\theta\text{-local function and }\psi_\theta\text{-operator}

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Abstract

In this paper, we introduce the notion of a \theta-local function and investigate some of their properties. Also, we define two operators \((\ast)^\theta\) and \(\psi_\theta\) in an ideal topological space.

Keywords: \theta-local function, \((\ast)^\theta\)-operator, \theta-compatible and \psi_\theta-operator.

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1 Introduction

In 1968, Velicko\cite{22} introduced the notions of \theta-open subsets, \theta-closed subsets and \theta-closure, for the sake of studying the important class of \(H\)-closed spaces in terms of arbitrary filterbases. In 1990, Jankovic and Hamlett\cite{7,8} defined the concept of I-open set via local function which was given by Vaidyanathaswamy. O.Njastad\cite{16,17} introduced the concept of compatible ideals in 1966. This ideal was also called as supercompact by Vaidyanathaswamy\cite{20,21}. In an ideal topological space, the local function was introduced by Kuratowski\cite{11}. After that so many mathematicians like Hayashi \cite{7}, Natkaniec\cite{15} and Modak and Bandyopadhay\cite{14} have studied this field and proved some new results in an ideal topological spaces. In 2009, Jeong Gi Kang and Chang Su Kim \cite{10} defined pre-local function, semi-local function and \(\alpha\)-local function. In 2011, Shyamapada Modak \cite{16} introduced \(\delta\)-local function and an operator \(\psi_\delta\) in the ideal topological spaces. In 2013, Arokia Rani and Nithya\cite{2} introduced precompatible ideals, Al-Omari and Noiri\cite{1} defined the local closure function and an operator \(\psi_\Gamma\) and K. Bhavani\cite{3,4} introduced \(g\)-local function and \(\psi_\theta\)-operator in the ideal topological spaces.

In this paper, we introduce the notion of a \theta-local function and investigate some of their properties. We also introduce two operators \((\ast)^\theta\) and \(\psi_\theta\) a \ast\theta-closure operator in lines with kuratowski. Also, we discuss \theta-compatibility of topological spaces.

2 Preliminaries

Let \((X, \tau)\) be a topological space with no separation properties assumed. For a subset \(A\) of a space \((X, \tau)\), \(\text{cl}(A)\) and \(\text{int}(A)\) denote the closure of \(A\) and the interior of \(A\) respectively. \((X, \tau)\) and \((Y, \sigma)\) will be replaced by \(X\) and \(Y\) if there is no chance of confusion. A subset \(A\) of \(X\) is said to be semi open\cite{9} (resp. pre open\cite{10} and \(\alpha\)-open\cite{13}) if \(A \subset \text{cl}(\text{int}(A))\) (resp. \(A \subset \text{int}(\text{cl}(A))\)) and \(A \subset \text{int}(\text{cl}(\text{int}(A)))\). The complement of semi open (resp. pre open and \(\alpha\)-open) is called semi closed (resp. pre closed and \(\alpha\)-closed).

A set \(A\) is said to be \theta\text{-open}\cite{1} if every point of \(A\) has an open neighborhood whose closure is contained in \(A\). It is very well known that the family of all \theta\text{-open subsets of }\((X, \tau)\) are topologies on \(X\) which we shall denote by \(\tau^\theta\). From the definitions it follows immediately that \(\tau^\theta \subset \tau\). A space \((X, \tau)\) is regular if and only if \(\tau^\theta = \tau\). A point \(x \in X\) is said to be in the \theta-closure of a subset \(A \subseteq X\)[6] if for each open neighbourhood \(U\) of \(x\)

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we have $\text{cl}(U) \cap A \neq \phi$. We shall denote $\theta$-closure by $\text{cl}_\theta(A)$. A subset $A \subseteq X$ is called $\theta$-closed if $A = \text{cl}_\theta(A)$. In general, the $\theta$-closure of a given set need not be a $\theta$-closed set. But it is always closed. A point $x \in A$ is said to be a $\theta$-limit point of $A$[5] in $X$ if for each $\theta$-open set $U$ containing $x$, such that $U \cap (A - \{x\}) \neq \phi$. The set all $\theta$-limit points of $A$ is called a $\theta$-derived set of $A$ and is denoted by $D_\theta(A)$.

An ideal $\mathcal{I}$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies (i) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $(A \cup B) \in \mathcal{I}$. A topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$ is called an ideal topological space and is denoted by $(X, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$ is called the local function of $A$ with respect to $\mathcal{I}$ and $\tau$[4]. We simply write $A^*$ in case there is no chance for confusion. A Kuratowski[11] closure operator $\text{cl}^*(\tau)$ for a topology $\tau^*(\mathcal{I})$ called the $\tau^*$-topology finer than $\tau$ is defined $\text{cl}^*(A) = A \cup A^*$. A subset $A$ of an ideal space $(X, \mathcal{I})$ is $\tau^*$-closed [18] (resp. $\tau^*$-dense in itself [18], $\tau^*$-perfect [18]) if $A^* \subseteq A$ (resp. $A \subset A^*$, $A = A^*$). Clearly, $A$ is $\tau^*$-perfect if and only if $A$ is $\tau^*$-closed and $\tau^*$-dense in itself. An ideal $\mathcal{I}$ in a space $(X, \tau)$ is said to be compatible with respect to $\tau[9]$, denoted by $\mathcal{I} \sim \tau$, if for every subset $A$ of $X$ and for each $x \in A$, there exists a neighborhood $U$ of $x$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. Let $(X, \tau)$ be a topological space with $\mathcal{I}$ an ideal on $X$, then $\tau$ is pre-compatible[2] with $\mathcal{I}$, if for every $A \subseteq X$, and for every $x \in A$, there exists a $U \in \text{PO}(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$ and is denoted by $\mathcal{I} \sim \text{po} \tau$. An operator[8] $\psi : \varphi(X) \rightarrow \tau$ is defined as: $\psi(A) = \{x \in X : \text{there exists an open set } O_x \text{ such that } O_x - A \in \mathcal{I}\}$, for every $A \in \varphi(X)$. Its equivalent definition is $\psi(A) = X - (X - A)^*$. Let $A$ be a subset of an ideal topological space $(X, \mathcal{I}, \tau)$. Then the set $(1) \ A^*_p(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau^p(x)\}$ is called the pre-local function with respect to $\mathcal{I}$ and $\tau$. (2) $A^*_s(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau^s(x)\}$ is called the semi-local function with respect to $\mathcal{I}$ and $\tau$. (3) $A^*_o(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau^o(x)\}$ is called the $\alpha$-local function with respect to $\mathcal{I}$ and $\tau$. Al-Omari and Noiri[1] defined the local closure function and an operator $\psi_\Gamma$ in an ideal topological spaces as follows:$\Gamma(A)(\mathcal{I}, \tau) = \{x \in X : A \cap \text{cl}(U) \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$ and $\psi_\Gamma(A) = X - \Gamma(X - A)$ where $\psi : \varphi(X) \rightarrow \tau$. K. Bhavani[3,4] introduced $g$-local function and $g$-operator in the ideal topological spaces as: $g(A^*(\mathcal{I}, \tau_g) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \text{ open set } U \text{ containing } x\}$ and $g(A) = \{x \in X : \text{there exists a } g\text{-open set } U_x \text{ containing } x \text{ such that } U_x - A \in \mathcal{I}\}$ for every $A \in \varphi(X)$ where $\psi_g : \varphi(X) \rightarrow \varphi(X)$.

**Result 2.1** Let $A$ be a subset of a topological space $(X, \tau)$. If $A \in \tau^\theta$, then $\text{cl}_\theta(A) = A$

**Lemma 2.1.** [1]. Let $A$ be a subset of a topological space $(X, \tau)$. Then

1. if $A$ is open, then $\text{cl}(A) = \text{cl}_\theta(A)$
2. if $A$ is closed, then $\text{int}(A) = \text{int}_\theta(A)$

**Lemma 2.2.** If $(X, \tau, \mathcal{I})$ is an ideal topological space, then $\mathcal{I}$ is codense[18] if and only in $A \subset A^*$ for every open set $A$ of $X$.

**Lemma 2.3.** [18]. If $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. If $A \subset A^*$, then $A^* = \text{cl}(A^*) = \text{cl}(A) = \text{cl}_\theta(A)$.

### 3 The Operator($\ast^\theta$)

In this section we shall introduce an operator ($\ast^\theta$) and discuss various properties of this operator.

**Definition 3.1.** Let $A$ be a subset of an ideal topological space $(X, \mathcal{I}, \tau)$. Then, the $\theta$-local function of $\mathcal{I}$ on $X$ is defined as $A^\ast\theta(\mathcal{I}, \tau) = \{x \in X : U_x \cap A \notin \mathcal{I} \text{ for every } U_x \in \theta \text{O}(X, x)\}$ with respect to $\mathcal{I}$ and $\tau$ and is denoted as $A^\ast\theta$ for $A^\ast\theta(\mathcal{I}, \tau)$.

**Lemma 3.1.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then every subset $A$ of $X$,

1. $A^*_p(\mathcal{I}, \tau) \subseteq A^\ast\theta(\mathcal{I}, \tau)$.
2. $A^*_s(\mathcal{I}, \tau) \subseteq A^\ast\theta(\mathcal{I}, \tau)$.
3. $A^*_o(\mathcal{I}, \tau) \subseteq A^\ast\theta(\mathcal{I}, \tau)$.
4. $\Gamma(A)(\mathcal{I}, \tau) \subseteq A^\ast\theta(\mathcal{I}, \tau)$.
5. $A^*_g(\mathcal{I}, \tau) \subseteq A^\ast\theta(\mathcal{I}, \tau)$. 

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Proof. Straight forward.

Remark 3.1. The converse of the Lemma 3.1 need not be true as seen in the following examples.

Example 3.1. Let \((X, \tau, I)\) be an ideal topological space with \(X = \{a, b, c, d, e\}, \tau = \{\phi, \{a\}, \{a, c\}, \{a, c, d\}, X\}\) and \(I = \{\phi, \{c\}\}. If A = \{a, b\}, then \(A^\theta = \{a, b, c\} \not\subset \{a, b\} = A^p.\)

Example 3.2. Let \((X, \tau, I)\) be an ideal topological space with \(X = \{a, b, c, d\}, \tau = \{\phi, \{a\}, \{a, c\}, \{a, c, d\}, \{a, b, c\}, \{a, b, c, d\}, \{a, b\}, \{a, c\}, \{a, c, d\}, \{a, b, c, d\}, X\} and I = \{\phi, \{b\}, \{c\}, \{b, c\}\}. If A = \{a, b, c\}, then \(A^\theta = \{a, c, d\} \not\subset \{a, d\} = A^s.\)

Example 3.3. In example 3.2, if \(A = \{b, c, d\}\) then, \(A^\theta = \{a, c, d\} \not\subset \{d\} = A^\alpha.\)

Example 3.4. Let \((X, \tau, I)\) be an ideal topological space with \(X = \{a, b, c, d\}, \tau = \{\phi, \{d\}, \{a, c\}, \{a, c, d\}, X\} and I = \{\phi, \{c\}\}. If A = \{a\}, then \(A^\theta = \{a, b, c\} \not\subset \{a, b, c\} = \Gamma(A).\)

Example 3.5. Let \((X, \tau, I)\) be an ideal topological space with \(X = \{a, b, c, d, e\}, \tau = \{\phi, \{a\}, \{a, c\}, \{a, c, d\}, \{a, b, c\}, \{a, b, c, d\}, \{b, c, d, e\}, X\} and I = \{\phi, \{b\}, \{c\}, \{b, c\}\}. If A = \{a, b, c, d\}, then \(A^\theta = X \not\subset \{a, b\} = A^g.\)

Remark 3.2. The above discussions are summarized in the following diagram.

\[ \begin{array}{ccc}
A^p & \subseteq & \Gamma(A) \\
& | & \\
& \downarrow & \\
A^\alpha & \subseteq & A^\theta \\
& | & \\
& \downarrow & \\
& A^g & \\
\end{array} \]

Remark 3.3. \(A \subset A^\theta \) and \(A^\theta \subset A\) are not true in general as shown in the following example.

Example 3.6. Let \((X, \tau, I)\) be an ideal topological space with \(X = \{a, b, c, d, e\}, \tau = \{\phi, \{a\}, \{a, c\}, \{a, c, d\}, \{b, c, d, e\}, X\} \) and \(I = \{\phi, \{b\}, \{c\}, \{b, c\}\}. (i) If \(A = \{a, b\}\), then \(A^\theta = \{a\}. Therefore, A \not\subset A^\theta.\) (ii) If \(A = \{a, b, d\}\), then \(A^\theta = X \not\subset \{a, b\} = A^g.\)

Remark 4.1. Let \((X, \tau, I)\) be an ideal topological space and \(A \subset X\). Then, \(cl^\theta(A) = A \cup A^\theta\) is a \(*\theta\)-closure operator.

Remark 4.2. Open sets of \(\tau^\theta\). Let \((X, \tau)\) be a topological space and \(I\) an ideal on \(X\) and observe that \(A\) is \(\tau^\theta\)-closed iff \(\tau^\theta A \subset A\). Now we have \(U \in \tau^\theta\) iff \(X - U\) is \(\tau^\theta\)-closed iff \(U \subset X - (X - U)^{\tau^\theta}\). Therefore, \(x \in U \Rightarrow x \notin (X - U)^{\tau^\theta}\) → there exists a \(\theta\)-neighbourhood \(V\) such that \(V \cap (X - U) \in I\). Now let \(I = V \cap (X - U)\) and we have \(x \in V - I \subseteq U\), where \(I \in I\). We shall denote \(\beta(I, \tau^\theta) = \{V - I : V \in \tau^\theta, I \in I\}.\)

Theorem 3.1. Let \((X, \tau)\) be a topological space and \(I\) an ideal on \(X\). Then \(\beta\) is a basis for \(\tau^\theta\).

Lemma 3.2. If \((X, \tau, I)\) and \(\beta\) be an ideal topological space and \(A \subset X\). If \(A \subset A^\theta\), then \(A^\theta = cl_\beta(A) = cl^{\theta^\beta}(A).\)

Proof. Always \(cl^\theta(A) \subset cl_\beta(A).\) Let \(x \notin cl^\theta(A).\) Then, there exists a \(\tau^\theta\)-open set \(G\) containing \(x\) such that \(G \cap A = \phi. By Remark 3.5, there exists \(V \in \tau^\theta\) and \(I \in I\) such that \(x \in V \cap I \subset G.\) Since \(G \cap A = \phi \Rightarrow (V \cap I) \cap A = \phi \Rightarrow (V \cap I) \cap A = \phi \Rightarrow \Gamma(A) \cap A = \phi \Rightarrow \Gamma(A) = A \subset A^\theta.\) Therefore, \(cl^\beta(A) \subset cl^{\theta^\beta}(A).\) Hence \(cl^{\theta^\beta}(A) = cl_\beta(A).\) From (1) and (2), \(A^\theta = cl^\theta(A) = cl^{\theta^\beta}(A).\)

Definition 3.2. Let \((X, \tau, I)\) be an ideal topological space and \(A \subset X.\) If \(A \subset A^\theta\), then \(A\) is said to be \(*\theta\)-dense in itself.

Definition 3.3. Let \((X, \tau, I)\) be an ideal topological space and \(A \subset X.\) If \(A^\theta \subset A,\) then \(A\) is said to be \(*\theta\)-closed.

Remark 3.6. Let \((X, \tau, I)\) be an ideal topological space and \(A \subset X.\) Then, \(\tau^\theta = \{X - A : cl^\theta(A) = A\}.\)
Proposition 3.1. Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \(A \subset X\). Then \(A\) is \(\tau^\theta\)-closed if and only if \(A^\theta \subset A\).

**Proof.** Let \(A\) be \(\tau^\theta\)-closed. Then, \(A = cl^\theta(A) \Rightarrow A = A \cup A^\theta \Rightarrow A^\theta \subset A\). Conversely, let \(A^\theta \subset A\). By assumption, \(A \cup A^\theta = A\). i.e. \(cl^\theta(A) = A\). Hence, \(A\) is \(\tau^\theta\)-closed. \(\square\)

Proposition 3.2. Let \((X, \tau, \mathcal{I})\) be an ideal topological space. Then the following hold for every subset \(A\) of \(X\), \(cl^\theta(A) \subset cl_\theta(A)\);

**Proof.** Let \(x \in cl^\theta(A)\). Then, \(x \in A\) or \(x \in A^\theta\). If \(x \in A^\theta\), then there exists a \(\theta\)-open set \(U_x\) containing \(x\) such that \(U_x \cap A \notin \mathcal{I}\). That is \(U_x \cap A \neq \emptyset\). This implies that \(x \in cl_\theta(A)\). Thus, \(cl^\theta(A) \subset cl_\theta(A)\). \(\square\)

Proposition 3.3. Let \(x \in cl^\theta(A)\) if and only if \(V \cap A \neq \emptyset\) for every \(\theta\)-open set \(V \subseteq X\).

Properties of \((\cdot)^\theta\) operator

**Theorem 3.2.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space and let \(A, B\) be subsets of \(X\). Then for \(\theta\)-local functions the following properties hold:

(i) \(\phi^\theta = \phi\).

(ii) \(A \subset B\) implies \(A^\theta \subset B^\theta\).

(iii) For an another ideal \(\mathcal{J} \supset \mathcal{I}\) on \(X\), \(A^\theta(\mathcal{J}) \subset A^\theta(\mathcal{I})\).

(iv) \(A^* \subset A^\theta\).

(v) \(A^\theta \subset cl_\theta(A)\).

(vi) \((A^\theta)^\theta \subset A^\theta\), if \(A\) is \(\theta\)-closed.

(vii) \(A^\theta \cup B^\theta = (A \cup B)^\theta\).

(viii) \((A \cap B)^\theta \subset A^\theta \cap B^\theta\).

(ix) for a \(\theta\)-open set \(U\), \(U \cap A^\theta = U \cap (U \cap A)^\theta \subset (U \cap A)^\theta\).

(x) For \(I \in \mathcal{I}\), \((A \cap I)^\theta = A^\theta = (A - I)^\theta\).

(xi) \((A - B)^\theta - B^\theta = (A^\theta - B^\theta) \subset (A - B)^\theta\).

(xii) \((A - A^\theta) \cap (A - A^\theta)^\theta = \emptyset\).

(xiii) If \(A \in \mathcal{I}\), then \(A^\theta = \emptyset\).

(xiv) \(A^\theta(\mathcal{I} \cap \mathcal{J}) \supset A^\theta(\mathcal{I}) \cup A^\theta(\mathcal{J})\).

**Proof.**

(i) From the definition of \(\theta\)-local function, \(\phi^\theta = \phi\) is obvious.

(ii) Let \(x \in A^\theta\). Then for every \(\theta\)-open set \(U_x\) containing \(x\), \(U_x \cap A \notin \mathcal{I}\). Since \(A \subset B\) implies that \(U_x \cap A \subset U_x \cap B \notin \mathcal{I}\). Therefore, \(U_x \cap B \notin \mathcal{I}\). This implies that \(x \in B^\theta\). Hence, \(A^\theta \subset B^\theta\).

(iii) Let \(x \in A^\theta(\mathcal{J})\). Then for every \(\theta\)-open set \(U_x\) containing \(x\), such that \(U_x \cap A \notin \mathcal{J}\). This implies that \(U_x \cap A \notin \mathcal{I}\), since \(\mathcal{I} \subset \mathcal{J}\). So, \(x \in A^\theta(\mathcal{I})\). Hence, \(A^\theta(\mathcal{J}) \subset A^\theta(\mathcal{I})\).

(iv) Let \(x \in A^*\). We assert that \(x \in A^\theta\). If not, then there is a \(\theta\)-open set \(U_x\) containing \(x\) such that \(U_x \cap A \in \mathcal{I}\). Since every \(\theta\)-open is open, \(U_x\) is open and since, \(U_x \cap A \in \mathcal{I}\) contradicts the assumption \(x \in A^*\). Therefore, \(x \in A^\theta\). This implies that \(A^* \subset A^\theta\).

(v) Let \(x \in A^\theta\). Then for every \(\theta\)-open set \(U_x\) containing \(x\), \(U_x \cap A \notin \mathcal{I}\). Since every \(\theta\)-open is open, \(U_x\) is open. This implies that \(U_x \cap A \neq \emptyset\) for every \(\theta\)-open set containing \(x\). Hence, \(x \in cl_\theta(A)\).

(vi) From (v) \(A^\theta \subset cl_\theta(A)\). \((A^\theta)^\theta \subset (cl_\theta(A))^\theta\). But \(A = cl_\theta(A)\), since \(A\) is \(\theta\)-closed. This implies that \((A^\theta)^\theta \subset A^\theta\).
(vii) Since \( A \subset A \cup B \) and \( B \subset A \cup B \). Then from (ii) \( A^* \subset (A \cup B)^* \) and \( B^* \subset (A \cup B)^* \). Hence, \( A^* \cup B^* \subset (A \cup B)^* \). Conversely suppose that (ii) \( A^* \subset (A \cup B)^* \) and \( B^* \subset (A \cup B)^* \). If \( x \notin A^* \), then there exists \( \theta \)-open set \( U_x \) containing \( x \) such that \( U_x \cap A \notin \mathcal{I} \). Similarly since \( x \notin B^* \), there exists \( \theta \)-open set \( V_x \) containing \( x \) such that \( V_x \cap A \in \mathcal{I} \). Then by the hereditary property of ideal, \( A \cap (U_x \cap V_x) \in \mathcal{I} \) and \( B \cap (U_x \cap V_x) \in \mathcal{I} \). Again, by the finite additivity of the ideal, \( (A \cup B) \cap (U_x \cap V_x) \in \mathcal{I} \). Hence, \( x \notin (A \cup B)^* \). So, \( (A \cup B)^* \subset A^* \cup B^* \). Hence \( A^* \cup B^* = (A \cup B)^* \).

(viii) Since \( A \cap B \subset A \) and \( A \cap B \subset B \), from (2), \( (A \cap B)^* \subset A^* \) and \( (A \cap B)^* \subset B^* \). Hence, \( (A \cap B)^* \subset A^* \cap B^* \).

(ix) Let \( x \in U \cap A^* \). Let \( V_x \) be a \( \theta \)-open set containing \( x \), then \( A \cap (U \cap V_x) \notin \mathcal{I} \), since \( x \in A^* \) and \( U \cap V_x \) is a \( \theta \)-open set containing \( x \). Hence, \( x \in \mathcal{I} \). Therefore, \( U \cap A^* \subset (U \cap A)^* \).

Remark 3.7. In Theorem 3.2, the reverse inclusions of (iii), (viii) are not valid as in the following example.

Example 3.7. Let \( \mathcal{X} = \{a, b, c, d\} \) with \( \tau = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\} \), \( \mathcal{J} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\} \) and \( \mathcal{I} = \{\emptyset, \{c\}\} \).

1. Let \( A = \{a, b\} \). Then, \( A^* = (\{a, b, c, d\} \cup \{a\}) = A^*(\mathcal{J}) \).

2. Let \( A = \{a, b, c, d\} \), \( A^* = X \), \( B = \{a, b, c, e\} \), \( B^* = X \), \( A \cap B = \{a, b, c\} \), \( (A \cap B)^* = \{a\} \). Therefore \( A^* \cap B^* = X \not\subset \{a\} = (A \cap B)^* \).

Proposition 3.4. Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \( A \subset X \) where \( \mathcal{I} = \{\emptyset\} \). Then \( A^* = \text{cl}_\theta(A) \).
Proof. Let \( I = \{ \phi \} \). We know that \( c_\theta(A) = A \cup D_\theta(A) \) where \( D_\theta(A) \) is the \( \theta \)-derived set of \( A \). Let \( x \in A \cup D_\theta(A) \) and let \( U_x \) be a \( \theta \)-open set containing \( x \). Then \( x \in A \) or \( x \in D_\theta(A) \). If \( x \in A \) then \( x \in U_x \cap A \) and so \( U_x \cap A \neq \phi \). If \( x \in D_\theta(A) \), then \( \phi \notin \{ U_x - \{ x \} \} \cap A \subset U_x \cap A \) and thus \( U_x \cap A \neq \phi \). Hence, \( c_\theta(A) = A \cup D_\theta(A) \subset A^\theta \). By Theorem 3.2(v), \( A^\theta \subset c_\theta(A) \). Therefore, \( A^\theta = c_\theta(A) \).

**Proposition 3.5.** Let \( (X, \tau, I) \) be an ideal topological space and \( A \subset X \) where \( I = \varnothing(X) \). Then \( A^\theta = \phi \) for every \( A \subset X \).

**Proof.** Since \( A^\theta = \{ x \in X : U_x \cap A \notin \varnothing(X) \} \) for every \( \theta \)-open set \( U_x \) containing \( x \) = \( \phi \). Therefore, \( A^\theta = \phi \) for every \( A \subset X \).

**Theorem 3.3.** Let \( (X, \tau, I) \) be an ideal topological space and let \( A, B \) be subsets of \( X \). Then for \( \theta \)-local functions the following properties hold:

1. \( A^\theta = c_\theta(A^\theta) \subseteq c_\theta(A) \) and \( A^\theta \) is \( \theta \)-closed.
2. If \( A \subseteq A^\theta \) and \( A^\theta \) is open, then \( A^\theta = c_\theta(A) \).

**Proof.**
1. Always \( A^\theta \subseteq c_\theta(A^\theta) \). Let \( x \in c_\theta(A^\theta) \). Then there exists some open set \( U_x \) containing \( x \) such that \( A^\theta \cap U_x \neq \phi \). Therefore, there exists some \( y \in A^\theta \cap U_x \) and \( U_x \in \varnothing(x) \). Since \( y \in A^\theta \), there exists some \( \theta \)-open set \( V_y \) such that \( A \cap V_y \cap U_x = A \cap V_x \notin I \). Therefore, \( x \in A^\theta \). Hence, \( A^\theta = c_\theta(A^\theta) \) and \( A^\theta = c_\theta(A^\theta) \subseteq c_\theta(A) \) by Theorem 3.2(v).

2. For any subset \( A \) of \( X \), by (1) we have \( A^\theta = c_\theta(A^\theta) \subseteq c_\theta(A) \). Since \( A \subseteq A^\theta \) and \( A^\theta \) is open, by Lemma 1.2, \( c_\theta(A) \subseteq c_\theta(A^\theta) = c_\theta(\varnothing(X)) = A^\theta \subseteq c_\theta(A) \) and hence, \( A^\theta = c_\theta(A) \).

**Theorem 3.4.** Let \( (X, \tau, I) \) be an ideal topological space. Then, \( A^\theta \supseteq A - \bigcup \{ U \subset X : U \in I \} \) for all \( A \subset X \).

**Proof.** Let \( B = \bigcup \{ U \subset X : U \in I \} \) and let \( x \in (A - B) \). Then \( x \in A \) and \( x \notin B \). This implies that \( x \notin U \) for all \( U \in I \) so that \( \{ x \} \cap A \notin I \) because \( x \in A \). For every \( G \in \varnothing(x) \), we have \( \{ x \} \cap A \subset G \cap A \notin I \) by the heredity of ideal. Hence, \( x \in A^\theta \).

**Remark 3.8.** The converse of the theorem 3.4 need not be true as seen in the following example.

**Example 3.8.** Let \( X = \{ a, b, c, d \} \) with \( \tau = \{ \phi, \{ a \}, \{ c, d \}, \{ a, c, d \}, \{ b, c, d \}, X \} \) and \( I = \{ \phi, \{ b \}, \{ c \}, \{ b, c \} \} \). Let \( A = \{ a, b, c, d \} \). \( B = \bigcup \{ U \subset X : U \in I \} = \{ b, c \} \). \( A - B = \{ a, d \} \). \( A^\theta = X \notin \{ a, d \} = A - B \).

**Theorem 3.5.** Let \( (X, \tau, I) \) be an ideal topological space and let \( B = \bigcup \{ U \subset X : U \in I \} \). If \( B \in I \) then \( (A^\theta)^* = A^\theta \) for all \( A \subset X \).

**Proof.** Let \( A \) be a subset of \( X \). Then, \( (A^\theta)^* \subset A^\theta \) by Theorem 3.2(vi). Furthermore, \( A^\theta \supseteq A - B \) by Theorem 3.4. It follows from Theorem 3.2(ii) that \( (A^\theta)^* \supseteq (A - B)^\theta \). Since \( B \in I \), by Theorem 3.2(x) it implies that \( (A^\theta)^* \supseteq (A - B)^\theta = A^\theta \). Therefore, \( (A^\theta)^* = A^\theta \).

**Theorem 3.6.** Let \( (X, \tau, I) \) be an ideal topological space in which \( \tau^\theta = \varnothing(X) \). Then \( A^\theta = A - \bigcup \{ U \subset X : U \in I \} \) for all \( A \subset X \).

**Proof.** Let \( B = A - \bigcup \{ U \subset X : U \in I \} \) and let \( x \in A^\theta \). Then \( \{ x \} \cap A \notin I \) because \( \{ x \} \in \varnothing(x) = \varnothing(X) \). Since ideal \( I \) always contains \( \phi \), \( \{ x \} \cap A \neq \phi \) and so \( x \in A \). It follows that \( \{ x \} \cap A \notin I \) so that \( x \notin U \) for all \( U \in I \). Hence, \( x \notin B \) and therefore, \( x \in A - B \). Hence, \( A^\theta \subset A - B \). The reverse inclusion is obvious by Theorem 3.4.

**Remark 3.9.** Let \( (X, \tau, I) \) be an ideal topological space in which every member of \( \tau \) is clopen. Then \( A^\theta = A - \bigcup \{ U \subset X : U \in I \} \) for all \( A \subset X \).

**Proof.** Let \( B = A - \bigcup \{ U \subset X : U \in I \} \) and \( A \in \varnothing(X) \). Then every clopen set is \( \theta \)-open. Hence \( A \in \varnothing(X) \), which means that \( \varnothing(X) \subset \tau^\theta \) so that \( \varnothing(X) = \tau^\theta \). By Theorem 3.6 \( A^\theta = A - B \).

**Theorem 3.7.** Let \( (X, \tau, I) \) be an ideal topological space. Then, the following properties holds.

1. If \( I = \{ \phi \} \), then \( cl^\theta(A) = c_\theta(A) \).
2. If \( I = \emptyset(X) \), then \( \text{cl}^{*\theta}(A) = A \).

3. If \( A \in I \), then \( \text{cl}^{*\theta}(A) = A \).

**Proof.** Obvious.

**Theorem 3.8.** Let \( (X, \tau, I) \) be an ideal topological space and let \( A, B \) be subsets of \( X \). Then for \( *\theta \)-local functions the following properties hold:

(i) \( \text{cl}^{*\theta}(\phi) = \phi \).

(ii) If \( A \subset B \), then \( \text{cl}^{*\theta}(A) \subset \text{cl}^{*\theta}(B) \).

(iii) For another ideal \( J \supseteq I \) on \( X \), \( \text{cl}^{*\theta}(A, \tau, J) \subset \text{cl}^{*\theta}(A, \tau, I) \).

(iv) \( \text{cl}^{*}(A) \subset \text{cl}^{*\theta}(A) \).

(v) \( \text{cl}^{*\theta}(A) \subset cl_{\theta}(A) \).

(vi) \( \text{cl}^{*\theta}(\text{cl}^{*\theta}(A)) \subset \text{cl}^{*\theta}(A) \) if \( A \) is \( \theta \)-closed.

(vii) \( \text{cl}^{*\theta}(A) \cup \text{cl}^{*\theta}(B) = \text{cl}^{*\theta}(A \cup B) \).

(viii) \( \text{cl}^{*\theta}(A \cap B) \subset \text{cl}^{*\theta}(A) \cap \text{cl}^{*\theta}(B) \).

**Proof.** It is obvious by using Remark 3.5 and Theorem 3.7.

**Remark 3.10.** In Theorem 3.8, the reverse inclusions of (iii), (iv), (v) and the converse of (iii) and (viii) are not valid as seen in the following examples.

**Example 3.9.** (iii) Let \( X = \{a, b, c, d\} \) with \( \tau = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, A, \emptyset\} \) and \( I = \{\emptyset\} \). Let \( \tau = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\} \) and \( J = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\} \). Let \( A = \{a, d\} \), \( \text{cl}^{*\theta}(A, \tau, J) = \{a, c\} \supset \{a, d\} = \text{cl}^{*\theta}(A, \tau, I) \) but \( J \not\subset I \).

**Example 3.10.** Let \( X = \{a, b, c, d\} \) with \( \tau = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, c, d\}, X\} \) and \( I = \{\emptyset\} \).

(i) Let \( A = \{a, b\} \). Then \( \text{cl}^{*\theta}(A) = \{a\} \subset X = \text{cl}^{*\theta}(B) \), but \( A \not\subset B \).

(ii) Let \( A = \{a\} \). Then \( \text{cl}^{*\theta}(A) = \{a, d\} \subset \{a, d\} = \text{cl}^{*}(A) \).

(viii) Let \( A = \{b, c\} \), \( B = \{b, d\} \). Then \( \text{cl}^{*\theta}(A) = \{b, c\}, \text{cl}^{*\theta}(B) = X, A \cap B = \emptyset \). \( \text{cl}^{*\theta}(A \cap B) = \emptyset \). So, \( \text{cl}^{*\theta}(A) \cap \text{cl}^{*\theta}(B) = \emptyset \).

(v) Let \( A = \{b, c\} \). Then, \( cl_{\theta}(A) = X \not\subset \{b, c\} = \text{cl}^{*\theta}(A) \).

**Remark 3.11.** \( D_{\theta}(A) \subset \text{cl}^{*\theta}(A) \) and \( \text{cl}^{*\theta}(A) \subset D_{\theta}(A) \) are not true in general as shown in the following example.

**Example 3.11.** Let \( (X, \tau, I) \) be an ideal topological space with \( X = \{a, b, c, d, e\} \), \( \tau = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{a, c, d\}, \{a, b, c, d, e\}, X\} \) and \( I = \{\emptyset\} \).

(i) If \( A = \{a, c, d\} \), then \( A^{*\theta} = \emptyset \). Therefore, \( D_{\theta}(A) = \{b, c, d, e\} \not\subset \{a, c, d\} = \text{cl}^{*\theta}(A) \).

(ii) If \( A = \{a, b, d\} \), Then \( A^{*\theta} = X \). Therefore, \( \text{cl}^{*\theta}(A) = X \not\subset \{b, c, d, e\} = D_{\theta}(A) \).

**Proposition 3.6.** Let \( (X, \tau, I) \) be an ideal topological space. For any subset \( A \) of \( X \), the following properties are hold.

(i) \( A^{*\theta} - A \subset \text{cl}_{\theta}(A) - A \subset D_{\theta}(A) \).

(ii) If \( I = \{\emptyset\} \), then \( A^{*\theta} = A = \text{cl}_{\theta}(A) - A = D_{\theta}(A) \).

(iii) If \( I = \emptyset(X) \), then \( A^{*\theta} = D_{\theta}(A) \).

**Proof.** (i) From Theorem 3.2(v), we have \( A^{*\theta} \subset \text{cl}_{\theta}(A) \). Then, \( A^{*\theta} - A \subset \text{cl}_{\theta}(A) - A \). Since \( \text{cl}_{\theta}(A) = A \cup D_{\theta}(A) \), \( \text{cl}_{\theta}(A) - A \subset D_{\theta}(A) \). It follows that \( A^{*\theta} - A \subset \text{cl}_{\theta}(A) - A \subset D_{\theta}(A) \).

(ii) and (iii) are straightforward by Proposition 3.4 and Proposition 3.5.
4 \( \theta \) - Compatibility

Definition 4.1. Let \((X, \tau, I)\) be an ideal topological space, then \(\tau\) is \(\theta\)-compatible with the ideal \(I\), if for every \(A \subseteq X\) and if for every \(x \in A\), there exists \(U \in \tau^\theta(x)\) such that \(U \cap A \in I\), then \(A \in I\) and it is denoted by \(\tau \sim^\theta I\).

Theorem 4.1. Let \((X, \tau, I)\) be an ideal topological space, then the following properties are equivalent:

1. \(\tau \sim^\theta I\);

2. If a subset \(A\) of \(X\) has a cover of \(\theta\)-open sets each of whose intersection with \(A\) is in \(I\), then \(A \in I\);

3. For every \(A \subseteq X\), \(A \cap A^{*\theta} = \phi\) implies that \(A \in I\);

4. For every \(A \subseteq X\), \(A - A^{*\theta} \in I\).

5. For every \(A \subseteq X\), if \(A\) contains no nonempty subset \(B\) with \(B \subseteq B^{*\theta}\), then \(A \in I\).

Proof. (1) \(\Rightarrow\) (2). The proof is obvious.

(2) \(\Rightarrow\) (3). Let \(A \subseteq X\) and \(x \in A\). Since \(A \cap A^{*\theta} = \phi\), \(x \notin A^{*\theta}\) and there exists some \(\theta\)-open set \(V_x \in \tau^\theta\) such that \(V_x \cap A \in I\). Therefore, we have \(A \subseteq \bigcup\{V_x : x \in A\}\) and \(V_x \in \tau^\theta\) and by (2) \(A \in I\).

(3) \(\Rightarrow\) (4). For any \(A \subseteq X\), \(A - A^{*\theta} \subseteq A\) and \((A - A^{*\theta}) \cap (A - A^{*\theta})^{*\theta} \subseteq (A - A^{*\theta}) \cap A^{*\theta} = \phi\). By (3), \(A - A^{*\theta} \in I\).

(4) \(\Rightarrow\) (5). By (4), for every \(A \subseteq X\), \(A - A^{*\theta} \in I\). Let \(A - A^{*\theta} = J \subseteq I\), \(A = J \cup (A \cap A^{*\theta})\) and by Theorem 3.17 (vii) and (xiii), \(A^{*\theta} = J^{*\theta} \cup (A \cap A^{*\theta})^{*\theta} = (A \cap A^{*\theta})^{*\theta}\). Therefore, we have \((A \cap A^{*\theta}) = A \cap (A \cap A^{*\theta})^{*\theta} \subseteq (A \cap A^{*\theta})^{*\theta}\) and \((A \cap A^{*\theta}) \subseteq A\). By the assumption \(A \cap A^{*\theta} = \phi\) and hence \(A = (A - A^{*\theta}) \subseteq I\).

(5) \(\Rightarrow\) (1). Let \(A \subseteq X\) and assume that for every \(x \in A\), there exists some \(\theta\)-open set \(U_x\) containing \(x\), \(U_x \cap A \in I\). Then \(A \cap A^{*\theta} = \phi\). Suppose that \(A\) contains \(B\) such that \(B \subseteq B^{*\theta}\). Then \(B = B \cap B^{*\theta} \subseteq A \cap A^{*\theta} = \phi\). Therefore, \(A\) contains no nonempty subset \(B\) with \(B \subseteq B^{*\theta}\). Hence \(A \in I\).

\[\square\]

Lemma 4.1. Let \((X, \tau, I)\) be an ideal topological space. If \(\tau \sim^\theta I\), then for every \(A \subseteq X\), \(A \cap A^{*\theta} = \phi\) implies that \(A^{*\theta} = \phi\).

Proof. Let \(A\) be any subset of \(X\) and \(A \cap A^{*\theta} = \phi\). By Theorem 4.1, \(A \in I\) and by Theorem 3.2 (xiii), \(A^{*\theta} = \phi\).

\[\square\]

Theorem 4.2. Let \((X, \tau, I)\) be an ideal topological space. If \(\tau \sim^\theta I\) then the following properties are equivalent:

1. For every \(A \subseteq X\), \(A \cap A^{*\theta} = \phi\) implies that \(A^{*\theta} = \phi\).

2. For every \(A \subseteq X\), \((A - A^{*\theta})^{*\theta} = \phi\).

3. For every \(A \subseteq X\), \((A \cap A^{*\theta})^{*\theta} = A^{*\theta}\).

Proof. (1) \(\Rightarrow\) (2). Assume that every \(A \subseteq X\), \(A \cap A^{*\theta} = \phi\) implies that \(A^{*\theta} = \phi\). Let \(B = A - A^{*\theta}\), then \(B \cap B^{*\theta} = (A - A^{*\theta}) \cap (A - A^{*\theta})^{*\theta} = (A \cap (X - A^{*\theta})) \cap (A \cap (X - A^{*\theta})^{*\theta}) \subseteq (A \cap (X - A^{*\theta})) \cap (A^{*\theta} \cap (X - A^{*\theta})^{*\theta}) = \phi\). By (1), we have \(B^{*\theta} = \phi\). Hence \((A - A^{*\theta})^{*\theta} = \phi\).

(2) \(\Rightarrow\) (3). Assume for every \(A \subseteq X\), \((A - A^{*\theta})^{*\theta} = \phi\). \(A = (A - A^{*\theta}) \cup (A \cap A^{*\theta})\). \(A^{*\theta} = [(A - A^{*\theta}) \cup (A \cap A^{*\theta})]^{*\theta} = (A - A^{*\theta})^{*\theta} \cup (A \cap A^{*\theta})^{*\theta} = (A \cap A^{*\theta})^{*\theta} = (A \cap A^{*\theta})^{*\theta} = A^{*\theta}\).

(3) \(\Rightarrow\) (1). Assume for every \(A \subseteq X\), \(A \cap A^{*\theta} = \phi\) and \((A \cap A^{*\theta})^{*\theta} = A^{*\theta}\). This implies that \(\phi = \phi^{*\theta} = A^{*\theta}\).

\[\square\]

Definition 4.2. If \((X, \tau, I)\) is an ideal topological space, then \(I\) is \(\theta\)-codense if and only if \(A \subseteq A^{*\theta}\) for every \(\theta\)-open set \(A\) of \(X\).

Characterization of \(\theta\) -local function in \(\theta\)-codense ideal topological space.

Theorem 4.3. Let \((X, \tau, I)\) be an ideal topological space. Then the following are equivalent:

1. \(X = X^{*\theta}\).

2. \(\tau^\theta \cap I = \{\phi\}\).
3. If \( I \in \mathcal{I} \), then \( \text{int}_\theta(I) = \phi \).

4. For every \( U \in \tau^\theta \), \( U \subset U^{*\theta} \).

Proof. (1) \( \Rightarrow \) (2): Let \( U \in \tau^\theta \cap \mathcal{I} \). Then \( U \in \tau^\theta \) and \( U \in \mathcal{I} \). Suppose that \( x \in U \). Since \( x \in X \), this implies \( x \in X^{*\theta} \). Since \( U \) is a \( \theta \)-open set containing \( x \), \( U \cap X \notin \mathcal{I} \) implies that \( U \notin \mathcal{I} \) which is a contradiction. Therefore, \( x \notin U \) for every \( x \in X \). This implies that \( U = \phi \) and so \( \tau^\theta \cap \mathcal{I} = \{ \phi \} \).

(2) \( \Rightarrow \) (3): Suppose that (2) holds. Let \( I \in \mathcal{I} \) be such that \( I \neq \phi \). Then \( \text{int}_\theta(I) \in \tau^\theta \) and \( \text{int}_\theta(I) \subset I \) implies that \( \text{int}_\theta(I) \in \mathcal{I} \). Therefore, by (2), \( \text{int}_\theta(I) = \phi \).

(3) \( \Rightarrow \) (4): \( U \in \tau^\theta \) and \( x \in U \). Suppose that \( x \notin U^{*\theta} \). Then there exists a \( \theta \)-open set \( V_x \) containing \( x \) such that \( V_x \cap U \notin \mathcal{I} \). Since \( U \cap V_x = \text{int}_\theta(U \cap V_x) = \phi \) by (3). Since \( x \in V_x \), \( x \notin U \). Thus \( U \subset U^{*\theta} \) for every \( U \in \tau^\theta \).

(4) \( \Rightarrow \) (1): Since \( X \) is \( \theta \)-open, by (4), \( X \subset X^{*\theta}, X = X^{*\theta} \).

\[ \square \]

**Theorem 4.4.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \( I \in \mathcal{I} \). Then, \( I \) is \( \tau^{*\theta} \)-closed.

Proof. Let \( I \in \mathcal{I} \). By Theorem 3.22 (x) \( I^{*\theta} = (I-I)^{*\theta} = \phi^{*\theta} = \phi \). Hence \( \tau^{*\theta}(I) = I \cup I^{*\theta} = \mathcal{I} \) which implies that \( I \) is \( \tau^{*\theta} \)-closed.

\[ \square \]

**Theorem 4.5.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \( A \subset X \). Then \( A^{*\theta}(\tau^{*\theta}, \mathcal{I}) \subset A^{*\theta}(\tau^{\theta}, \mathcal{I}) \).

Proof. Let \( x \in A^{*\theta}(\tau^{*\theta}, \mathcal{I}) \). Suppose that \( x \notin A^{*\theta}(\tau^{*\theta}, \mathcal{I}) \). Then there exists a \( \theta \)-open set \( U_x \) containing \( x \), such that \( A \cap U_x \notin \mathcal{I} \). Since \( U_x \subset \tau^{*\theta} \subset \tau^{\theta} \), \( A \cap U_x \in \mathcal{I} \) for a \( \tau^{*\theta} \)-open set \( U_x \) containing \( x \). Therefore, \( x \notin A^{*\theta}(\tau^{*\theta}, \mathcal{I}) \) which implies that \( A^{*\theta}(\tau^{*\theta}, \mathcal{I}) \subset A^{*\theta}(\tau^{\theta}, \mathcal{I}) \).

\[ \square \]

**Theorem 4.6.** Let \((X, \tau)\) be an ideal topological space where \( \mathcal{I} \) and \( \mathcal{J} \) are ideals on \( X \) and \( A \subset X \). Then the following hold:

(i) \( A^{*\theta}(\mathcal{I} \cap \mathcal{J}) = A^{*\theta}(\mathcal{I}) \cup A^{*\theta}(\mathcal{J}) \).

(ii) If \( \mathcal{I} \subset \mathcal{J} \), then \( \tau^{*\theta}(\mathcal{I}) \subset \tau^{*\theta}(\mathcal{J}) \).

(iii) \( \tau^{*\theta}(\mathcal{I} \cap \mathcal{J}) = \tau^{*\theta}(\mathcal{I}) \cap \tau^{*\theta}(\mathcal{J}) \).

Proof. (i) Let \( x \notin A^{*\theta}(\mathcal{I} \cap \mathcal{J}) \) if and only if there exists a \( \theta \)-open set \( U_x \) containing \( x \), such that \( A \cap U_x \notin \mathcal{I} \cap \mathcal{J} \) if and only if \( A \cap U_x \notin \mathcal{I} \) and \( A \cap U_x \notin \mathcal{J} \) if and only if \( x \notin A^{*\theta}(\mathcal{I}) \) and \( x \notin A^{*\theta}(\mathcal{J}) \) if and only if \( x \notin A^{*\theta}(\mathcal{I}) \cup A^{*\theta}(\mathcal{J}) \). Hence, \( A^{*\theta}(\mathcal{I} \cap \mathcal{J}) = A^{*\theta}(\mathcal{I}) \cup A^{*\theta}(\mathcal{J}) \) for every subset \( A \subset X \).

(ii) Let \( \mathcal{I} \subset \mathcal{J} \). Now if \( X - A \in \tau^{*\theta}(\mathcal{I}) \), then \( A \cap A^{*\theta}(\mathcal{I}) = A \) which implies that \( A^{*\theta}(\mathcal{I}) \subset A \). Since \( \mathcal{I} \subset \mathcal{J} \), \( A^{*\theta}(\mathcal{J}) \subset A^{*\theta}(\mathcal{I}) \subset A \) by Theorem 3.17 (iii). Therefore, \( X - A \in \tau^{*\theta}(\mathcal{J}) \) which implies that \( \tau^{*\theta}(\mathcal{I} \cap \mathcal{J}) \subset \tau^{*\theta}(\mathcal{J}) \).

(iii) Let \( A \subset X \) and \( X - A \in \tau^{*\theta}(\mathcal{I} \cap \mathcal{J}) \). Since \( \mathcal{I} \cap \mathcal{J} \) is a subset of \( \mathcal{I} \) and \( \mathcal{J} \), \( X - A \in \tau^{*\theta}(\mathcal{I}) \) and \( X - A \in \tau^{*\theta}(\mathcal{J}) \) if and only if \( A \) is \( \tau^{*\theta}(\mathcal{I}) \)-closed and \( \tau^{*\theta}(\mathcal{J}) \)-closed if and only if \( A^{*\theta}(\mathcal{I}) \subset A \) and \( A^{*\theta}(\mathcal{J}) \subset A \). Hence, \( A^{*\theta}(\mathcal{I}) \cup A^{*\theta}(\mathcal{J}) \subset A \) if and only if \( A^{*\theta}(\mathcal{I} \cap \mathcal{J}) \subset A \) by (i). This implies that \( A \) is \( \tau^{*\theta}(\mathcal{I} \cap \mathcal{J}) \)-closed. Therefore, \( \tau^{*\theta}(\mathcal{I} \cap \mathcal{J}) = \tau^{*\theta}(\mathcal{I}) \cap \tau^{*\theta}(\mathcal{J}) \).

\[ \square \]

5 The operator \( \psi_\theta \)

**Definition 5.1.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. An operator \( \psi_\theta : \mathcal{P}(X) \rightarrow \tau \) is defined as \( \psi_\theta(A) = \{ x \in X \): there exists a \( \theta \)-open set \( U_x \) containing \( x \) such that \( U_x - A \notin \mathcal{I} \}, \) for every \( A \in \mathcal{P}(X) \). We observe that \( \psi_\theta(A) = X - (X - A)^{*\theta} \).

**Theorem 5.1.** Let \((X, \tau, \mathcal{I})\) be a ideal topological space. Then, for \( A \in \mathcal{P}(X) \), \( \psi_\theta(A) = X - (X - A)^{*\theta} \).
Proof. Let \( x \in \psi_0(A) \). Then there exists a \( \theta \)-open set \( U_x \) containing \( x \) such that \( U_x - A \in \mathcal{I} \). Then \( X \cap (U_x - A) \in \mathcal{I} \), implies that \( U_x \cap (X - A) \in \mathcal{I} \). So \( x \notin (X - A)^{\theta*} \) and hence, \( x \in X - (X - A)^{\theta*} \). Therefore, \( \psi_0(A) \subset X - (X - A)^{\theta*} \). For reverse inclusion, if \( x \in X - (X - A)^{\theta*} \), then \( x \notin (X - A)^{\theta*} \) and so there exists a \( \theta \)-open set \( U_x \) containing \( x \) such that \( U_x \cap (X - A) \in \mathcal{I} \) which implies that \( U_x - A \in \mathcal{I} \). Hence \( x \in \psi_0(A) \). Thus \( X - (X - A)^{\theta*} \subset \psi_0(A) \) and so \( \psi_0(A) = X - (X - A)^{\theta*} \).

**Theorem 5.2.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space and let \( A, B \) be subsets of \( X \), then the following hold:

(i) If \( A \subseteq B \), then \( \psi_0(A) \subseteq \psi_0(B) \).

(ii) If \( A, B \in \mathfrak{g}(X) \), then \( \psi_0(A) \cup \psi_0(B) \subset \psi_0(A \cup B) \).

(iii) If \( A, B \in \mathfrak{g}(X) \), then \( \psi_0(A) \cap \psi_0(B) = \psi_0(A \cap B) \).

(iv) If \( A \subseteq X \), \( \psi_0(A) \subset \psi(A) \).

(v) If \( U \in \tau^\theta \), then \( U \subseteq \psi(U) \). Also, if \( U \in \tau^{\theta*} \), then \( U \subseteq \psi(U) \).

(vi) If \( A \subseteq X \), then \( \psi_0(A) \subseteq \psi(\psi_0(A)) \).

(vii) If \( A \subseteq X \), then \( \psi_0(A) = \psi(\psi_0(A)) \) if and only if \((X - A)^{\theta*} = (X - A)^{\theta*} \).

(viii) If \( A \subseteq X \) and \( I \in \mathcal{I} \), then \( \psi_0(A - I) = \psi_0(A) = \psi_0(A \cup I) \).

(ix) If \((A - B) \cup (B - A) \in \mathcal{I} \), then \( \psi_0(A) = \psi_0(B) \).

**Proof.**

(i) Since \( A \subseteq B \), then \( (X - A) \supseteq (X - B) \). Then by Theorem 3.22 (ii), \((X - A)^{\theta*} \supseteq (X - B)^{\theta*} \) and hence \( \psi_0(A) \subseteq \psi_0(B) \).

(ii) Since \( A \subset A \cup B \) and \( B \subset A \cup B \), by (i) \( \psi_0(A) \cup \psi_0(B) \subset \psi_0(A \cup B) \).

(iii) \( \psi_0(A \cap B) = X - (X - (A \cap B))^{\theta*} = X - ((X - A) \cup (X - B))^{\theta*} \). This implies that \( \psi_0(A \cap B) = X - ((X - A)^{\theta*} \cup (X - B)^{\theta*}) \), from Theorem 3.22 (xi). Therefore, \( \psi_0(A \cap B) = (X - (X - A)^{\theta*}) \cup (X - (X - B)^{\theta*}) \) and hence, \( \psi_0(A \cap B) = \psi_0(A) \cap \psi_0(B) \).

(iv) From Theorem 3.17 (iv), we have that \( (X - A)^* \subset (X - A)^{\theta*} \). This implies that \( X - (X - A)^* \subset X - (X - A)^{\theta*} \) and \( \psi_0(A) \subset \psi(A) \).

(v) Since \( U \in \tau^\theta \), then \( X - U \) is a \( \theta \)-closed set. So, \( \text{cl}_\theta(X - U) = X - U \). By theorem 3.22 (vi), \((X - U)^{\theta*} \subseteq \text{cl}_\theta(X - U) = (X - U) \). Then, \( U \subseteq X - (X - U)^{\theta*} = \psi_0(U) \) for every \( U \in \tau^\theta \). If \( U \in \tau^{\theta*} \), then \( X - U \) is a \( \tau^{\theta*} \)-closed which implies that \((X - U)^{\theta*} \subseteq (X - U) \) and so, \( U \subseteq X - (X - U)^{\theta*} = \psi_0(U) \).

(vi) This follows from (i) and (v).

(vii) Since \( \psi_0(\psi_0(A)) = X - (X - \psi_0(A))^{\theta*} = X - ((X - (X - A)^{\theta*}))^{\theta*} = X - ((X - A)^{\theta*})^{\theta*} = X - (X - A)^{\theta*} \),

(viii) We know that \( X - (X - (A - I))^* = X - ((X - A) \cup I)^{\theta*} = X - (X - A)^{\theta*} \), (Theorem 3.22 (xvii)). So, \( \psi_0(A - I) = \psi_0(A) \). Also, we know that \( X - (X - (A \cup I))^* = X - ((X - A) - I)^{\theta*} = X - (X - A)^{\theta*} \), (from Theorem 3.22 (xvi)). So, \( \psi_0(A - I) = \psi_0(A) \). Also, \( \psi_0(A \cup I) = \psi_0(A) \).

(ix) Given that \((A - B) \cup (B - A) \in \mathcal{I} \), and let \( A - B = I_1 \), \( B - A = I_2 \). We observe that \( I_1 \) and \( I_2 \) \( \in \mathcal{I} \) by heredity. Also, observe that, \( B = ((A - I_1) \cup I_2) \). Thus, \( \psi_0(A) = \psi_0((A - I_1) \cup I_2) = \psi_0(B) \).

**Corollary 5.1.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. Then \( U \subseteq \psi_0(U) \) for every \( \theta \)-open set \( U \subseteq X \).

**Proof.** We know that \( \psi_0(U) = X - (X - U)^{\theta*} \). Now \( (X - U)^{\theta*} \subseteq \text{cl}_\theta(X - U) = X - U \), since \( X - U \) is \( \theta \)-closed. Therefore, \( U = X - (X - U) \subseteq X - (X - U)^{\theta*} = \psi_0(U) \).

**Remark 5.1.** The following example shows that a set \( A \) is not \( \theta \)-open but satisfies \( A \subseteq \psi_0(A) \).
Example 5.1. Let \( X = \{a, b, c, d\} \), \( \tau = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{a, c, d\}, X \} \) and \( \mathcal{I} = \{\phi, \{c\}, \{c, d\}\} \). Let \( A = \{b\} \). Then \( \psi_\theta(\{b\}) = X - (X - \{b\})^{\tau_\theta} = X - ((a, c, d)^{\tau_\theta} = X - \{a\} = \{b, c, d\} \). Therefore, \( A \subseteq \psi_\theta(A) \). But \( A \) is not \( \theta \)-open.

Theorem 5.3. Let \((X, \tau, \mathcal{I})\) be an ideal topological space. If \( A \subseteq X \), then \( A \cap \psi_\theta(A) = \text{int}_\theta(A) \).

Proof. If \( x \in A \cap \psi_\theta(A) \), then \( x \in A \) and there exists a \( \theta \)-open set \( U_x \) containing \( x \), such that \( U_x - A \in \mathcal{I} \). Then, by Remark 3.5, \( U_x - (U_x - A) \in \tau^\theta \)-open neighborhood of \( x \) and \( x \in \text{int}_\theta(A) \). On the other hand, if \( x \in \text{int}_\theta(A) \) there exists a basic \( \tau^\theta \)-open neighborhood \( V_x - A \) of \( x \), where \( V_x - A \in \tau \) and \( \mathcal{I} \in \mathcal{I} \), such that \( x \in V_x - I \subseteq A \) which implies \( V_x - A \subseteq I \) and hence \( V_x - A \in \mathcal{I} \). Hence, \( x \in A \cap \psi_\theta(A) \).

Theorem 5.4. Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \( A \subseteq X \). Then the following properties hold:

1. \( \psi_\theta(A) = \bigcup\{U \in \tau^\theta : U - A \in \mathcal{I}\} \).
2. \( \psi_\theta(A) \supseteq \bigcup\{U \in \tau^\theta : (U - A) \cup (A - U) \in \mathcal{I}\} \).

Proof. (1) This follows immediately from the definition of \( \psi_\theta \)-operator.

(2) Since \( \mathcal{I} \) is heredity, it is obvious that \( \bigcup\{U \in \tau^\theta : (U - A) \cup (A - U) \in \mathcal{I}\} \subseteq \bigcup\{U \in \tau^\theta : U - A \in \mathcal{I}\} = \psi_\theta(A) \) for every \( A \subseteq X \).

Theorem 5.5. Let \((X, \tau, \mathcal{I})\) be an ideal topological space. Then \( \tau \sim^\theta \mathcal{I} \) if and only if \( \psi_\theta(A) - A \in \mathcal{I} \) for every \( A \subseteq X \).

Proof. Necessity:

Assume \( \tau \sim^\theta \mathcal{I} \) and let \( A \subseteq X \). Observe that \( x \in \psi_\theta(A) - A \in \mathcal{I} \) if and only if \( x \notin A \) and \( x \notin (X - A)^{\tau_\theta} \) if and only if \( x \notin A \) and there exists some \( \theta \)-open set \( U_x \in \tau^\theta(x) \) such that \( U_x - A \in \mathcal{I} \) if and only if there exists some \( \theta \)-open set \( U_x \in \tau^\theta(x) \) such that \( x \in U_x - A \in \mathcal{I} \). Now, \( A \subseteq \psi_\theta(X - X) - (X - A) \in \mathcal{I} \) and hence, \( A \in \mathcal{I} \) by heredity of \( \mathcal{I} \).

Sufficiency:

Let \( A \subseteq X \) and assume that for each \( x \in A \) there exists some \( \theta \)-open set \( U_x \in \tau^\theta(x) \) such that \( U_x \cap A \in \mathcal{I} \). Observe that \( \psi_\theta(X - A) - (X - A) = A - A^{\tau_\theta} = \{x : \text{there exists some} \ \theta \)-open set \( U_x \in \tau^\theta(x) \) such that \( U_x \cap A \in \mathcal{I}\}. Thus, we have \( A \subseteq \psi_\theta(X - X) - (X - A) \in \mathcal{I} \) and hence, \( A \in \mathcal{I} \) by heredity of \( \mathcal{I} \).

Theorem 5.6. Let \((X, \tau, \mathcal{I})\) be an ideal topological space with \( \tau \sim^\theta \mathcal{I} \), \( A \subseteq X \). If \( N \) is a nonempty \( \theta \)-open subset of \( A^{\tau_\theta} \cap \psi_\theta(A) \), then \( N - A \in \mathcal{I} \) and \( N \cap A \notin \mathcal{I} \).

Proof. If \( N \subseteq A^{\tau_\theta} \cap \psi_\theta(A) \), then \( N - A \subseteq \psi_\theta(A) - A \in \mathcal{I} \) by Theorem 5.5 and hence \( N - A \in \mathcal{I} \) by heredity. Since \( N \in \tau^\theta - \{\phi\} \) and \( N \subseteq A^{\tau_\theta} \), we have \( N \cap A \notin \mathcal{I} \) by the definition of \( A^{\tau_\theta} \).

Remark 5.2. Let \((X, \tau, \mathcal{I})\) be an ideal topological space with \( \tau \sim^\theta \mathcal{I} \). Then \( \psi_\theta(A) = \psi_\theta(\psi_\theta(A)) \) for every \( A \subseteq X \).

Proof. \( \psi_\theta(A) \subseteq \psi_\theta(\psi_\theta(A)) \) follows from Theorem 5.2(vi). Since \( \tau \sim^\theta \mathcal{I} \), it follows from Theorem 5.5 that \( \psi_\theta(A) \subseteq A \cup \mathcal{I} \) for some \( I \in \mathcal{I} \), and hence \( \psi_\theta(A) = \psi_\theta(\psi_\theta(A)) \) by Theorem 5.2(viii).

Theorem 5.7. Let \((X, \tau, \mathcal{I})\) be an ideal topological space with \( \tau \sim^\theta \mathcal{I} \). Then \( \psi_\theta(A) = \bigcup\{\psi_\theta(U) : U \in \tau^\theta, \psi_\theta(U) - A \in \mathcal{I}\} \).

Proof. Let \( \Phi(A) = \bigcup\{\psi_\theta(U) : U \in \tau^\theta, \psi_\theta(U) - A \in \mathcal{I}\} \). Clearly \( \Phi(A) \subseteq \psi_\theta(A) \). Now let \( x \in \psi_\theta(A) \). Then, there exists a \( \theta \)-open set \( U \) such that \( U - A \in \mathcal{I} \). By Corollary 5.1, \( U \subseteq \psi_\theta(U) \) and \( \psi_\theta(U) - A \subseteq [\psi_\theta(U) - U] \cup [U - A] \). By Theorem 5.5 \( \psi_\theta(U) - U \in \mathcal{I} \). Hence, \( x \in \Phi(A) \) and \( \Phi(A) \supseteq \psi_\theta(A) \). Consequently, we obtain \( \Phi(A) = \psi_\theta(A) \).

Theorem 5.8. Let \((X, \tau, \mathcal{I})\) be an ideal topological space with \( \tau \sim^\theta \mathcal{I} \), where \( \tau^\theta \cap \mathcal{I} = \phi \). Then for \( A \subseteq X \), \( \psi_\theta(A) \subseteq A^{\tau_\theta} \).

Proof. Suppose \( x \in \psi_\theta(A) \) and \( x \notin A^{\tau_\theta} \). Then, there exists a \( \theta \)-open set \( U_x \in \tau(x) \) such that \( U_x \cap A \in \mathcal{I} \). Since \( x \in \psi_\theta(A) \), by Theorem 5.4 \( x \in \bigcup\{U \in \tau^\theta : U - A \in \mathcal{I}\} \) and there exists a \( \theta \)-open set \( V_x \in \tau^\theta(x) \) such that \( V_x - A \in \mathcal{I} \). Now, we have \( U_x \cap V_x \in \tau^\theta(x) \), \( U_x \cap V_x \cap A \in \mathcal{I} \), and \( U_x \cap V_x - A \in \mathcal{I} \) by heredity. Hence, by finite additivity, we have \((U_x \cap V_x \cap A) \cup (U_x \cap V_x - A) = U_x \cap V_x \in \mathcal{I} \). Since \( U_x \cap V_x \in \tau^\theta \), this is contrary to \( \tau^\theta \cap \mathcal{I} = \phi \). Therefore, \( x \in A^{\tau_\theta} \). This implies that \( \psi_\theta(A) \subseteq A^{\tau_\theta} \).
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References


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