

**g**^*ωα*-Separation Axioms in Topological Spaces

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Abstract

In this paper, we introduce and study the new separation axioms called **g**^*ωα*-T_i (i= 0,1,2) and weaker forms of regular and normal spaces called **g**^*ωα*-normal and **g**^*ωα*-regular spaces using **g**^*ωα*-closed sets in topological spaces.

**Keywords:** **g**^*ωα*-closed sets, **g**^*ωα*-T_0 spaces, **g**^*ωα*-T_1 spaces, **g**^*ωα*-T_2 spaces, **g**^*ωα*-regular spaces, **g**^*ωα*-normal spaces.

2010 MSC: 54C05, 54C08, 54D10.

1 Introduction

General Topology plays an important role in many fields of applied sciences as well as branches of mathematics. More importantly, generalized closed sets suggest some new separation axioms which have been found to be very useful in the study of certain objects of digital topology.


In this paper, we introduce new weaker forms of separation axioms called **g**^*ωα*-T_0, **g**^*ωα*-T_1, **g**^*ωα*-T_2 spaces and new class of spaces namely **g**^*ωα*-regular and **g**^*ωα*-normal spaces and their characterizations are obtained.

2 Preliminary

Throughout this paper space (X, τ) and (Y, σ) (or simply X and Y) always denote topological spaces on which no separation axioms are assumed unless explicitly stated.

For a subset A of a space X, the closure (resp. _α_-closure [5]) and interior (resp. _α_-interior) of A is denoted by cl(A) (resp. _α_-cl(A)) and int(A) (resp. _α_-int(A)).

**Definition 2.1.** [9] A subset A of a topological space X is said to be a generalized star _ωα_-closed (briefly **g**^*ωα*-closed) if cl(A) ⊆ U whenever A ⊆ U and U is _ωα_-open in X.

The family of all **g**^*ωα*-closed (resp. **g**^*ωα*-open) subsets of a space X is denoted by **G**^*ωα_C(X) (resp. **G**^*ωα_O(X)).

**Definition 2.2.** [10] The intersection of all **g**^*ωα*-closed sets containing a subset A of X is called **g**^*ωα*-closure of A and is denoted by **g**^*ωα_-cl(A).

A set A is **g**^*ωα*-closed if and only if **g**^*ωα_-cl(A) = A.

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Definition 2.3. The union of all g*-ωα-open sets contained in a subset A of X is called g*-ωα-interior of A and it is denoted by g*-ωα-int(A). A set A is called g*-ωα-open if and only if g*-ωα-int(A) = A.

Definition 2.4. A function f : X → Y is called a
(i) g*-ωα-continuous if f⁻¹(V) is g*-ωα-closed in X for every closed set V in Y.
(ii) g*-ωα-irresolute if f⁻¹(V) is g*-ωα-closed in X for every g*-ωα-closed set V in Y.
(iii) pre g*-ωα-open if f(V) is g*-ωα-open in Y for every open set V in X.
(iv) pre g*-ωα-closed if f(V) is g*-ωα-closed set V in Y.

Definition 2.5. A topological space X is said to be a Tg*-ωα-space if every g*-ωα-closed set is closed.

3 g*-ωα-Separation Axioms

In this section, we introduce weaker forms of separation axioms such as g*-ωα-T₀, g*-ωα-T₁ and g*-ωα-T₂ spaces and obtain their properties.

Definition 3.1. A topological space X is said to be a g*-ωα-T₀ if for each pair of distinct points in X, there exists a g*-ωα-open set containing one point but not other.

Example 3.2. Let X = {a, b, c} and τ = {{X, φ}, {a}}. Then the space (X, τ) is g*-ωα-T₀ space.

Theorem 3.3. A space X is g*-ωα-T₀ if and only if g*-ωα-closures of distinct points are distinct.

Proof: Let x, y ∈ X with x ≠ y and X be g*-ωα-T₀ space. Since, X is g*-ωα-T₀, there exists g*-ωα-open set G such that x ∈ G but y /∈ G. Also, x /∈ X-G and y ∈ X-G where X-G is g*-ωα-closed in X. Since g*-ωα-cl(x) is the intersection of all g*-ωα-closed sets which contains y and hence y ∈ g*-ωα-cl(x). But x /∈ g*-ωα-cl(x) as x /∈ X-G. Therefore g*-ωα-cl(x) ≠ g*-ωα-cl(y).

Conversely, suppose for any pair of distinct points, x, y ∈ X, g*-ωα-cl(x) ≠ g*-ωα-cl(y). Then, there exists at least one point z ∈ X such that z ∈ g*-ωα-cl(x) but z /∈ g*-ωα-cl(y). We claim that x /∈ g*-ωα-cl(y). If x ∈ g*-ωα-cl(y), then g*-ωα-cl(x) ⊆ g*-ωα-cl(y), so z ∈ g*-ωα-cl(y) which is contradiction. Hence x /∈ g*-ωα-cl(y).

Then implies x ∈ X - g*-ωα-cl(y), which is g*-ωα-open set in X containing x but not y. Hence X is g*-ωα-T₀-space.

Theorem 3.4. Every subspace of a g*-ωα-T₀ space is g*-ωα-T₀ space.

Proof: Let y₁, y₂ be two distinct points of Y then y₁ and y₂ are also distinct points of X. Since X is g*-ωα-T₀, there exists g*-ωα-open set G such that y₁ ∈ G, y₂ /∈ G. Then G ∩ Y is g*-ωα-open set in Y containing y₁ but not y₂. Hence Y is g*-ωα-T₀-space.

Definition 3.5. A mapping f : X → Y is said to be a pre g*-ωα-open if the image of every g*-ωα-open set of X is g*-ωα-open in Y.

Lemma 3.6. The property of a space being g*-ωα-T₀ space is preserved under bijective and pre g*-ωα-open.

Proof: Let X be a g*-ωα-T₀-space and f : X → Y be bijective, pre g*-ωα-open. Let y₁, y₂ ∈ Y with y₁ ≠ y₂. Since f is bijective, there exist x₁, x₂ ∈ X such that f(x₁) = y₁ and f(x₂) = y₂. Also, since X is g*-ωα-T₀, there exists g*-ωα-open set G in X such that x₁ ∈ G but x₂ /∈ G. Then f(G) is g*-ωα-open set containing f(x₁) but not f(x₂) as X is g*-ωα-open. Thus, there exists g*-ωα-open set f(G) in Y such that y₁ ∈ f(G) and y₂ /∈ f(G). Hence Y is g*-ωα-T₀ space.
intersection of all $g^*\alpha$-open sets containing $A$ is the set $A$ itself.

c) $\Rightarrow$ (d): Obvious.

d) $\Rightarrow$ (a): Let us assume that the intersection of all $g^*\alpha$-open sets containing the point $x \in X$ is $\{x\}$. Let $x, y \in X$ with $x \neq y$. By hypothesis, there exists $g^*\alpha$-open set $G_x$ such that $x \in G_x$ and $y \notin G_x$. That is, $X$ is $g^*\alpha$-T$_0$ space.

**Theorem 3.8.** If $X$ is $g^*\alpha$-T$_0$, $T_{g^*\omega}$-space and $Y$ is $g^*\alpha$-closed subspace of $X$, then $Y$ is $g^*\alpha$-T$_0$-space.

**Theorem 3.9.** If $f : X \to Y$ is bijective, pre $g^*\omega$-open and $X$ is $g^*\omega$-T$_0$ space, then $Y$ is also $g^*\omega$-T$_0$ space.

**Proof:** Let $y_1$ and $y_2$ be two distinct points of $Y$. Then there exist $x_1$ and $x_2$ of $X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. As $X$ is $g^*\omega$-T$_0$, there exists $g^*\omega$-open set $G$ such that $x_1 \in G$ and $x_2 \notin G$. Therefore, $y_1 = f(x_1) \in f(G)$, $y_2 = f(x_2) \notin f(G)$. Then $f(G)$ is $g^*\omega$-open in $Y$. Thus, there exists $g^*\omega$-open set $f(G)$ in $Y$ such that $y_1 \in f(G)$ and $y_2 \notin f(G)$. Therefore $Y$ is $g^*\omega$-T$_0$ space.

**Definition 3.10.** A topological space $X$ is said to be a $g^*\omega$-T$_1$ if for each pair of distinct points $x, y$ in $X$, there exist a pair of $g^*\alpha$-open sets, one containing $x$ but not $y$ and the other containing $y$ but not $x$.

**Remark 3.11.** Every $T_1$-space is $g^*\omega$-T$_1$-space.

**Example 3.12.** $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Then $(X, \tau)$ is $g^*\omega$-T$_1$ space but not $T_1$-space.

**Remark 3.13.** Every $g^*\omega - T_1$ space is $g^*\omega - T_0$ space.

**Example 3.14.** Let $X = \{a, b\}$ and $\tau = \{X, \phi, \{a\}\}$. Then the space $X$ is $g^*\omega$-T$_0$ but not $g^*\omega$-T$_1$ space.

**Theorem 3.15.** A space $X$ is $g^*\omega$-T$_1$ if and only if every singleton subset $\{x\}$ of $X$ is $g^*\omega$-closed in $X$.

**Proof:** Let $x, y$ be two distinct points of $X$ such that $\{x\}$ and $\{y\}$ are $g^*\omega$-closed. Then $\{x\}^c$ and $\{y\}^c$ are $g^*\omega$-open in $X$ such that $y \in \{x\}^c$ but $x \notin \{y\}^c$ and $x \in \{y\}^c$ but $y \notin \{x\}^c$. Hence $X$ is $g^*\omega$-T$_1$-space.

Conversely, let $x$ be any arbitrary point of $X$. If $y \in \{x\}^c$, then $y \neq x$. Now the space being $g^*\omega$-T$_1$ and $y$ is different from $x$, there must exists $g^*\omega$-open set $G_y$ such that $y \in G_y$ but $x \notin G_y$. Thus for each $y \in \{x\}^c$, there exists a $g^*\omega$-open set $G_y$ such that $y \in G_y \subseteq \{x\}^c$. Therefore $\bigcup\{G_y : y \neq x\} \subseteq \{x\}^c$ which implies that $\{x\}^c \subseteq \bigcup\{G_y : y \neq x\} \subseteq \{x\}^c$. Therefore $\{x\}^c = \bigcup\{G_y : y \neq x\} \subseteq \{x\}^c$. Since, $G_y$ is $g^*\omega$-open set in $X$ and the union of $g^*\omega$-open set is again $g^*\omega$-open in $X$, so $\{x\}^c$ is $g^*\omega$-open in $X$. Hence $\{x\}$ is $g^*\omega$-closed in $X$.

**Corollary 3.16.** A space $X$ is $g^*\omega$-T$_1$ if and only if every finite subset of $X$ is $g^*\omega$-closed.

**Theorem 3.17.** Let $f : X \to Y$ be bijective and $g^*\omega$-open. If $X$ is $g^*\omega$-T$_1$ and $T_{g^*\omega}$-space then, $Y$ is $g^*\omega$-T$_1$-space.

**Proof:** Let $y_1$ and $y_2$ be any two distinct points of $Y$. Since $f$ is bijective, then there exist distinct points $x_1$ and $x_2$ of $X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then there exist $g^*\omega$-open sets $G$ and $H$ such that $x_1 \in G$, $x_2 \notin G$ and $x_1 \notin H$, $x_2 \in H$. Therefore $y_1 = f(x_1) \in f(G)$ but $y_2 = f(x_2) \notin f(G)$ and $y_2 = f(x_2) \in f(H)$ and $y_1 = f(x_1) \notin f(H)$. As $X$ is $T_{g^*\omega}$-space, $G$ and $H$ are open sets in $X$ and as $f$ is $g^*\omega$-open, $f(G)$ and $f(H)$ are $g^*\omega$-open subsets of $Y$. Thus, there exist $g^*\omega$-open sets such that $y_1 \in f(G)$, $y_2 \notin f(G)$ and $y_2 \in f(H)$, $y_1 \notin f(H)$. Hence $Y$ is $g^*\omega$-T$_1$-space.

**Theorem 3.18.** Let $f : X \to Y$ be $g^*\omega$-irresolute and injective. If $Y$ is $g^*\omega$-T$_1$ then $X$ is $g^*\omega$-T$_1$.

**Proof:** Let $x, y \in X$ such that $x \neq y$. Then there exist $g^*\omega$-open sets $U$ and $V$ in $X$ such that $f(x) \in U$, $f(y) \in V$ and $f(x) \notin V$, $f(y) \notin U$. Then $x \in f^{-1}(U)$, $y \in f^{-1}(V)$ and $x \notin f^{-1}(V)$, $y \notin f^{-1}(U)$, since $f$ is $g^*\omega$-irresolute. Hence $X$ is $g^*\omega$-T$_1$ space.

**Theorem 3.19.** If $f : X \to Y$ is $g^*\omega$-continuous, injective and $Y$ is $T_1$ then, $X$ is $g^*\omega$-T$_1$ space.

**Proof:** For any two distinct points $x_1$ and $x_2$ of $X$ there exist disjoint points $y_1$ and $y_2$ of $Y$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $Y$ is $T_1$, there exist open sets $U$ and $V$ in $Y$ such that $y_1 \in U$, $y_2 \notin U$ and $y_1 \notin V$, $y_2 \in V$. That is, $x_1 \in f^{-1}(U)$, $x_2 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V)$, $x_2 \notin f^{-1}(U)$. Again, since $f$ is $g^*\omega$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $g^*\omega$-open sets in $X$. Thus, for two distinct points $x_1$ and $x_2$ of $X$, there exist $g^*\omega$-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U)$, $x_1 \notin f^{-1}(V)$ and $x_2 \notin f^{-1}(V)$, $x_2 \notin f^{-1}(U)$. Therefore $X$ is $g^*\omega$-T$_1$ space.

**Definition 3.20.** A space $X$ is said to be $g^*\omega$-T$_2$ if for each pair of distinct points $x, y$ of $X$, there exist disjoint $g^*\omega$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$.

**Example 3.21.** Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the space $(X, \tau)$ is $g^*\omega$-T$_2$ space, but not $g^*\omega$-T$_1$ and $g^*\omega$-T$_0$ space.
Theorem 3.22. Let X be a topological space. Then X is $g^*\omega\alpha$-$T_2$ if and only if the intersection of all $g^*\omega\alpha$-closed neighborhood of each point of X is singleton set.

Proof: Let x and y be any two distinct points of X. Since, X is $g^*\omega\alpha$-$T_2$, there exist $g^*\omega\alpha$-open sets G and H such that $x \in G$, $y \in H$ and $G \cap H = \phi$. Since, $G \cap H = \phi$, $x \in G \subseteq X - H$, so X-H is $g^*\omega\alpha$-closed neighborhood of x which does not contains y. Thus y does not belong to the intersection of all $g^*\omega\alpha$-closed neighborhood of x. Since y is arbitrary, the intersection of all $g^*\omega\alpha$-closed neighborhood of x is the singleton set.

Conversely, let $\{x\}$ be the intersection of all $g^*\omega\alpha$-closed neighborhood of an arbitrary point x in X and y be a point of X different from x. Since y does not belong to the intersection, there exists $g^*\omega\alpha$-closed neighborhood N of x, such that $y \notin N$. Since, N is $g^*\omega\alpha$ neighborhood of x there exists $g^*\omega\alpha$-open set G such that x $\in G \subseteq N$. Thus G and X-N are $g^*\omega\alpha$-open sets such that $x \in G$, y $\in X - N$ and $G \cap (X - N) = \phi$. Hence X is $g^*\omega\alpha$-$T_2$ space.

Theorem 3.23. If $f : X \rightarrow Y$ is an injective, $g^*\omega\alpha$-irresolute and Y is $g^*\omega\alpha$-$T_2$ then, X is $g^*\omega\alpha$-$T_2$.

Proof: Let $x_1$ and $x_2$ be any two distinct points in X. So, $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$ as f is bijective. Then $y_1$ and $y_2$ $\in Y$ such that $y_1 \neq y_2$. Since, Y is $g^*\omega\alpha$-$T_2$, there exist $g^*\omega\alpha$-open sets G and H such that $y_1 \in G$, $y_2 \in H$ and $G \cap H = \phi$. Then $f^{-1}(G)$ and $f^{-1}(H)$ are $g^*\omega\alpha$-open sets of X as f is $g^*\omega\alpha$-irresolute. Now $f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\phi) = \phi$. Then $y_1 \in G$ implies $f^{-1}(y_1) \in f^{-1}(G)$ and $x_1 \in f^{-1}(G)$, $y_2 \in H$ that is, $f^{-1}(y_2) \in f^{-1}(H)$ so $x_2 \in f^{-1}(H)$. Thus for every pair of distinct points $x_1$ and $x_2$ of X, there exist disjoint $g^*\omega\alpha$-open sets $f^{-1}(G)$ and $f^{-1}(H)$ such that $x_1 \in f^{-1}(G)$, $x_2 \in f^{-1}(H)$. Hence X is $g^*\omega\alpha$-$T_2$ space.

Theorem 3.24. If $f : X \rightarrow Y$ is $g^*\omega\alpha$-continuous and injective and Y is $T_2$ then X is $g^*\omega\alpha$-$T_2$ space.

Proof: For any two distinct points $x_1$ and $x_2$ of X, there exist disjoint points $y_1$ and $y_2$ of Y such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since Y is $T_2$, there exist disjoint open sets U and V in Y such that $y_1 \in U$ and $y_2 \in V$, that is, $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. Again, since f is $g^*\omega\alpha$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $g^*\omega\alpha$-open sets in X. Further $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\phi) = \phi$. Thus for two disjoint points $x_1$ and $x_2$ of X, there exist disjoint $g^*\omega\alpha$-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. Therefore X is $g^*\omega\alpha$-$T_2$ space.

Theorem 3.25. The following properties are equivalent for any topological space X:

(a) $g^*\omega\alpha$-$T_2$ space

(b) for each $x \neq y$, there exists $g^*\omega\alpha$-open set U such that $x \in U$ and $y \notin g^*\omega\alpha$-cl(U)

(c) for each $x \in X$, $\{x\} = \cap\{g^*\omega\alpha$-cl(U): U is $g^*\omega\alpha$-open in X and $x \in U\}.$

Proof: (a) $\Rightarrow$ (b): Let $x \in X$ and $x \neq y$, then there exist disjoint $g^*\omega\alpha$-open sets U and V such that $x \in U$ and $y \in V$. Then $X - V$ is $g^*\omega\alpha$-closed. Since $U \cap V = \phi$, $U \subseteq X - V$. Therefore $g^*\omega\alpha$-cl(U) $\subseteq g^*\omega\alpha$-cl(X - V) = X - V. Now $y \notin X - V$ implies that $y \notin g^*\omega\alpha$-cl(U).

(b) $\Rightarrow$ (c): For each $x \neq y$, there exists $g^*\omega\alpha$-open set U such that $x \in U$ and $y \notin g^*\omega\alpha$-cl(U). So $y \notin \cap\{g^*\omega\alpha$-cl(U): U is $g^*\omega\alpha$-open in X, $x \in U\} = \{x\}$.

(c) $\Rightarrow$ (a): Let $x, y \in X$ and $x \neq y$. Then by hypothesis, there exists $g^*\omega\alpha$-open set U such that $x \in U$ and $y \notin g^*\omega\alpha$-cl(U). This implies that, there exists $g^*\omega\alpha$-closed set V such that $y \notin V$. Therefore $y \in X - V$ and $X - V$ is $g^*\omega\alpha$-open set. Thus, there exist two disjoint $g^*\omega\alpha$-open sets X and $X - V$ such that $x \in U$ and $y \in X - V$. Therefore X is $g^*\omega\alpha$-$T_2$ space.

4 $g^*\omega\alpha$-Normal Spaces

In this section, the concept of $g^*\omega\alpha$-normal spaces are introduced and obtained their characterizations.

Definition 4.1. A space X is said to be a $g^*\omega\alpha$-normal if for any pair of disjoint $g^*\omega\alpha$-closed sets A and B in X, there exist disjoint open sets U and V in X such that $A \subseteq U$, $B \subseteq V$.

Remark 4.2. Every $g^*\omega\alpha$-normal space normal.

However, the converse is not true in general as seen from the following example.

Example 4.3. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Then the space $(X, \tau)$ is normal but not $g^*\omega\alpha$-normal.

Remark 4.4. If X is normal and $T_{g^*\omega\alpha}$-space then X is $g^*\omega\alpha$-normal.

Theorem 4.5. The following are equivalent for a space X:

(a) X is normal

(b) for any disjoint closed sets A and B, there exist disjoint $g^*\omega\alpha$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$
(c) for any closed set A and any open set V containing A, there exists $g^*\omega\alpha$-open set U in X such that $A \subseteq U \subseteq \text{cl}(U) \subseteq V$.

**Proof:** (a) $\Rightarrow$ (b): Follows from [9].

(b) $\Rightarrow$ (c): Let A be a closed and V be an open set containing A. Then A and X-V are disjoint closed sets in X. There exist $g^*\omega\alpha$-open sets U and W such that $A \subseteq U$ and $X-V \subseteq W$. Since X-V is closed, X-V is $g^*\omega\alpha$-closed [9]. We have, $X-V \subseteq \text{int}(W)$ and $U \cap \text{int}(W) = \phi$ and so, $\text{cl}(U) \cap \text{int}(W) = \phi$ and hence $A \subseteq U \subseteq \text{cl}(U) \subseteq X-\text{int}(W) \subseteq V$.

(c) $\Rightarrow$ (a): Let A, B be disjoint closed sets in X. Thus Y is $g^*\omega\alpha$-normal. We have A $\subseteq$ int(G) put $U = \text{int}(G)$ and $V = \text{int}(X-G)$ . Then U and V are disjoint open sets of X such that $A \subseteq U$ and $B \subseteq V$. Therefore X is normal.

**Theorem 4.6.** The following statements are equivalent for a topological space X:

(a) X is $g^*\omega\alpha$-normal

(b) for each closed set A and for each open set U containing A, there exists $g^*\omega\alpha$-open set V containing A such that $g^*\omega\alpha\text{-cl}(V) \subseteq U$

(c) for each pair of disjoint closed sets A and B there exists $g^*\omega\alpha$-open set U containing A such that $g^*\omega\alpha\text{-cl}(U) \cap B = \phi$.

**Proof** (a) $\Rightarrow$ (b): Let A be closed and U be an open set containing A. Then $A \cap (X \setminus U) = \phi$ and therefore disjoint closed sets in X. Since X is $g^*\omega\alpha$-normal, there exist disjoint $g^*\omega\alpha$-open sets V and W such that $A \subseteq U$, $X-U \subseteq W$, that is $X-W \subseteq U$. Now $V \cap W = \phi$, implies $V \subseteq X-W$. Therefore $g^*\omega\alpha\text{-cl}(V) \subseteq g^*\omega\alpha\text{-cl}(X-W) = X-W$ since $X-W$ is $g^*\omega\alpha$-closed. Thus, $A \subseteq V \subseteq g^*\omega\alpha\text{-cl}(V) \subseteq X-W \subseteq U$. That is $A \subseteq V \subseteq g^*\omega\alpha\text{-cl}(V) \subseteq U$.

(b) $\Rightarrow$ (c): Let A and B be disjoint closed sets in X then A $\subseteq X-B$ and $X-B$ is an open set containing A. Then there exists $g^*\omega\alpha$-open set U such that $A \subseteq U$ and $g^*\omega\alpha\text{-cl}(U) \subseteq X-B$, which implies $g^*\omega\alpha\text{-cl}(U) \cap B = \phi$.

(c) $\Rightarrow$ (a): Let A and B be disjoint closed sets in X. Then there exists $g^*\omega\alpha$-open set U such that $A \subseteq U$ and $g^*\omega\alpha\text{-cl}(U) \subseteq X-B$. Hence X is $g^*\omega\alpha$-normal.

**Theorem 4.7.** If X is normal and $F \cap A = \phi$ where $F$ is $\omega\alpha$-closed and A is $g^*\omega\alpha$-closed then there exist open sets U and V such that $F \subseteq U$ and $A \subseteq V$.

**Proof:** Let X be a normal and $F \cap A = \phi$. Since, $F$ is $\omega\alpha$-closed and A is $g^*\omega\alpha$-closed such that $A \subseteq X-F$ and $X-F$ is $\omega\alpha$-open. Therefore $\text{cl}(A) \subseteq X-F$ implies that $\text{cl}(A) \cap F = \phi$. Now F is closed, so F and cl(A) are disjoint closed sets in X. As X is a normal, there exist disjoint open sets U and V of X such that $F \subseteq U$ and $A \subseteq V$.

**Theorem 4.8.** If X is $g^*\omega\alpha$-normal and Y is $g^*\omega\alpha$-closed subset of X then, the subspace Y is also $g^*\omega\alpha$-normal.

**Proof:** Let A and B be any two disjoint $g^*\omega\alpha$-closed sets in Y, then A and B are $g^*\omega\alpha$-closed sets in X by [9]. Since X is $g^*\omega\alpha$-normal, there exist disjoint open sets U and V in X such that $A \subseteq U$, $B \subseteq V$. Therefore $U \cap Y$ and $V \cap Y$ are disjoint open subsets of the subspace Y such that $A \subseteq U \cap Y$ and $B \subseteq V \cap Y$. Hence the subspace Y is $g^*\omega\alpha$-normal.

**Remark 4.9.** The property of being $g^*\omega\alpha$-normal is closed hereditary.

**Theorem 4.10.** If $f : X \rightarrow Y$ is pre $g^*\omega\alpha$-closed, continuous injective and Y is $g^*\omega\alpha$-normal then, X is $g^*\omega\alpha$-normal.

**Proof:** Let A and B be disjoint $g^*\omega\alpha$-closed sets in X. Since, f is pre $g^*\omega\alpha$-closed, $f(A)$ and $f(B)$ are disjoint $g^*\omega\alpha$-closed sets in Y. Again, since Y is $g^*\omega\alpha$-normal there exist disjoint open sets U and V such that $f(A) \subseteq U$, $f(B) \subseteq V$. Thus $A \subseteq f^{-1}(U)$, $B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets in X as f is continuous. Hence X is $g^*\omega\alpha$-normal.

**Theorem 4.11.** If $f : X \rightarrow Y$ is $g^*\omega\alpha$-irresolute, bijective, open map from a $g^*\omega\alpha$-normal space X on to a space Y then Y is $g^*\omega\alpha$-normal.

**Proof:** Let A and B be two disjoint $g^*\omega\alpha$-closed sets in Y. Since, f is $g^*\omega\alpha$-irresolute and bijective, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $g^*\omega\alpha$-closed sets in X. As X is $g^*\omega\alpha$-normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$, that is $A \subseteq f(U)$ and $B \subseteq f(V)$. Then $f(U)$ and $f(V)$ are open sets in Y and $f(U) \cap f(V) = \phi$. Thus Y is $g^*\omega\alpha$-normal.

5 $g^*\omega\alpha$-Regular Spaces

The concept of $g^*\omega\alpha$-regular spaces and their properties are studied in this section.

**Definition 5.1.** A topological space X is said to be a $g^*\omega\alpha$-regular if for each $g^*\omega\alpha$-closed set F and each point $x \notin F$ there exist disjoint open sets U and V in X such that $x \in U$ and $F \subseteq V$. 


Remark 5.2. Every $g^\omega\alpha$-regular space is regular. However, the converse need not be true as seen from the following example.

Example 5.3. From Example 4.3, the space $(X, \tau)$ is regular but not $g^\omega\alpha$-regular.

Theorem 5.4. Every $g^\omega\alpha$-regular $T_0$-space is $g^\omega\alpha$-$T_2$.

Proof: Let $x$ and $y$ be any two points in $X$ such that $x \neq y$. Let $V$ be an open set containing $x$ but not $y$. Then, $X - V$ is a closed set containing $y$ but not $x$. Hence, $X$ is $g^\omega\alpha$-regular.

Remark 5.5. From Example 4.3, the space $(X, \tau)$ is regular but not $g^\omega\alpha$-regular.

Theorem 5.5. In a topological spaces $X$, the following properties are equivalent:

(a) $X$ is $g^\omega\alpha$-regular space

(b) for each point $x \in X$ and each $g^\omega\alpha$-open neighborhood $A$ of $x$, there exists open neighborhood $V$ of $X$ such that $cl(V) \subseteq A$.

Proof: (a) $\Rightarrow$ (b): Suppose $X$ is $g^\omega\alpha$-open neighborhood of $x$. Then there exists $g^\omega\alpha$-open set $G$ such that $x \in G \subseteq A$. Since $X - G$ is $g^\omega\alpha$-closed and $x \notin X - G$. By hypothesis, there exist open sets $U$ and $V$ such that $X - G \subseteq U$, $x \notin V$ and $U \cap V = \phi$ and so $V \subseteq X - U$. Now $cl(V) \subseteq cl(X - U) = X - U$ and $X - G \subseteq U$ implies $X - U \subseteq G \subseteq A$. Therefore $cl(V) \subseteq A$.

(b) $\Rightarrow$ (a): Let $F$ be a closed set in $X$ with $x \notin F$. Then $x \in X - F$ and $X - F$ is $g^\omega\alpha$-open and so $X - F$ is $g^\omega\alpha$-neighborhood of $X$. By hypothesis, there exists open neighborhood $V$ of $X$ such that $x \in V$ and $cl(V) \subseteq X - F$, which implies $F \subseteq X - cl(V)$. Then $X - cl(V)$ is an open set containing $F$ and $V \cap (X - cl(V)) = \phi$. Therefore $X$ is $g^\omega\alpha$-regular.

Theorem 5.6. If $X$ is $g^\omega\alpha$-regular and $Y$ is open, $g^\omega\alpha$-closed subspace of $X$, then the subspace $Y$ is $g^\omega\alpha$-regular.

Proof: Let $A$ be $g^\omega\alpha$-closed subspace of $Y$ and $y \notin A$ then $A$ is $g^\omega\alpha$-closed in $X$. Since $X$ is $g^\omega\alpha$-regular there exist open sets $U$ and $V$ in $X$ such that $y \in U$ and $A \subseteq V$. Therefore $U \cap Y$ and $V \cap Y$ are disjoint open sets of the subspace $Y$, such that $y \in U \cap Y$ and $A \subseteq V \cap Y$. Hence $Y$ is $g^\omega\alpha$-regular.

Theorem 5.7. Let $f : X \rightarrow Y$ be bijective, $g^\omega\alpha$-irresolute and open. If $X$ is $g^\omega\alpha$-regular then $Y$ is also $g^\omega\alpha$-regular.

Proof: Let $F$ be $g^\omega\alpha$-closed set of $Y$ and $y \notin F$. Since $f$ is $g^\omega\alpha$-irresolute, $f^{-1}(F)$ is $g^\omega\alpha$-closed in $X$. Let $f(x) = y$, so $x = f^{-1}(y)$ and $x \notin f^{-1}(F)$. Again, $X$ is $g^\omega\alpha$-regular there exist open sets $U$ and $V$ such that $x \in U$ and $f^{-1}(F) \subseteq V$, $U \cap V = \phi$. Since, $f$ is open and bijective, so $y \in f(U)$, $f \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$. Hence $Y$ is $g^\omega\alpha$-regular.

Theorem 5.8. If $f : X \rightarrow Y$ is bijective, pre $g^\omega\alpha$-closed and open map from a space $X$ in to a $g^\omega\alpha$-regular space $Y$. If $X$ is $T_{g^\omega\alpha\omega}$ space then $X$ is $g^\omega\alpha$-regular.

Proof: Let $x \in X$ and $F$ be a $g^\omega\alpha$-closed set in $X$ with $x \notin F$. Since $X$ is $T_{g^\omega\alpha\omega}$ space so, $F$ is closed in $X$. Then $f(F)$ is $g^\omega\alpha$-closed with $f(x) \notin f(F)$ in $Y$ as $f$ is pre $g^\omega\alpha$-closed. Again, since $Y$ is $g^\omega\alpha$-regular there exist open sets $U$ and $V$ such that $f(x) \in U$ and $f(F) \subseteq V$. Therefore $x \in f^{-1}(U)$ and $F \subseteq f^{-1}(V)$. Hence $X$ is $g^\omega\alpha$-regular.

Theorem 5.9. Every subspace of a $g^\omega\alpha$-regular space is $g^\omega\alpha$-regular.

Proof: Let $Y$ be a $g^\omega\alpha$-regular space. Let $x \in Y$ and $F$ be a $g^\omega\alpha$-closed set in $Y$ such that $x \notin F$. Then there exists $g^\omega\alpha$-closed set $A$ of $X$ with $F = Y \cap A$ and $x \notin A$. Therefore, we have $x \in X$, $A$ is $g^\omega\alpha$-closed in $X$ such that $x \notin A$. Since, $X$ is $g^\omega\alpha$-regular, there exist open sets $G$ and $H$ such that $x \in G$, $A \subseteq H$ and $G \cap H = \phi$. Note that $Y \cap G$ and $Y \cap H$ are open sets in $Y$. Also $x \in G$ and $x \in Y$ which implies $x \in Y \cap G$ and $A \subseteq H$ implies $Y \cap G \subseteq Y \cap H$, $F \subseteq Y \cap H$. Also $Y \cap G \cap (Y \cap H) = \phi$. Hence $Y$ is $g^\omega\alpha$-regular.

Theorem 5.10. Let $f : X \rightarrow Y$ be continuous, $g^\omega\alpha$-closed, surjective and open map. If $X$ is regular then $Y$ is also regular.

Proof: Let $y \in Y$ and $V$ be an open set containing $y$ in $Y$. Let $x$ be a point of $X$ such that $y = f(x)$. Since, $X$ is regular and $f$ is continuous there exists open set $U$ such that $x \in U \subseteq cl(U) \subseteq f^{-1}(V)$. Hence $y \in f(U) \subseteq f(cl(U)) \subseteq V$. Again, since $f$ is $g^\omega\alpha$-closed map, $f(cl(U))$ is $g^\omega\alpha$-closed set contained in the open set $V$. Hence $cl(f(cl(U))) \subseteq V$. Therefore $y \in f(U) \subseteq f(cl(U)) \subseteq cl(f(cl(U))) \subseteq V$. This implies $y \in f(U) \subseteq f(V) \subseteq V$ and $f(U)$ is open. Hence $Y$ is regular.

Theorem 5.11. If $f : X \rightarrow Y$ is $g^\omega\alpha$-irresolute, open, bijective and $X$ is $g^\omega\alpha$-regular then, $Y$ is $g^\omega\alpha$-regular.

Proof: Let $F$ be a $g^\omega\alpha$-closed set in $Y$ and $y \notin F$. Take $y = f(x)$ for some $x \in X$. Since, $f$ is $g^\omega\alpha$-irresolute, $f^{-1}(F)$ is $g^\omega\alpha$-closed in $X$ and $x \neq f^{-1}(F)$. Then there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $f^{-1}(F) \subseteq V$, that is $y = f(x) \in f(U)$, $f \subseteq f(V)$ and $f(U) \cap f(V) = \phi$. Therefore $Y$ is $g^\omega\alpha$-regular.
Theorem 5.12. If \( f : X \to Y \) be pre \( g^*\omega\alpha \)-open, closed, injective and \( Y \) is \( g^*\omega\alpha \)-regular then, \( X \) is \( g^*\omega\alpha \)-regular.

Proof: Let \( F \) be a \( g^*\omega\alpha \)-closed set in \( X \) and \( x \notin F \). Since, \( f \) is pre \( g^*\omega\alpha \)-closed, \( f(F) \) is \( g^*\omega\alpha \)-closed in \( Y \) such that \( f(x) \notin f(F) \). Now \( Y \) is \( g^*\omega\alpha \)-regular, there exist open sets \( G \) and \( H \) such that \( f(x) \in G \) and \( f(H) \subseteq H \). This implies that \( x \in f^{-1}(G) \) and \( F \subseteq f^{-1}(H) \). Further \( f^{-1}(G) \cap f^{-1}(H) = \phi \). Hence \( X \) is \( g^*\omega\alpha \)-regular.

6 Conclusion

The research in topology over last two decades has reached a high level in many directions. By researching generalizations of closed sets, some new separation axioms have been founded and they turn out to be useful in the study of digital topology. Therefore, \( g^*\omega\alpha \)-separation axioms are defined by using \( g^*\omega\alpha \)-closed sets will have many possibilities of applications in digital topology and computer graphics.

7 Acknowledgment

The first and second authors are grateful to the University Grants Commission, New Delhi, India for financial support under UGC SAP DRS-III : F-510/3/DRS-III/2016(SAP-I) dated 29th February 2016 to the Department of Mathematics, Karnataka University, Dharwad, India. Also, the third author is thankful to Karnataka University, Dharwad, India for financial support under No.KU/Sch/UGC-UPE/2014-15/893 dated 24th November 2014.

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Received: January 07, 2017; Accepted: March 17, 2017