Malaya J. Mat. 3(4)(2015) 580–586

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

# Some multiple series identities and their hypergeometric forms

Ranjana Shrivastava[a,*] Usha Gill[b], Kaleem A. Quraishi[c] and Z. A. Taqvi[d]

[a,b,d] Department of Applied Sciences and Humanities, Al-Falah School of Engineering and Technology, Dhauj, Faridabad, Haryana-121004, India.

[c] Department of Applied Sciences and Humanties, Mewat Engineering College (Waqf), Palla, Nuh, Mewat, Haryana-122107, India.

**Abstract**

In this paper, we obtain solutions of some multiple series identities involving bounded multiple sequences. We also derive hypergeometric forms of these identities involving Kampé de Fériet double hypergeometric function, Srivastava's triple hypergeometric function.

*Keywords:* Pochhammer Symbol; Bounded Sequences; Multiple Series Identities; Srivastava's triple double Hypergeometric Function ; Kampé de Fériet double Hypergeometric Function.

*2010 MSC:* 33B10, 33C05, 33C65.


# 1 Introduction

**Pochhammer's Symbol**
The Pochhammer's symbol or Appell's symbol or shifted factorial or rising factorial or generalized factorial function is defined by

$$(b,\ k)=(b)_k=\frac{\Gamma(b+k)}{\Gamma(b)}=\begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } k=1,2,3,\cdots \\ 1 & ; \text{ if } k=0 \\ k! & ; \text{ if } b=1,\ k=1,2,3,\cdots \end{cases} \tag{1.1}$$

where $b$ is neither zero nor negative integer and the notation $\Gamma$ stands for Gamma function.

**Generalized Gaussian Hypergeometric Function[17,p.42(1)]**
Generalized ordinary hypergeometric function of one variable is defined by

$${}_AF_B\left[\begin{array}{lll} a_1,a_2,\cdots,a_A & ; & \\ & & z \\ b_1,b_2,\cdots,b_B & ; & \end{array}\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_A)_k z^k}{(b_1)_k(b_2)_k\cdots(b_B)_k k!}$$

or

$${}_AF_B\left[\begin{array}{lll} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array}\right]\equiv{}_AF_B\left[\begin{array}{lll} (a_j)_{j=1}^{A} & ; & \\ & & z \\ (b_j)_{j=1}^{B} & ; & \end{array}\right]=\sum_{k=0}^{\infty}\frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1.2}$$

where denominator parameters $b_1,b_2,\cdots,b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.
If $A\leq B$, then series ${}_AF_B$ is always convergent for all finite values of $z$(real or complex).
If $A=B+1$, then series ${}_AF_B$ is convergent when $|z|<1$.

$$((a_A))_{2m}=4^{Am}\Big(\Big(\frac{a_A}{2}\Big)\Big)_m\Big(\Big(\frac{1+a_A}{2}\Big)\Big)_m$$

*Corresponding author.
*E-mail address*: ranjanashri@gmail.com (Ranjana Shrivastava), ushagill79@gmail.com(Usha Gill).

$$((a_A))_{1+2m} = 4^{Am}\Big(\Big(\frac{1+a_A}{2}\Big)\Big)_m\Big(\Big(\frac{2+a_A}{2}\Big)\Big)_m \prod_{i=1}^{A}(a_i)$$

where $m = 0, 1, 2, 3, \ldots$

**Kampé de Fériet's General Double Hypergeometric Function[17,p.63(16); see also 16]**

In 1921, Appell's four double hypergeometric functions $F_1, F_2, F_3, F_4$ and their confluent forms $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$ were unified and generalized by Kampé de Fériet.

We recall the definition of general double hypergeometric function of Kampé de Fériet in slightly modified notation of H.M.Srivastava and R.Panda:

$$F^{A:B;D}_{E:G;H}\left[\begin{array}{lc} (a_A):(b_B);(d_D) \quad ; & \\ & x,\, y \\ (e_E):(g_G);(h_H) \quad ; & \end{array}\right] = \sum_{m,n=0}^{\infty} \frac{((a_A))_{m+n}\,((b_B))_m\,((d_D))_n\, x^m\, y^n}{((e_E))_{m+n}\,((g_G))_m\,((h_H))_n\, m!\, n!} \tag{1.3}$$

where for convergence

(i) $A+B < E+G+1,\ A+D < E+H+1 \ \ ; |x| < \infty,\ \ |y| < \infty$, or

(ii) $A+B = E+G+1,\ A+D = E+H+1$, and

$$\begin{cases} |x|^{\frac{1}{(A-E)}} + |y|^{\frac{1}{(A-E)}} < 1 & ,\text{if } E < A \\ \max\ \{|x|, |y|\} < 1 & ,\text{if } E \geq A \end{cases}$$

**Srivastava's Triple Hypergeometric Function[17,p.69(39,40)]**

In 1967, H. M. Srivastava defined a general triple hypergeometric function $F^{(3)}$ in the following form

$$F^{(3)}\left[\begin{array}{lc} (a_A) :: (b_B); (d_D); (e_E) : (g_G); (h_H); (l_L); & \\ & x,\, y,\, z \\ (m_M) :: (n_N); (p_P); (q_Q) : (r_R); (s_S); (t_T); & \end{array}\right]$$

$$= \sum_{i,j,k=0}^{\infty} \frac{((a_A))_{i+j+k}\ ((b_B))_{i+j}\ ((d_D))_{j+k}\ ((e_E))_{k+i}\ ((g_G))_i\ ((h_H))_j\ ((l_L))_k\ x^i\, y^j\, z^k}{((m_M))_{i+j+k}\ ((n_N))_{i+j}\ ((p_P))_{j+k}\ ((q_Q))_{k+i}\ ((r_R))_i\ ((s_S))_j\ ((t_T))_k\ i!\, j!\, k!} \tag{1.4}$$

**Some Series Identities**

We recall the following identities which are potentially useful in the series rearrangement techniques.

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m+n-1}\Psi(m,\, n,\, r) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(0,\, n+r+1,\, r) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m+r+1,\, n,\, r) +$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m+1,\, n+r+1,\, r+m+1) \tag{1.5}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m+n}\Psi(m,\, n,\, r) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m+r,\, n,\, r) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(m,\, n+r+1,\, r+m+1) \tag{1.6}$$

where $\{\Psi(m,n,r)\}_{m,n,r=0}^{\infty}$ are suitably bounded double and triple sequences of essentially arbitrary(real or complex) parameters respectively.

**Some Useful Indefinite Integrals**

When $m = 0, 1, 2, 3, \cdots,$ then

$$\int \sinh^{2m}\theta\, d\theta = \left\{\frac{-(\frac{1}{2})_m(-1)^m\ \sinh\theta\cosh\theta}{(1)_m}\sum_{r=0}^{m-1}\frac{(1)_r(-1)^r\ \sinh^{2r}\theta}{(\frac{3}{2})_r}\right\} + \left\{\frac{\theta\,(-1)^m\,(\frac{1}{2})_m}{(1)_m}\right\} + Constant \tag{1.7}$$

$$\int \cosh^{2m}\theta\, \mathrm{d}\theta = \left\{ \frac{(\frac{1}{2})_m \sinh\theta\cosh\theta}{(1)_m} \sum_{r=0}^{m-1} \frac{(1)_r \cosh^{2r}\theta}{(\frac{3}{2})_r} \right\} + \left\{ \frac{\theta\,(\frac{1}{2})_m}{(1)_m} \right\} + Constant \tag{1.8}$$

$$\int \sinh^{2m+1}\theta\, \mathrm{d}\theta = \frac{(1)_m\,(-1)^m \cosh\theta}{(\frac{3}{2})_m} \sum_{r=0}^{m} \frac{(\frac{1}{2})_r\,(-1)^r \sinh^{2r}\theta}{(1)_r} + Constant \tag{1.9}$$

$$\int \cosh^{2m+1}\theta\, \mathrm{d}\theta = \frac{(1)_m \sinh\theta}{(\frac{3}{2})_m} \sum_{r=0}^{m} \frac{(\frac{1}{2})_r \cosh^{2r}\theta}{(1)_r} + Constant \tag{1.10}$$

Above formulas (1.7)-(1.10) can be verified for $m = 0, 1, 2, 3, \dots$ and it is the convention that the empty sum $\sum\limits_{r=0}^{-1} F(r)$ is treated as zero.

## 2 A family of multiple-series identities

**Theorem 1.** *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \left( \int_0^{\gamma} \sinh^{2m+2n}\theta\, \mathrm{d}\theta \right) \frac{y^m z^n}{m!\, n!}$$

$$= \gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \frac{(\frac{1}{2})_{m+n}(-y)^m(-z)^n}{(m+n)!\, m!\, n!} + \frac{z\sinh\gamma\cosh\gamma}{2} \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{0,n+r+1} \frac{(\frac{3}{2})_{n+r}\,(1)_r\,(-z)^n\,(z\sinh^2\gamma)^r}{(2)_{n+r}\,(2)_{n+r}\,(\frac{3}{2})_r} +$$

$$+ \frac{y\sinh\gamma\cosh\gamma}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+r+1,n} \frac{(\frac{3}{2})_{m+n+r}\,(1)_r\,(-y)^m\,(-z)^n\,(y\sinh^2\gamma)^r}{(2)_{m+n+r}\,(2)_{m+r}\,(\frac{3}{2})_r\,(1)_n} +$$

$$+ \frac{y\,z\sinh^3\gamma\cosh\gamma}{4} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+1,n+r+1} \frac{(\frac{5}{2})_{m+n+r}\,(2)_{m+r}\,(y\sinh^2\gamma)^m\,(-z)^n\,(z\sinh^2\gamma)^r}{(3)_{m+n+r}(\frac{5}{2})_{m+r}(2)_{n+r}(2)_m} \tag{2.1}$$

*provided that each of the series involved is absolutely convergent.*

**Theorem 2.** *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \left( \int_0^{\gamma} \cosh^{2m+2n}\theta\, \mathrm{d}\theta \right) \frac{y^m z^n}{m!\, n!}$$

$$= \gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \frac{(\frac{1}{2})_{m+n}\, y^m z^n}{(m+n)!\, m!\, n!} + \frac{z\sinh\gamma\cosh\gamma}{2} \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{0,n+r+1} \frac{(\frac{3}{2})_{n+r}\,(1)_r\, z^n\,(z\cosh^2\gamma)^r}{(2)_{n+r}\,(2)_{n+r}\,(\frac{3}{2})_r} +$$

$$+ \frac{y\sinh\gamma\cosh\gamma}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+r+1,n} \frac{(\frac{3}{2})_{m+n+r}\,(1)_r\, y^m\, z^n\,(y\cosh^2\gamma)^r}{(2)_{m+n+r}\,(2)_{m+r}\,(\frac{3}{2})_r\,(1)_n} +$$

$$+ \frac{y\,z\sinh\gamma\cosh^3\gamma}{4} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+1,n+r+1} \frac{(\frac{5}{2})_{m+n+r}\,(2)_{m+r}\,(y\cosh^2\gamma)^m\, z^n\,(z\cosh^2\gamma)^r}{(3)_{m+n+r}\,(\frac{5}{2})_{m+r}(2)_{n+r}(2)_m} \tag{2.2}$$

*provided that each of the series involved is absolutely convergent.*

**Theorem 3.** *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \left( \int_0^{\gamma} \sinh^{2m+2n+1}\theta\, \mathrm{d}\theta \right) \frac{y^m z^n}{m!\, n!}$$

$$= \cosh\gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+r,n} \frac{(1)_{m+n+r}\,(\frac{1}{2})_r\,(-y)^m\,(-z)^n (y\sinh^2\gamma)^r}{(\frac{3}{2})_{m+n+r}\,(1)_{m+r}\, n!\, r!} +$$

$$+ \frac{z\sinh^2\gamma\cosh\gamma}{3} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m,n+r+1} \frac{(2)_{m+n+r}\,(\frac{3}{2})_{m+r}\,(y\sinh^2\gamma)^m\,(-z)^n\,(z\sinh^2\gamma)^r}{(\frac{5}{2})_{m+n+r}\,(2)_{n+r}\,(2)_{m+r}\, m!} -$$

$$- \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \frac{(1)_{m+n}\,(-y)^m\,(-z)^n}{(\frac{3}{2})_{m+n}\; m!\,n!} \tag{2.3}$$

*provided that each of the series involved is absolutely convergent.*

**Theorem 4.** *Let $\{\Lambda_{m,n}\}_{m,n=0}^{\infty}$ be a suitably bounded double sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \left(\int_0^{\gamma} \cosh^{2m+2n+1}\theta\, d\theta\right) \frac{y^m\, z^n}{m!\; n!}$$

$$= \sinh\gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+r,n} \frac{(1)_{m+n+r}\,(\frac{1}{2})_r\, y^m\, z^n (y\cosh^2\gamma)^r}{(\frac{3}{2})_{m+n+r}\,(1)_{m+r}\; n!\, r!} +$$

$$+ \frac{z\sinh\gamma\cosh^2\gamma}{3} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m,n+r+1} \frac{(2)_{m+n+r}\,(\frac{3}{2})_{m+r}\,(y\cosh^2\gamma)^m\, z^n\,(z\cosh^2\gamma)^r}{(\frac{5}{2})_{m+n+r}\,(2)_{n+r}\,(2)_{m+r}\; m!} \tag{2.4}$$

*provided that each of the series involved is absolutely convergent.*

## 3 Derivations

Suppose left hand side of (2.1) is denoted by "$T$" and using the integral (1.7), then we get

$$T = -\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{m+n-1} \Lambda_{m,n} \frac{\sinh\gamma\cosh\gamma\,(\frac{1}{2})_{m+n}\,(1)_r\,(-1)^{m+n}\,(-1)^r\, y^m\, z^n (\sinh^2\gamma)^r}{(1)_{m+n}(1)_m(1)_n(\frac{3}{2})_r} +$$

$$+\gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \frac{(\frac{1}{2})_{m+n}\,(-1)^{m+n}\, y^m\, z^n}{(1)_{m+n}\; m!\, n!}$$

$$= -\sinh\gamma\cosh\gamma \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{0,n+r+1} \frac{(\frac{1}{2})_{n+r+1}\,(1)_r\,(-1)^{n+r+1}\,(-1)^r\; z^{n+r+1}(\sinh^2\gamma)^r}{(1)_{n+r+1}\,(1)_{n+r+1}(\frac{3}{2})_r} -$$

$$-\sinh\gamma\cosh\gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+r+1,n} \frac{(\frac{1}{2})_{m+n+r+1}\,(1)_r\,(-1)^{m+n+r+1}\,(-1)^r\, y^{m+r+1}\, z^n (\sinh^2\gamma)^r}{(1)_{m+n+r+1}(1)_{m+r+1}(1)_n(\frac{3}{2})_r} -$$

$$-\sinh\gamma\cosh\gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+1,n+r+1} \frac{(\frac{1}{2})_{m+n+r+2}\,(1)_{m+r+1}\,(-1)^{m+n+r+2}\,(-1)^{m+r+1}}{(1)_{m+n+r+2}(1)_{m+1}(1)_{n+r+1}} \times$$

$$\times \frac{y^{m+1}\, z^{n+r+1}(\sinh^2\gamma)^{m+r+1}}{(\frac{3}{2})_{m+r+1}} + \gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \frac{(\frac{1}{2})_{m+n}\,(-1)^{m+n}\, y^m\, z^n}{(1)_{m+n}\; m!\, n!}$$

$$= \frac{z\sinh\gamma\cosh\gamma}{2} \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{0,n+r+1} \frac{(\frac{3}{2})_{n+r}\,(1)_r\,(-z)^n (z\sinh^2\gamma)^r}{(2)_{n+r}\,(2)_{n+r}(\frac{3}{2})_r} +$$

$$+\frac{y\sinh\gamma\cosh\gamma}{2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+r+1,n} \frac{(\frac{3}{2})_{m+n+r}\,(1)_r\,(-y)^m\,(-z)^n (y\sinh^2\gamma)^r}{(2)_{m+n+r}(2)_{m+r}(1)_n(\frac{3}{2})_r} +$$

$$+\frac{yz\sinh^3\gamma\cosh\gamma}{4} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Lambda_{m+1,n+r+1} \frac{(\frac{5}{2})_{m+n+r}\,(2)_{m+r}\,(y\sinh^2\gamma)^m\,(-z)^n (z\sinh^2\gamma)^r}{(3)_{m+n+r}(2)_m(2)_{n+r}(\frac{5}{2})_{m+r}} \times$$

$$+\gamma \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Lambda_{m,n} \frac{(\frac{1}{2})_{m+n}\,(-y)^m\,(-z)^n}{(1)_{m+n}\; m!\, n!}$$

which is the right hand side of (2.1).

Similarly we can derive (2.2) to (2.4) by means of series identities (1.5) and (1.6).

## 4 Hypergeometric generalizations of integrals and their solutions

Setting $\Lambda_{m,n} = \frac{((a_A))_{m+n}((d_D))_m((g_G))_n}{((b_B))_{m+n}((e_E))_m((h_H))_n}$, in theorems (2.1) to (2.4), using some algebraic properties of Pochhammer symbol and multiple power series in hypergeometric notations given by (1.2) to (1.4), we get the analytical solutions of following integrals.

$$\int_0^{\gamma} F^{A:D;G}_{B:E;H}\left[\begin{array}{l} (a_j)_{j=1}^A : (d_j)_{j=1}^D ; (g_j)_{j=1}^G \;; \\ (b_j)_{j=1}^B : (e_j)_{j=1}^E ; (h_j)_{j=1}^H \;; \end{array} y\sinh^2\theta,\, z\sinh^2\theta\right] \mathrm{d}\theta$$

$$= \gamma\, F^{A+1:D;G}_{B+1:E;H}\left[\begin{array}{l} (a_j)_{j=1}^A,\, \tfrac{1}{2} : (d_j)_{j=1}^D ; (g_j)_{j=1}^G \;; \\ (b_j)_{j=1}^B,\, 1 : (e_j)_{j=1}^E ; (h_j)_{j=1}^H \;; \end{array} -y,\, -z\right] + \frac{z\sinh\gamma\cosh\gamma \prod_{i=1}^{A}(a_i) \prod_{i=1}^{G}(g_i)}{2\prod_{i=1}^{B}(b_i)\prod_{i=1}^{H}(h_i)} \times$$

$$\times\, F^{A+G+1:1;2}_{B+H+2:0;1}\left[\begin{array}{l} \tfrac{3}{2},\, (1+a_j)_{j=1}^A,\, (1+g_j)_{j=1}^G \;: 1 ; 1, 1 \;; \\ 2,\, 2,\, (1+b_j)_{j=1}^B,\, (1+h_j)_{j=1}^H : - ;\, \tfrac{3}{2} \;; \end{array} -z,\, z\sinh^2\gamma\right] +$$

$$+ \frac{y\sinh\gamma\cosh\gamma \prod_{i=1}^{A}(a_i) \prod_{i=1}^{D}(d_i)}{2\prod_{i=1}^{B}(b_i)\prod_{i=1}^{E}(e_i)} \times$$

$$\times\, F^{(3)}\left[\begin{array}{l} \tfrac{3}{2}, (1+a_j)_{j=1}^A :: - ; - ; (1+d_j)_{j=1}^D \;: 1 ; (g_j)_{j=1}^G ; 1, 1 \;; \\ 2,\, (1+b_j)_{j=1}^B :: - ; - ; (1+e_j)_{j=1}^E, 2 : - ; (h_j)_{j=1}^H ; \tfrac{3}{2} \;; \end{array} -y,\, -z,\, y\sinh^2\gamma\right] +$$

$$+ \frac{yz\sinh^3\gamma\cosh\gamma \prod_{i=1}^{A}(a_i)\prod_{i=1}^{A}(1+a_i)\prod_{i=1}^{D}(d_i)\prod_{i=1}^{G}(g_i)}{4\prod_{i=1}^{B}(b_i)\prod_{i=1}^{B}(1+b_i)\prod_{i=1}^{E}(e_i)\prod_{i=1}^{H}(h_i)} \times$$

$$\times F^{(3)}\left[\begin{array}{l} \tfrac{5}{2}, (2+a_j)_{j=1}^A :: - ; (1+g_j)_{j=1}^G \;; 2 : 1, (1+d_j)_{j=1}^D ; 1 ; 1 \;; \\ 3,\, (2+b_j)_{j=1}^B :: - ; (1+h_j)_{j=1}^H, 2 ; \tfrac{5}{2} : 2, (1+e_j)_{j=1}^E ; - ; - \;; \end{array} y\sinh^2\gamma,\, -z,\, z\sinh^2\gamma\right] \qquad (4.1)$$

$$\int_0^{\gamma} F^{A:D;G}_{B:E;H}\left[\begin{array}{l} (a_j)_{j=1}^A : (d_j)_{j=1}^D ; (g_j)_{j=1}^G \;; \\ (b_j)_{j=1}^B : (e_j)_{j=1}^E ; (h_j)_{j=1}^H \;; \end{array} y\cosh^2\theta,\, z\cosh^2\theta\right] \mathrm{d}\theta$$

$$= \gamma\, F^{A+1:D;G}_{B+1:E;H}\left[\begin{array}{l} (a_j)_{j=1}^A,\, \tfrac{1}{2} : (d_j)_{j=1}^D ; (g_j)_{j=1}^G \;; \\ (b_j)_{j=1}^B,\, 1 : (e_j)_{j=1}^E ; (h_j)_{j=1}^H \;; \end{array} y,\, z\right] + \frac{z\sinh\gamma\cosh\gamma \prod_{i=1}^{A}(a_i) \prod_{i=1}^{G}(g_i)}{2\prod_{i=1}^{B}(b_i)\prod_{i=1}^{H}(h_i)} \times$$

$$\times\, F^{A+G+1:1;2}_{B+H+2:0;1}\left[\begin{array}{l} \tfrac{3}{2},\, (1+a_j)_{j=1}^A,\, (1+g_j)_{j=1}^G \;: 1 ; 1, 1 \;; \\ 2,\, 2,\, (1+b_j)_{j=1}^B,\, (1+h_j)_{j=1}^H : - ;\, \tfrac{3}{2} \;; \end{array} z,\, z\cosh^2\gamma\right] +$$

$$+ \frac{y\sinh\gamma\cosh\gamma \prod_{i=1}^{A}(a_i) \prod_{i=1}^{D}(d_i)}{2\prod_{i=1}^{B}(b_i)\prod_{i=1}^{E}(e_i)} \times$$

$$\times F^{(3)}\left[\begin{array}{l}\frac{3}{2},(1+a_j)_{j=1}^{A}::-;-;(1+d_j)_{j=1}^{D}\;:1\;;(g_j)_{j=1}^{G};1,1\;;\\ 2,\,(1+b_j)_{j=1}^{B}::-;-;(1+e_j)_{j=1}^{E},2:-;(h_j)_{j=1}^{H};\frac{3}{2}\;;\end{array}\;y,\,z,\,y\cosh^2\gamma\right]+$$

$$+\frac{y\,z\,\sinh\gamma\,\cosh^3\gamma\,\prod_{i=1}^{A}(a_i)\,\prod_{i=1}^{A}(1+a_i)\,\prod_{i=1}^{D}(d_i)\,\prod_{i=1}^{G}(g_i)}{4\prod_{i=1}^{B}(b_i)\,\prod_{i=1}^{B}(1+b_i)\,\prod_{i=1}^{E}(e_i)\,\prod_{i=1}^{H}(h_i)}\times$$

$$\times F^{(3)}\left[\begin{array}{l}\frac{5}{2},(2+a_j)_{j=1}^{A}::-;(1+g_j)_{j=1}^{G}\;;2:1,(1+d_j)_{j=1}^{D};1\,;1\;;\\ 3,\,(2+b_j)_{j=1}^{B}::-;(1+h_j)_{j=1}^{H},2;\frac{5}{2}:2,(1+e_j)_{j=1}^{E};-;-\;;\end{array}\;y\cosh^2\gamma,\,z,\,z\cosh^2\gamma\right] \tag{4.2}$$

$$\int_0^{\gamma}\sinh\theta\,F^{A:D;G}_{B:E;H}\left[\begin{array}{l}(a_j)_{j=1}^{A}:(d_j)_{j=1}^{D};(g_j)_{j=1}^{G}\;;\\ (b_j)_{j=1}^{B}:(e_j)_{j=1}^{E};(h_j)_{j=1}^{H}\;;\end{array}\;y\sinh^2\theta,\,z\sinh^2\theta\right]d\theta$$

$$=\cosh\gamma\,F^{(3)}\left[\begin{array}{l}1,\,(a_j)_{j=1}^{A}::-;-;(d_j)_{j=1}^{D}\;:1\;;(g_j)_{j=1}^{G};\,\frac{1}{2}\;;\\ \frac{3}{2},\,(b_j)_{j=1}^{B}::-;-;1,\,(e_j)_{j=1}^{E}:-;(h_j)_{j=1}^{H};-\;;\end{array}\;-y,\,-z,\,y\sinh^2\gamma\right]+$$

$$+\frac{z\,\sinh^2\gamma\,\cosh\gamma\,\prod_{i=1}^{A}(a_i)\,\prod_{i=1}^{G}(g_i)}{3\prod_{i=1}^{B}(b_i)\,\prod_{i=1}^{H}(h_i)}\times$$

$$\times F^{(3)}\left[\begin{array}{l}2,\,(1+a_j)_{j=1}^{A}::-;(1+g_j)_{j=1}^{G}\;;\frac{3}{2}:(d_j)_{j=1}^{D};\,1\,;\,1\;;\\ \frac{5}{2},\,(1+b_j)_{j=1}^{B}::-;2,\,(1+h_j)_{j=1}^{H};2:(e_j)_{j=1}^{E};-\,;-\;;\end{array}\;y\sinh^2\gamma,\,-z,\,z\sinh^2\gamma\right]+$$

$$+F^{A+1:D;G}_{B+1:E;H}\left[\begin{array}{l}1,\,(a_j)_{j=1}^{A}:(d_j)_{j=1}^{D};(g_j)_{j=1}^{G}\;;\\ \frac{3}{2},\,(b_j)_{j=1}^{B}:(e_j)_{j=1}^{E};(h_j)_{j=1}^{H}\;;\end{array}\;-y,\,-z\right] \tag{4.3}$$

$$\int_0^{\gamma}\cosh\theta\,F^{A:D;G}_{B:E;H}\left[\begin{array}{l}(a_j)_{j=1}^{A}:(d_j)_{j=1}^{D};(g_j)_{j=1}^{G}\;;\\ (b_j)_{j=1}^{B}:(e_j)_{j=1}^{E};(h_j)_{j=1}^{H}\;;\end{array}\;y\cosh^2\theta,\,z\cosh^2\theta\right]d\theta$$

$$=\sinh\gamma\,F^{(3)}\left[\begin{array}{l}1,\,(a_j)_{j=1}^{A}::-;-;(d_j)_{j=1}^{D}\;:1\;;(g_j)_{j=1}^{G};\,\frac{1}{2}\;;\\ \frac{3}{2},\,(b_j)_{j=1}^{B}::-;-;1,\,(e_j)_{j=1}^{E}:-;(h_j)_{j=1}^{H};-\;;\end{array}\;y,\,z,\,y\cosh^2\gamma\right]+$$

$$+\frac{z\,\sinh\gamma\,\cosh^2\gamma\,\prod_{i=1}^{A}(a_i)\,\prod_{i=1}^{G}(g_i)}{3\prod_{i=1}^{B}(b_i)\,\prod_{i=1}^{H}(h_i)}\times$$

$$\times F^{(3)}\left[\begin{array}{l}2,\,(1+a_j)_{j=1}^{A}::-;(1+g_j)_{j=1}^{G}\;;\frac{3}{2}:(d_j)_{j=1}^{D};\,1\,;\,1\;;\\ \frac{5}{2},\,(1+b_j)_{j=1}^{B}::-;2,\,(1+h_j)_{j=1}^{H};2:(e_j)_{j=1}^{E};-\,;-\;;\end{array}\;y\cosh^2\gamma,\,z,\,z\cosh^2\gamma\right] \tag{4.4}$$

*provided that each of the series as well as associated integrals involved are convergent.*

These solutions are not found in Ramanujan's notebooks[11-13], Five notebooks of B. C. Berndt[5-9], Three volumes of R. P. Agarwal[1-3] and other literature[4;10;14;15] on special functions.

## References

[1] R. P. Agarwal; *Resonance of Ramanujan's Mathematics,* Vol. I, New Age International(P) Ltd., New Delhi, 1996.

[2] R. P. Agarwal; *Resonance of Ramanujan's Mathematics,* Vol. II, New Age International(P) Ltd., New Delhi, 1996.

[3] R. P. Agarwal; *Resonance of Ramanujan's Mathematics,* Vol. III, New Age International(P) Ltd., New Delhi, 1999.

[4] G. E. Andrews and B. C. Berndt; *Ramanujan's Lost Notebook*, Part I, Springer-Verlag, New York, 2005.

[5] B. C. Berndt; *Ramanujan's Notebooks,* Part I, Springer-Verlag, New York, 1985.

[6] B. C. Berndt; *Ramanujan's Notebooks,* Part II, Springer-Verlag, New York 1989.

[7] B. C. Berndt; *Ramanujan's Notebooks,* Part III, Springer-Verlag, New York, 1991.

[8] B. C. Berndt; *Ramanujan's Notebooks,* Part IV, Springer-Verlag, New York 1994.

[9] B. C. Berndt; *Ramanujan's Notebooks,* Part V, Springer-Verlag, New York 1998.

[10] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev; *Integrals and Series*, Vol 3., More Special Functions, Nauka, Moscow, 1986; Translated from the Russian by G. G. Gould; Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo, Melbourne, 1990.

[11] S. Ramanujan; *Notebooks of Srinivasa Ramanujan,* Vol. I, Tata Institute of Fundamental Research, Bombay, 1957; Reprinted by Narosa Publishing House, New Delhi, 1984.

[12] S. Ramanujan; *Notebooks of Srinivasa Ramanujan,* Vol. II, Tata Institute of Fundamental Research, Bombay, 1957; Reprinted by Narosa Publishing House, New Delhi, 1984.

[13] S. Ramanujan; *The Lost Notebook and Other Unpublished Papers,* Narosa Publishing House, New Delhi, 1988.

[14] M. I. Qureshi, C. W. Mohd, M. P. Chaudhary, K. A. Quraishi and I. H. Khan; Analytical Solutions of Incomplete Elliptic Integrals Motivated by the Work of Ramanujan, *Malaya Journal of Matematik,* 2(4) (2014), 376-391.

[15] L. J. Slater; *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, London and New York, 1966.

[16] H. M. Srivastava and P. W. Karlsson; *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Ltd., Chichester, Brisbane, U. K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[17] H. M. Srivastava and H. L. Manocha; *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Ltd., Chichester, Brisbane, U. K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

*Received*: September 12, 2015; *Accepted*: October 03, 2015

**UNIVERSITY PRESS**

Website: http://www.malayajournal.org/