), (-\gamma, k); (\alpha + \beta + \rho, k); 1, \frac{1}{k} \right).
\]
Finally use theorem (4.1) and rearrange terms, expression (4.5) yields
\[
(I_{-k}^{a,b,\gamma} t^{\frac{p}{k}})(x) = \sum_{n=0}^{\infty} k^n \frac{\Gamma_k(\rho + \beta) \Gamma_k(\rho + \gamma)}{\Gamma_k(\rho) \Gamma_k(\alpha + \beta + \rho + \gamma)} x^{\frac{p}{k} - 1}.
\]
\(\square\)
5 Left side Saigo k-Fractional Integration of the generalized k-Mittag-Leffler function

In this section we have discussed the left-sided Saigo k-fractional integration formula of the generalized k-Mittag-Leffler function.

**Theorem 5.1.** Let \( \alpha, \beta, \gamma, \rho, \delta, \xi \in C \) and \( k \in R^+ \) be such that \( \text{Re}(\alpha) > 0, \text{Re}(\rho + \gamma - \beta) > 0, p, q > 0, q \leq \text{Re}(v) + p \). Also, let \( c \in R \) and \( v > 0 \). If condition (2.15) is satisfied and \( I^{\alpha,\beta,\gamma}_{0+} \) be the left sided operator of the generalized k-fractional integration associated with k-Gauss hypergeometric function, then there holds the following relationship

\[
\left( I^{\alpha,\beta,\gamma}_{0+} \left( t^{\frac{v}{k}} \frac{E_{\gamma,\delta,\xi,\rho}^{\delta,\xi,\rho} \left[ ct^r \right]}{\Gamma_k(\delta)} \right) \right) (x) = x^{\frac{v}{k} - 1} \frac{\Gamma_k(\xi)}{\Gamma_k(\delta)} \psi_3 \left[ \begin{array}{c} (\rho + \gamma - \beta, v), (\delta, qk), (k, k) \\ (\rho - \beta, v), (\rho + \alpha + \gamma, v), (\xi, pk) \end{array} \right] ; cx^\frac{v}{k} \tag{5.1}
\]

**Proof.** Applying (2.16) and (4.2) in the left-side of (5.1), we have

\[
\left( I^{\alpha,\beta,\gamma}_{0+} \left( t^{\frac{v}{k}} \frac{E_{\gamma,\delta,\xi,\rho}^{\delta,\xi,\rho} \left[ ct^r \right]}{\Gamma_k(\delta)} \right) \right) (x) = I^{\alpha,\beta,\gamma}_{0+} \left( t^{\frac{v}{k}} \sum_{n=0}^{\infty} \frac{(\delta)_{\eta,\xi} \left( ct^r \right)^n}{\Gamma_k(vn + \rho) (\xi)^{n+1}} \right) (x) = \sum_{n=0}^{\infty} \frac{(\delta)_{\eta,\xi} \left( ct^r \right)^n}{\Gamma_k(vn + \rho) (\xi)^{n+1}} \left( I^{\alpha,\beta,\gamma}_{0+} \left( t^{\frac{v}{k}} \right) \right)^n (x) = x^{\frac{v}{k} - 1} \sum_{n=0}^{\infty} \frac{\Gamma_k(\rho + \gamma - \beta + vn) \Gamma_k(\delta + qkn) \Gamma_k(\xi) \Gamma_k(1 + n) \Gamma_k(\rho - \beta + vn)}{n!} (cx^\frac{v}{k})^n.
\]

On using \( \Gamma(n + 1) = k^{-n} \Gamma_k(nk + k) \), we get required result

\[
\left( I^{\alpha,\beta,\gamma}_{0+} \left( t^{\frac{v}{k}} \frac{E_{\gamma,\delta,\xi,\rho}^{\delta,\xi,\rho} \left[ ct^r \right]}{\Gamma_k(\delta)} \right) \right) (x) = x^{\frac{v}{k} - 1} \frac{\Gamma_k(\xi)}{\Gamma_k(\delta)} \psi_3 \left[ \begin{array}{c} (\rho + \gamma - \beta, v), (\delta, qk), (k, k) \\ (\rho - \beta, v), (\rho + \alpha + \gamma, v), (\xi, pk) \end{array} \right] ; cx^\frac{v}{k} \tag{5.1}
\]

\[\square\]

**Remark 5.1.** If we put \( k = 1 \) in equation (5.1), we arrive at the result [6, p.140, Eq.2.1].

**Remark 5.2.** If we set \( p = q = k = \xi = 1 \) in our formula (5.1), we get the result [1, p.116, Eq.3.1].

6 Right side Saigo k-Fractional Integration of the generalized k-Mittag-Leffler function

In this section we have discussed the right-sided Saigo k-fractional integration formula of the generalized k-Mittag-Leffler function.

**Theorem 6.1.** Let \( \alpha, \beta, \gamma, \rho, \delta, \xi \in C \) and \( k \in R^+ \) be such that \( \text{Re}(\alpha) > 0, \text{Re}(\alpha + \rho) > \max[\text{Re}(\beta), \text{Re}(\gamma)] \) with condition \( \text{Re}(\beta) \neq \text{Re}(\gamma), \nu > 0, p, q > 0, q \leq \text{Re}(v) + p \). Also, let \( c \in R, v \in R, \nu > 0 \) and \( I^{\alpha,\beta,\gamma}_{-k} \) be the right sided operator of the generalized k-fractional integration associated with k-Gauss hypergeometric function, then there holds the following formula:

\[
\left( I^{\alpha,\beta,\gamma}_{-k} \left( t^{\frac{v}{k}} \frac{E_{\gamma,\delta,\xi,\rho}^{\delta,\xi,\rho} \left[ ct^r \right]}{\Gamma_k(\delta)} \right) \right) (x) = x^{\frac{v}{k} - 1} \frac{\Gamma_k(\xi)}{\Gamma_k(\delta)} \psi_4 \left[ \begin{array}{c} (\alpha + \beta + \rho, v), (\alpha + \gamma + \rho, v), (\delta, qk), (k, k) \\ (\alpha + \rho, v), (2\alpha + \beta + \rho + \gamma, v), (\xi, pk), (\rho, v) \end{array} \right] ; cx^\frac{v}{k} \tag{6.1}
\]
Proof. Applying (2.16) and (4.3) in the left-side of (6.1), we have
\[ \left( I_0^a r_t t^{\frac{\alpha}{2}} \right)(x) = \Gamma_{\alpha} \frac{\Gamma_k(\frac{\alpha}{2} + \gamma + n)}{\Gamma_k(\rho + \alpha + \gamma + n)} \Gamma_{\frac{\alpha}{2}} \Gamma_{\frac{\alpha + \gamma + n}{2}} x^{\frac{\alpha}{2} + \gamma + n} - 1. \]

(b) If \( \text{Re} \beta \rightarrow 0 \), then
\[ \left( I_0^a r_t t^{\frac{\alpha}{2}} \right)(x) = \Gamma_{\alpha} \frac{\Gamma_k(\frac{\alpha}{2} + \gamma + n)}{\Gamma_k(\rho + \alpha + \gamma + n)} \Gamma_{\frac{\alpha}{2}} \Gamma_{\frac{\alpha + \gamma + n}{2}} x^{\frac{\alpha}{2} + \gamma + n} - 1. \]

Remark 6.1. If \( p = q = \xi = 1 \) in equation (6.1), we can obtain the result [6, p.141, Eq.2.3].

Remark 6.2. If \( p = q = \xi = 1 \) in (6.1), we get the required result [1, p.117, Eq.4.1].

Lemma 6.2. Let \( \alpha, \beta, \gamma, \rho \in \mathbb{C}, n = [\text{Re}(\alpha)] + 1, k \in \mathbb{R}^+(0, \infty) \)

(a) If \( \text{Re}(\rho) \geq 0, \text{Re}(\alpha - \beta - \gamma) \), then
\[ (D_{0+}^{a, \beta, \gamma} - t^{\frac{\alpha}{2}} - 1)(x) = \frac{\Gamma_k(\rho + \alpha + \gamma + n)}{\Gamma_k(\rho + \alpha + \gamma + n)} \Gamma_{\frac{\alpha}{2}} \Gamma_{\frac{\alpha + \gamma + n}{2}} x^{\frac{\alpha}{2} + \gamma + n} - 1. \]

(b) If \( \text{Re}(\rho) > \max[\text{Re}(\alpha - \beta - \gamma), \text{Re}(\beta - nk + n)] \), then
\[ (D_{0+}^{a, \beta, \gamma} - t^{\frac{\alpha}{2}} - 1)(x) = \frac{\Gamma_k(\rho + \alpha + \gamma + n)}{\Gamma_k(\rho + \alpha + \gamma + n)} \Gamma_{\frac{\alpha}{2}} \Gamma_{\frac{\alpha + \gamma + n}{2}} x^{\frac{\alpha}{2} + \gamma + n} - 1. \]
On using (2.3), we have

\[
= \sum_{n=0}^{\infty} (-1)^n k^n \frac{\Gamma_k(\rho + \alpha + \gamma)}{\Gamma_k(\rho - \beta + \gamma)} \frac{k^{\frac{\rho-\beta-n}{k}} \Gamma(\frac{\rho-\beta-n}{k}) \Gamma\left(\frac{-\rho+\beta+n}{k}+1\right)}{\Gamma\left(\frac{-\rho+\beta+n-nk+k}{k}\right)} x^{-\frac{\rho+\beta+n}{k}}. \tag{6.4}
\]

The reflection formula for gamma function see [3]

\[
\Gamma\left(\frac{\beta-\beta-n}{k}\right) \Gamma\left(1-\left(\frac{\beta-\beta-n}{k}\right)\right) = \frac{\pi}{\sin\left(\frac{\beta-\beta-n}{k}\right)\pi}, \tag{6.5}
\]

and

\[
\frac{1}{\Gamma\left(1-\left(\frac{\beta-\beta-n+nk}{k}\right)\right)} = \frac{\Gamma\left(\frac{\rho-\beta-n+nk}{k}\right)}{\Gamma\left(\frac{\rho-\beta-n+nk}{k}\right) \Gamma\left(1-\left(\frac{\rho-\beta-n+nk}{k}\right)\right)} \Gamma\left(\frac{\rho-\beta-n}{k}\right) \sin\left(\frac{\rho-\beta-n}{k}\right) \pi
\]

\[
= \frac{\sin\left(\frac{\rho-\beta-n}{k}\right) \pi \cos n\pi}{\Gamma\left(\frac{\rho-\beta-n}{k}\right) \sin\left(\frac{\rho-\beta-n}{k}\right) \pi}
\]

\[
= k^{1-\left(\frac{\rho-\beta-n+nk}{k}\right)} \Gamma_k(\rho - \beta - n + nk) \frac{\sin\left(\frac{\rho-\beta-n}{k}\right) \pi \cos n\pi}{\pi}. \tag{6.6}
\]

using (6.5) and (6.6) in (6.4), we obtain required result

\[
(D_{-\alpha}^{\rho,\beta,\gamma, c} f)(x) = \frac{\Gamma_k(\rho - \beta - n + nk) \Gamma(\rho + \alpha + \gamma)}{\Gamma(\rho - \beta + \gamma)} x^{-\frac{\rho+\beta+n}{k}}. \tag{6.7}
\]

\[
\square
\]

7. Left and right side Saigo k-Fractional Differentiation of the generalized k-Mittag-Leffler function

In this section we have discussed the left and right sided Siago k-fractional differentiation formula of the generalized k-Mittag-Leffler function.

Theorem 7.1. Let \( \alpha, \beta, \gamma, \rho, \delta, \xi \in \mathbb{C} \) and \( k \in \mathbb{R}^+ \) be such that \( \Re(\alpha) > 0, \Re(\rho + \beta + \gamma) > 0, \nu > 0, p, q > 0, \nu \leq \Re(\nu) + p, c \in \mathbb{R} \) and \( D_{0+}^{\alpha,\beta,\gamma} \) be the left sided operator of the generalized k-fractional differentiation then there holds the formula:

\[
\left(D_{0+}^{\alpha,\beta,\gamma} (I_{0+}^{\xi-1} E^{\delta,q,\zeta}_{\nu,p} [ct^\xi])\right)(x) = \frac{x^{\frac{\rho+\beta-n}{k}} \Gamma_k(\xi)}{\Gamma_k(\delta)} \left[ \begin{array}{c}
(\rho + \beta + \gamma + \alpha, \nu), (\delta, qk), (k, k) \\
(\rho + \gamma, \nu), (\rho + \beta, \nu + 1 - k), (\xi, pk)
\end{array} \right] \tag{7.1}
\]
Proof. Applying (2.16) and (6.2) in the left-side of (7.1), we have
\[
\left( D^{\alpha,\beta}_{0+k} \left( t^{\frac{\xi}{n}} E_{\nu}^{\delta,\rho} [ct^{\frac{1}{n}}] \right) \right) (x) 
= D^{\alpha,\beta}_{0+k} \left( t^{\frac{\xi}{n}} \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}(ct^{\frac{1}{n}})^n}{\Gamma_k(\nu + n + \rho)(\xi)_{pn,k}} \right) (x) 
= \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}c^n}{\Gamma_k(\nu + n + \rho)(\xi)_{pn,k}} \left( D^{\alpha,\beta}_{0+k} t^{\frac{n+\beta-1}{n}} \right) (x) 
= x^{\frac{n+\beta-1}{n}} \Gamma_k(\xi) \sum_{n=0}^{\infty} \frac{\Gamma_k(\rho + \beta + \gamma + n)\Gamma_k(\delta + qn)\Gamma_k(1 + n)}{\Gamma_k(\xi + pnk)\Gamma_k(\gamma + vn)\Gamma_k(\rho + \beta + vn + n - nk)n!} 
\]

On applying \( n + 1 = k \cdot n \Gamma_k(nk + k) \), we get required result
\[
\left( D^{\alpha,\beta}_{0+k} \left( t^{\frac{\xi}{n}} E_{\nu}^{\delta,\rho} [ct^{\frac{1}{n}}] \right) \right) (x) 
= x^{\frac{n+\beta-1}{n}} \Gamma_k(\xi) \sum_{n=0}^{\infty} \frac{\Gamma_k(\rho + \beta + \gamma + n, \delta, qk, (k,k)}{\Gamma_k(\rho + \gamma, (k,k)} \right) \quad \text{\( k^{-1} c x^{\frac{n+\beta-1}{n}} \) (7.2) }
\]

Remark 7.1. If we put \( k = 1 \) in equation (7.1), we get the result [6, p.142, Eq.2.4].

Remark 7.2. If we put \( p = q = k = \xi = 1 \) in our formula (7.1), we get the result [1, p.119, Eq.5.1].

Theorem 7.2. Let \( \alpha, \beta, \gamma, \rho, \delta, \xi \in C \) and \( k \in R^+ \) be such that \( \text{Re}(\alpha) > 0, \text{Re}(\rho) > \max[\text{Re}(\alpha + \beta) + n - \text{Re}(\gamma)], v > 0, p, q > 0, q \leq \text{Re}(v) + p \) and \( c \in R, \text{Re}(\alpha + \beta - \gamma) + n \neq 0 \), (where \( n = [\text{Re}(\alpha) + 1] \)) and \( D^{\alpha,\beta}_{\xi} \) be the right sided operator of the generalized k-fractional differentiation then there holds the formula:
\[
\left( D^{\alpha,\beta}_{\xi} \left( t^{\frac{\xi}{n}} E_{\nu}^{\delta,\rho} [ct^{\frac{1}{n}}] \right) \right) (x) 
= \frac{x^{\frac{\alpha+\beta-\rho}{n}} \Gamma_k(\xi)}{\Gamma_k(\delta)} \sum_{n=0}^{\infty} \frac{\Gamma_k(\rho - \alpha - \beta, v - 1 + k, \rho + \gamma, v, \delta, qk, (k,k)}{\Gamma_k(\rho, v, (k,k)} \quad \text{\( k^{-1} c x^{\frac{n+\beta-1}{n}} \) (7.2) }
\]

Proof. Applying (2.16) and (6.3) in the left-side of (7.2), we have
\[
\left( D^{\alpha,\beta}_{\xi} \left( t^{\frac{\xi}{n}} E_{\nu}^{\delta,\rho} [ct^{\frac{1}{n}}] \right) \right) (x) 
= D^{\alpha,\beta}_{\xi} \left( \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}(ct^{\frac{1}{n}})^n}{\Gamma_k(\nu + n + \rho)(\xi)_{pn,k}} \right) (x) 
= \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}c^n}{\Gamma_k(\nu + n + \rho)(\xi)_{pn,k}} \left( D^{\alpha,\beta}_{\xi} t^{\frac{n+\beta-1}{n}} \right) (x) 
\]

Implying the simplification process used for providing preceding theorems, we obtain
\[
\left( D^{\alpha,\beta}_{\xi} \left( t^{\frac{\xi}{n}} E_{\nu}^{\delta,\rho} [ct^{\frac{1}{n}}] \right) \right) (x) 
= \frac{x^{\frac{\alpha+\beta-\rho}{n}} \Gamma_k(\xi)}{\Gamma_k(\delta)} \sum_{n=0}^{\infty} \frac{\Gamma_k(\rho - \alpha - \beta, v - 1 + k, \rho + \gamma, v, \delta, qk, (k,k)}{\Gamma_k(\rho, v, (k,k)} \quad \text{\( k^{-1} c x^{\frac{n+\beta-1}{n}} \) (7.2) }
\]

Remark 7.3. On taking \( k = 1 \) in equation (7.2), we can produce the result [6, p.143, Eq.(2.5)].

Remark 7.4. on setting \( p = q = k = \xi = 1 \) in equation (7.2), we obtained the result [1, p.120, Eq.(6.1)].
References


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