Stability of $n$-Dimensional Additive Functional Equation: Direct and Fixed Point Methods

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Abstract

In this paper, the authors established the generalized Ulam - Hyers stability of $n$-dimensional additive functional equation

$$\sum_{i=1}^{n} f \left( \sum_{j=1}^{n} x_{ij} \right) = (n - 2) \sum_{j=1}^{n} f \left( x_{j} \right) \quad \text{where} \quad x_{ij} = \begin{cases} -x_{j} & \text{if} \quad i = j \\ x_{j} & \text{if} \quad i \neq j \end{cases}$$

and $n$ is a positive integer with $n \neq 2$ using direct and fixed point methods.

Keywords: Additive functional equation, Generalized Ulam-Hyers stability, Fixed point method.

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1 Introduction

The stability of functional equations had been first raised by S.M. Ulam [25].

In 1941, D. H. Hyers [11] remarked a positive answer to the question of Ulam regards to Banach spaces. In 1950, T. Aoki [2] was considered as the second author to handle this problem for additive mappings. Eventually Th.M. Rassias [19] succeeded in extending the result of Hyers’ Theorem by weakening the condition for the Cauchy difference controlled by $(||x||^p + ||y||^p)$, $p \in [0,1)$ to be unbounded. While considering the influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Th.M. Rassias is called Hyers-Ulam-Rassias stability one can refer [1, 7, 12, 13].

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In 1982, J.M. Rassias [17] followed the innovative approach of the Th.M. Rassias theorem [19] in which he replaced the factor $||x||^p + ||y||^p$ by $||x||^p||y||^q$ for $p, q \in R$ with $p + q \neq 1$. The comprehensive state of above results were obtained by P. Gavruta [9] in 1994 by replacing the unbounded Cauchy difference by a general control function $\phi(x, y)$ in the spirit of Rassias approach.

In 2008, a special case of Gavruta’s theorem for the unbounded Cauchy difference was procured by Ravi etal., [23] in view of the summation of both the sum and the product of two $p-$ norms in the sprit of Rassias approach. The stability problems of several functional equations have been examined by a number of authors which produced numerous compelling results regarding to this problem (see [3, 4, 5, 6, 8, 14, 18, 23] and reference cited there in.

In this paper, the authors established generalized Ulam-Hyers stability of $n-$dimensional additive functional equation of the form

$$\sum_{i=1}^{n} f \left( \sum_{j=1}^{n} x_{ij} \right) = (n-2) \sum_{j=1}^{n} f(x_j) \quad \text{where} \quad x_{ij} = \begin{cases} -x_j & \text{if} \quad i = j \\ x_j & \text{if} \quad i \neq j \end{cases}$$

and $n$ is a positive integer with $n \neq 2$ using direct and fixed point methods.

## Stability Results: Direct Method

In this section, the authors presented the generalized Ulam - Hyers stability of the functional equation (1.1).

Through out this paper, let $X$ be a normed space and $Y$ be a Banach space respectively. Define a mapping $D f : X \to Y$ by

$$D f(x_1, x_2, \ldots, x_n) = f(-x_1 + x_2 + \ldots + x_n) + f(x_1 - x_2 + \ldots + x_n) + \ldots \quad \text{for all} \quad x_1, x_2, \ldots, x_n \in X.$$

**Theorem 2.1.** Let $j \in \{-1, 1\}$. Let $\varphi : X^n \to [0, \infty)$ and $f : X \to Y$ be functions which satisfies

$$\sum_{k=0}^{\infty} \frac{\varphi \left( (n-2)^{k}x, (n-2)^{k}x_2, \ldots, (n-2)^{k}x_n \right)}{(n-2)^{kj}} \text{ converges to } R \text{ and}$$

$$\lim_{k \to \infty} \frac{\varphi \left( (n-2)^{k}x, (n-2)^{k}x_2, \ldots, (n-2)^{k}x_n \right)}{(n-2)^{kj}} = 0 \quad (2.1)$$

and

$$\|D f(x_1, x_2, \ldots, x_n)\| \leq \varphi(x_1, x_2, \ldots, x_n) \quad (2.2)$$

for all $x_1, x_2, \ldots, x_n \in X$. Then there exists a unique function $A : X \to Y$ of the form

$$A(x) = \lim_{k \to \infty} \frac{f((n-2)^{k}x)}{(n-2)^{kj}} \quad (2.3)$$

for all $x \in X$, which satisfies (1.1) and

$$\|f(x) - A(x)\| \leq \frac{1}{(n-2)} \sum_{i=j}^{\infty} \Phi_D\left( (n-2)^{ij}x \right) \quad (2.4)$$
where
\[
\Phi_D((n-2)^j x) = \frac{1}{n} \varphi((n-2)^j x, (n-2)^j x, \ldots, (n-2)^j x)
\]
for all \(x \in X\).

Proof. Assume \(j = 1\). Replacing \((x_1, x_2, \ldots, x_n)\) by \((x, x, \ldots, x)\) and divided by \((n-2)\), we get
\[
\left\| \frac{f((n-2)x)}{(n-2)} - f(x) \right\| \leq \frac{1}{n(n-2)} \varphi(x, x, \ldots, x)
\]
(2.6)

Define \(\Phi_D(x) = \frac{1}{n} \varphi(x, x, \ldots, x)\) in (2.6), we arrive
\[
\left\| \frac{f((n-2)x)}{(n-2)} - f(x) \right\| \leq \Phi_D(x) \tag{2.7}
\]
for all \(x \in X\). Replacing \(x\) by \((n-2)x\) in (2.7) and divided by \((n-2)\), we get
\[
\left\| \frac{f((n-2)^2x)}{(n-2)^2} - \frac{f((n-2)x)}{(n-2)} \right\| \leq \Phi_D ((n-2)x) \tag{2.8}
\]
for all \(x \in X\). Combining (2.7) and (2.8), we obtain
\[
\left\| \frac{f((n-2)^2x)}{(n-2)^2} - f(x) \right\| \leq \frac{1}{(n-2)} \left[ \Phi_D (x) + \Phi_D ((n-2)x) \right]
\]
(2.9)
for all \(x \in X\). One can easy to verify for any positive integer \(k\), we reach
\[
\left\| \frac{f((n-2)^kx)}{(n-2)^k} - f(x) \right\| \leq \frac{1}{(n-2)} \sum_{i=0}^{k-1} \Phi_D ((n-2)^i x) \tag{2.10}
\]
for all \(x \in X\). In order to prove the convergence of the sequence \(\left\{ \frac{f((n-2)^kx)}{(n-2)^k} \right\}\), replace \(x\) by \((n-2)^m x\) and divided by \((n-2)^m\) in (2.9), for any \(m, k > 0\), we arrive
\[
\left\| \frac{f((n-2)^k(n-2)^m x)}{(n-2)^{k+m}} - \frac{f((n-2)^m x)}{(n-2)^m} \right\| = \frac{1}{(n-2)^m} \left\| \frac{f((n-2)^k(n-2)^m x)}{(n-2)^k} - \frac{f((n-2)^m x)}{(n-2)^m} \right\|
\]
(2.11)
\[
\leq \frac{1}{(n-2)} \sum_{i=0}^{k-1} \Phi_D ((n-2)^i (n-2)^m x) \leq \frac{1}{(n-2)} \sum_{i=0}^{\infty} \Phi_D ((n-2)^i+m x) \tag{2.12}
\]
for all \(x \in X\). The right hand side of the inequality (2.11) tends to 0 as \(m \to \infty\), the sequence \(\left\{ \frac{f((n-2)^kx)}{(n-2)^k} \right\}\) is a Cauchy sequence. Since \(Y\) is complete, there exists a mapping \(A: X \to Y\) such that
\[
A(x) = \lim_{k \to \infty} \frac{f((n-2)^kx)}{(n-2)^k}, \quad \forall \ x \in X.
\]
Letting \(k \to \infty\) in (2.10), we see that (2.4) holds for all \(x \in X\). Now, we need to prove \(A\) satisfies (1.1), replacing \((x_1, x_2, \ldots, x_n)\) by \((n-2)^k x_1, (n-2)^k x_2, \ldots, (n-2)^k x_n)\) and divided by \((n-2)^k\) in (2.2), we arrive
\[
\frac{1}{(n-2)^k} \left\| Df \left((n-2)^k x_1, \ldots, (n-2)^k x_n\right) \right\| \leq \varphi \left((n-2)^k x_1, \ldots, (n-2)^k x_n) \right\| \tag{2.13}
\]
for all \( x_1, x_2, \ldots, x_n \in X \). Taking \( k \to \infty \) on both sides of (2.13) and using (2.1), we arrive
\[
\| DA \left( (n-2)^k x_1, (n-2)^k x_2, \ldots, (n-2)^k x_n \right) \| = 0.
\]
Hence \( A \) satisfies (1.1) for all \( x_1, x_2, \ldots, x_n \in X \). In order to prove \( A \) is unique, let \( A'(x) \) be another additive mapping satisfying (2.4) and (1.1). Then
\[
\| A(x) - A'(x) \| = \frac{1}{(n-2)^k} \| A((n-2)^k x) - A'((n-2)^k x) \|
\leq \frac{1}{(n-2)^k} \left\{ \| A((n-2)^k x) - f((n-2)^k x) \| + \| f((n-2)^k x) - A'((n-2)^k x) \| \right\}
\leq \frac{2}{(n-2)} \sum_{i=0}^{\infty} \frac{\Phi_D((n-2)^{k+i} x)}{(n-2)^{(k+i)}}
\rightarrow 0 \text{ as } k \to \infty
\]
for all \( x \in X \). Hence \( A \) is unique.

For \( j = -1 \), we can prove the similar stability result. This completes the Proof of the theorem.

The following corollary is an immediate consequence of Theorem 2.1 concerning the stability of (1.1).

**Corollary 2.1.** Let \( \lambda \) and \( s \) be nonnegative real numbers and \( f : X \to Y \) be a function satisfies the inequality
\[
\| Df(x_1, x_2, \ldots, x_n) \| \leq \begin{cases} 
\lambda, \\
\lambda \sum_{i=1}^{n} ||x_i||^s, & s \neq 1; \\
\lambda \prod_{i=1}^{n} ||x_i||^s, & s \neq 1; \\
\lambda \left\{ \prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns} \right\}, & s \neq \frac{1}{n}; 
\end{cases}
\]
(2.14)
for all \( x_1, x_2, \ldots, x_n \) in \( X \). Then there exists a unique additive function \( A : X \to Y \) such that
\[
\| f(x) - A(x) \| \leq \begin{cases} 
\frac{\lambda}{n(n-3)}, \\
\frac{\lambda||x||^s}{n(n-2)-(n-2)^s}, \\
\frac{\lambda||x||^{ns}}{n||x||^{ns} - (n-2)^{ns}}, \\
\frac{\lambda||x||^{ns}}{(1+\frac{n}{s})||x||^{ns} - (n-2)^{ns}}, \\
\frac{\lambda||x||^{ns}}{||x||^{ns} - (n-2)^{ns}}.
\end{cases}
\]
(2.15)
for all \( x \in X \).

### 3 Stability Results: Fixed Point Method

In this section, the authors present the generalized Ulam - Hyers stability of the functional equation (1.1) in Banach space using fixed point method.

Now we will recall the fundamental results in fixed point theory.

**Theorem 3.1.** (The alternative of fixed point) Suppose that for a complete generalized metric space \( (X, d) \) and a strictly contractive mapping \( T : X \to X \) with Lipschitz constant \( L \). Then, for each given element \( x \in X \), either
\[ d(T^nx, T^{n+1}x) = \infty \quad \forall \ n \geq 0, \]

or

(A) there exists a natural number \( n_0 \) such that:

(A1) \( d(T^nx, T^{n+1}x) < \infty \) for all \( n \geq n_0 \);

(A2) The sequence \( (T^n x) \) is convergent to a fixed point \( y^* \) of \( T \);

(A3) \( y^* \) is the unique fixed point of \( T \) in the set \( Y = \{ y \in X : d(T^0 x, y) < \infty \} \);

(A4) \( d(y^*, y) \leq \frac{1}{1-L} d(y, Ty) \) for all \( y \in Y \).

Using (3.1) to prove the stability result we define the following:

\[ \mu_i \]

is a constant such that

\[ \mu_i = \begin{cases} 
  n - 2 & \text{if } i = 0, \\
  \frac{1}{n - 2} & \text{if } i = 1 
\end{cases} \]

and \( \Omega \) is the set such that

\[ \Omega = \{ g : g : X \to Y, g(0) = 0 \} . \]

**Theorem 3.2.** Let \( Df : X \to Y \) and \( \varphi : X^n \to [0, \infty) \) be mappings satisfies the following

\[
\lim_{k \to \infty} \frac{1}{\mu_i^k} \varphi(\mu_i^k x_1, \mu_i^k x_2, \ldots, \mu_i^k x_n) = 0 \quad (3.1)
\]

and

\[
\| Df(x_1, x_2, \ldots, x_n) \| \leq \varphi(x_1, x_2, \ldots, x_n) \quad (3.2)
\]

for all \( x_1, x_2, \ldots, x_n \in X \). If there exists \( L = L(i) < 1 \) such that the function

\[
x \to \Phi_F(x) = \frac{1}{n} \varphi \left( \frac{x}{n-2}, \frac{x}{n-2}, \ldots, \frac{x}{n-2} \right), \quad (3.3)
\]

has the property

\[
\Phi_F(x) = L \frac{1}{\mu_i} \Phi_F(\mu_i x). \quad (3.4)
\]

for all \( x \in X \). Then there exists a unique additive mapping \( A : X \to Y \) satisfying the functional equation (1.1) and

\[
\| f(x) - A(x) \| \leq \frac{L^{1-i}}{1-L} \Phi_F(x) \quad (3.5)
\]

for all \( x \in X \).

**Proof.** Consider the set

\[ \Omega = \{ p / p : X \to Y, p(0) = 0 \} \]

and introduce the generalized metric on \( \Omega \),

\[
d(p, q) = d(p, q) = \inf \{ K \in (0, \infty) : \| p(x) - q(x) \| \leq K \Phi_F(x), x \in X \}.
\]

It is easy to see that \( (\Omega, d) \) is complete.

Define \( T : \Omega \to \Omega \) by

\[
Tp(x) = \frac{1}{\mu_i} p(\mu_i x),
\]
for all \( x \in X \). This implies \( d(Tp, Tq) \leq Ld(p, q) \), for all \( p, q \in \Omega \). i.e., \( T \) is a strictly contractive mapping on \( \Omega \) with Lipschitz constant \( L \).

From (2.6), we arrive

\[
\| f(x) - \frac{f((n - 2)x)}{n - 2} \| \leq \frac{1}{n(n - 2)} \varphi(x, x, \ldots, x) \tag{3.6}
\]

for all \( x \in X \). Using (3.3) and (3.4) for the case \( i = 0 \) it reduces to

\[
\| f(x) - \frac{f((n - 2)x)}{n - 2} \| \leq L \Phi_F(x)
\]

for all \( x \in U \), i.e., \( d\varphi(f, Tf) \leq L \Rightarrow d(f, Tf) \leq L \leq L^1 < \infty \).

Again replacing \( x = \frac{x}{n - 2} \) in (3.6), we get

\[
\| (n - 2)f\left( \frac{x}{n - 2} \right) - f(x) \| \leq \frac{1}{n} \varphi\left( \frac{x}{n - 2}, \frac{x}{n - 2}, \ldots, \frac{x}{n - 2} \right) \tag{3.7}
\]

for all \( x \in X \). Using (3.3) and (3.4) for the case \( i = 1 \) it reduces to

\[
\| f(x) - (n - 2)f\left( \frac{x}{n - 2} \right) \| \leq \Phi_F(x)
\]

for all \( x \in X \), i.e., \( d\varphi(f, Tf) \leq 1 \Rightarrow d(f, Tf) \leq 1 \leq L^0 < \infty \).

In both cases, we arrive

\[
d(f, Tf) \leq L^{1-i}.
\]

Therefore (A1) holds.

By (A2), it follows that there exists a fixed point \( A \) of \( T \) in \( \Omega \) such that

\[
A(x) = \lim_{k \to \infty} \frac{1}{\mu_i^k} \left( f(\mu_i^k x) \right) \tag{3.8}
\]

for all \( x \in X \).

To prove \( A : X \to Y \) is additive. Replacing \( (x_1, x_2, \ldots, x_n) \) by \( (\mu_i^k x_1, \mu_i^k x_2, \ldots, \mu_i^k x_n) \) in (3.2) and dividing by \( \mu_i^k \), it follows from (3.1) that

\[
\| DA(x_1, x_2, \ldots, x_n) \| = \lim_{k \to \infty} \frac{\| D f(\mu_i^k x_1, \mu_i^k x_2, \ldots, \mu_i^k x_n) \|}{\mu_i^k} \leq \lim_{k \to \infty} \frac{\varphi(\mu_i^k x_1, \mu_i^k x_2, \ldots, \mu_i^k x_n)}{\mu_i^k} = 0
\]

for all \( x_1, x_2, \ldots, x_n \in X \). i.e., \( A \) satisfies the functional equation (1.1).

By (A3), \( A \) is the unique fixed point of \( T \) in the set \( \Delta = \{ A \in \Omega : d(f, A) < \infty \} \), \( A \) is the unique function such that

\[
\| f(x) - A(x) \| \leq K \Phi_F(x)
\]

for all \( x \in X \) and \( K > 0 \). Finally by (A4), we obtain

\[
d(f, A) \leq \frac{1}{1 - L} d(f, Tf)
\]
this implies
\[ d(f, A) \leq \frac{L^{1-i}}{1-L} \]
which yields
\[ \| f(x) - A(x) \| \leq \frac{L^{1-i}}{1-L} \Phi_F(x) \]
this completes the Proof of the theorem.

The following corollary is an immediate consequence of Theorem 3.2 concerning the stability of (1.1).

**Corollary 3.2.** Let \( Df : X \to Y \) be a mapping and there exists real numbers \( \lambda \) and \( s \) such that

\[
\| Df(x_1, x_2, \ldots, x_n) \| \leq \left\{ \begin{array}{ll}
\lambda, & s \neq 1; \\
\lambda \sum_{i=1}^{n} ||x_i||^s, & s \neq \frac{1}{n}; \\
\lambda \prod_{i=1}^{n} ||x_i||^s, & s \neq \frac{1}{n}; \\
\lambda \left\{ \prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns} \right\}, & s \neq \frac{1}{n};
\end{array} \right.
(3.9)
\]

for all \( x_1, x_2, \ldots, x_n \in X \), then there exists a unique additive function \( A : X \to Y \) such that

\[
\| f(x) - A(x) \| \leq \left\{ \begin{array}{ll}
\frac{\lambda}{n|n-3|}, & \lambda = 1; \\
\frac{\lambda ||x||^s}{|n-2| + |n-2|s}, & \lambda = 1; \\
\frac{n|n-2| + (n-2)|ns|}{1 + \frac{1}{n}} \lambda ||x||^{ns}, & \lambda = 1;
\end{array} \right.
(3.10)
\]

for all \( x \in X \).

**Proof.** Replacing \( \phi(x_1, x_2, \ldots, x_n) = \left\{ \begin{array}{ll}
\lambda, & s \neq 1; \\
\lambda \sum_{i=1}^{n} ||x_i||^s, & s \neq \frac{1}{n}; \\
\lambda \prod_{i=1}^{n} ||x_i||^s, & s \neq \frac{1}{n}; \\
\lambda \left\{ \prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns} \right\}, & s \neq \frac{1}{n};
\end{array} \right. \)

for all \( x_1, x_2, \ldots, x_n \in X \). Now,

\[
\phi(\mu_1^{k}x_1, \mu_2^{k}x_2, \ldots, \mu_n^{k}x_n) = \left\{ \begin{array}{ll}
\lambda \mu_1^{-k}, & \lambda \mu_1^{k} \\
\lambda \mu_1^{k} \prod_{i=1}^{n} ||x_i||^s, & \lambda \mu_1^{k} \\
\lambda \prod_{i=1}^{n} ||x_i||^s, & \lambda \mu_1^{k} (ns-1) \\
\lambda \prod_{i=1}^{n} ||x_i||^s, & \lambda \mu_1^{k} (ns-1) \\
\lambda \prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}, & \lambda \mu_1^{k} (ns-1)
\end{array} \right. = \left\{ \begin{array}{ll}
\to 0 \text{ as } k \to \infty, & \lambda \mu_1^{k} \\
\to 0 \text{ as } k \to \infty, & \lambda \mu_1^{k} \\
\to 0 \text{ as } k \to \infty, & \lambda \mu_1^{k} \\
\to 0 \text{ as } k \to \infty. & \lambda \mu_1^{k}
\end{array} \right.
\]

Thus, (3.1) is holds.
But we have $\Phi_F(x) = \frac{1}{n} \varphi \left( \frac{x}{n-2}, \frac{x}{n-2}, \cdots, \frac{x}{n-2} \right)$ has the property $\Phi_F(x) = L \cdot \frac{1}{\mu_i} \Phi_F(\mu_i x)$ for all $x \in X$. Hence

$$
\Phi_F(x) = \frac{1}{n} \varphi \left( \frac{x}{n-2}, \frac{x}{n-2}, \cdots, \frac{x}{n-2} \right) = \left\{ \begin{array}{ll}
\frac{\lambda}{n \mu_i} & \frac{1}{\mu_i} \phi(x) \\
\frac{\lambda}{n \mu_i (n-2)^s} ||x||^s & \frac{1}{\mu_i} \phi(x) \\
\frac{\lambda}{n \mu_i (n-2)^{ns}} ||x||^{ns} & \frac{1}{\mu_i} \phi(x) \\
\frac{\lambda}{\mu_i (n-2)^{ns}} ||x||^{ns} & \phi(x) \\
\end{array} \right.
$$

Now,

$$
\frac{1}{\mu_i} \Phi_F(\mu_i x) = \left\{ \begin{array}{ll}
\mu_i^{-1} \Phi_F(x) \\
\mu_i^{1-s} \Phi_F(x) \\
\mu_i^{ns-1} \Phi_F(x) \\
\end{array} \right.
$$

From (3.5), we prove the following cases.

**Case 1** $L = (n-2)^{-1}$ if $i = 0$,

$$
||f(x) - A(x)|| \leq \lambda \frac{((n-2)^{-1})^{1-0}}{1 - (n-2)^{-1}} = \frac{\lambda}{n(n-3)}
$$

**Case 2** $L = (n-2)^1$ if $i = 0$,

$$
||f(x) - A(x)|| \leq \lambda \frac{((n-2)^{1})^{1-1}}{1 - (n-2)^1} = \frac{\lambda}{n(n-3)}
$$

**Case 3** $L = (n-2)^{s-1}$ for $s < 1$ if $i = 0$,

$$
||f(x) - A(x)|| \leq \frac{\lambda}{(n-2)^s} \left( \frac{((n-2)^{(s-1)})^{1-0}}{1 - (n-2)^{(s-1)}} \right) ||x||^s = \frac{\lambda}{(n-2) - (n-2)^s} ||x||^s.
$$

**Case 4** $L = (n-2)^{1-s}$ for $s > 1$ if $i = 1$,

$$
||f(x) - A(x)|| \leq \frac{\lambda}{(n-2)^s} \left( \frac{((n-2)^{(1-s)})^{1-1}}{1 - (n-2)^{(1-s)}} \right) ||x||^s = \frac{\lambda}{(n-2)^s - (n-2)} ||x||^s.
$$

**Case 5** $L = (n-2)^{ns-1}$ for $s < \frac{1}{n}$ if $i = 0$,

$$
||f(x) - A(x)|| \leq \frac{\lambda}{n(n-2)^s} \left( \frac{((n-2)^{(ns-1)})^{1-0}}{1 - (n-2)^{(ns-1)}} \right) ||x||^{ns} = \frac{\lambda}{n((n-2) - (n-2)^{ns})} ||x||^{ns}.
$$

**Case 6** $L = (n-2)^{1-ns}$ for $s > \frac{1}{n}$ if $i = 1$,

$$
||f(x) - A(x)|| \leq \frac{\lambda}{n(n-2)^s} \left( \frac{((n-2)^{(1-ns)})^{1-0}}{1 - (n-2)^{(1-ns)}} \right) ||x||^{ns} = \frac{\lambda}{n((n-2)^{ns} - (n-2))} ||x||^{ns}.
$$

Hence the Proof is complete
4 Counter Examples for Non Stability Cases

In this section, authors discussed the counter examples for non stable cases for the coollaries 2.1 and 3.2.

Now we will provide an example to illustrate that the functional equation (1.1) is not stable for \( s = 1 \) in Condition (ii) of coollary 2.1.

**Example 4.1.** Let \( \phi : \mathbb{R} \to \mathbb{R} \) be a function defined by

\[
\phi(x) = \begin{cases} 
\mu x, & \text{if } |x| < 1 \\
\mu, & \text{otherwise} 
\end{cases}
\]

where \( \mu > 0 \) is a constant, and define a function \( f : \mathbb{R} \to \mathbb{R} \) by

\[
f(x) = \sum_{n=0}^{\infty} \frac{\phi((n-2)^k x)}{(n-2)^k} \quad \text{for all } x \in \mathbb{R}.
\]

Then \( f \) satisfies the functional inequality

\[
|D f(x_1, x_2, \ldots, x_n)| \leq \frac{n(n-1)(n-2)}{n-3} \mu (|x_1| + |x_2| + \cdots + |x_n|)
\]

for all \( x_1, x_2, \ldots, x_n \in \mathbb{R} \). Then there does not exists a mapping \( A : \mathbb{R} \to \mathbb{R} \) and a constant \( \beta > 0 \) such that

\[
|f(x) - A(x)| \leq \beta |x| \quad \text{for all } x \in \mathbb{R}.
\]

**Proof.** Now

\[
|f(x)| \leq \sum_{n=0}^{\infty} \frac{|\phi((n-2)^k x)|}{|(n-2)^k|} = \sum_{k=0}^{\infty} \frac{\mu}{(n-2)^k} = \left( \frac{n-2}{n-3} \right)^k \mu.
\]

Therefore we see that \( f \) is bounded. We are going to prove that \( f \) satisfies (4.1).

If \( x_1 = x_2 = \cdots = x_n = 0 \) then (4.1) is trivial. If \( |x_1| + |x_2| + \cdots + |x_n| \geq 1 \) then the left hand side of (4.1) is less than \( \left( \frac{n(n-1)(n-2)}{n-3} \right)^k \mu \). Now suppose that \( 0 < |x_1| + |x_2| + \cdots + |x_n| < 1 \). Then there exists a positive integer \( \ell \) such that

\[
\frac{1}{(n-2)^\ell} \leq |x_1| + |x_2| + \cdots + |x_n| < \frac{1}{(n-2)^{\ell-1}},
\]

so that \( (n-2)^{\ell-1}|x_1| < 1, (n-2)^{\ell-1}|x_2| < 1, \ldots, (n-2)^{\ell-1}|x_n| < 1 \) and consequently

\[
(n-2)^{\ell-1}(-x_1 + x_2 + \cdots + x_n), (n-2)^{\ell-1}(x_1 - x_2 + \cdots + x_n), \ldots, (n-2)^{\ell-1}(x_1 + x_2 + \cdots - x_n),
\]

\[
- (n-2)^{\ell-1}(x_1), -(n-2)^{\ell-1}(x_2), \ldots, -(n-2)^{\ell-1}(x_n) \in (-1, 1).
\]

Therefore for each \( k = 0, 1, \ldots, \ell - 1 \), we have

\[
(n-2)^{\ell-1}(-x_1 + x_2 + \cdots + x_n), (n-2)^{\ell-1}(x_1 - x_2 + \cdots + x_n), \ldots, (n-2)^{\ell-1}(x_1 + x_2 + \cdots - x_n),
\]

\[
- (n-2)^{\ell-1}(x_1), -(n-2)^{\ell-1}(x_2), \ldots, -(n-2)^{\ell-1}(x_n) \in (-1, 1).
\]

and

\[
\phi \left( (n-2)^k(-x_1 + x_2 + \cdots + x_n) \right) + \cdots + \phi \left( (n-2)^k(x_1 + x_2 + \cdots - x_n) \right)
\]

\[
- (n-2) \left( \phi \left( (n-2)^k x_1 \right) + \cdots + \phi \left( (n-2)^k x_n \right) \right) = 0
\]
for $k = 0, 1, \ldots, \ell - 1$. From the definition of $f$ and (4.3), we obtain that

$$
|D f(x_1, x_2, \ldots, x_n)| \\
\leq \sum_{k=0}^{\infty} \frac{1}{(n-2)^k} \left| \phi \left( (n-2)^k (-x_1 + x_2 + \cdots + x_n) \right) \right| + \cdots + \left| \phi \left( (n-2)^k (x_1 + x_2 + \cdots - x_n) \right) \right| \\
- (n-2) \left| \phi \left( (n-2)^k x_1 \right) \right| + \cdots + \left| \phi \left( (n-2)^k (x_n) \right) \right|
$$

Thus $f$ satisfies (4.1) for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$ with $0 < |x_1| + |x_2| + \cdots + |x_n| < 1$.

We claim that the additive functional equation (1.1) is not stable for $s = 1$ in condition (ii) of corollary 2.1. Suppose on the contrary that there exist a mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying (4.2). Since $f$ is bounded and continuous for all $x \in \mathbb{R}$, $A$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 2.1, $A$ must have the form $A(x) = cx$ for any $x \in \mathbb{R}$. Thus we obtain that

$$
|f(x)| \leq (\beta + |c|) |x|.
$$

But we can choose a positive integer $m$ with $m\mu > \beta + |c|$. If $x \in \left( 0, \frac{1}{(n-2)^m} \right)$, then $(n-2)^k x \in (0, 1)$ for all $k = 0, 1, \ldots, m - 1$. For this $x$, we get

$$
f(x) = \sum_{n=0}^{\infty} \frac{\phi((n-2)^k x)}{(n-2)^k} \geq \sum_{n=0}^{m-1} \frac{\mu((n-2)^k x)}{(n-2)^k} = m\mu x > (\beta + |c|) x
$$

which contradicts (4.4). Therefore the additive functional equation (1.1) is not stable in sense of Ulam, Hyers and Rassias if $s = 1$, assumed in the inequality (2.2).

A counter example to illustrate the non stability in Condition (iii) of corollary 2.1.

**Example 4.2.** Let $s$ be such that $0 < s < \frac{1}{n}$. Then there is a function $f : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying

$$
|D f(x_1, x_2, \ldots, x_n)| \leq \lambda \left( |x_1|^s |x_2|^s \cdots |x_{n-1}|^s |x_n|^{1-(n-1)p} \right)
$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and

$$
\sup_{x \neq 0} \frac{|f(x) - A(x)|}{|x|} = +\infty
$$

for every mapping $A : \mathbb{R} \to \mathbb{R}$.

**Proof.** If we take

$$
f(x) = \left\{ \begin{array}{ll}
    x \ln |x|, & \text{if } x \neq 0, \\
    0, & \text{if } x = 0.
\end{array} \right.
$$

Then from the relation (4.6), it follows that

$$
\sup_{x \neq 0} \frac{|f(x) - A(x)|}{|x|} \geq \sup_{n \in \mathbb{N}} \frac{|f(n) - A(n)|}{|n|} = \sup_{n \in \mathbb{N}} \frac{|n \ln |n| - n A(1)|}{|n|} = \sup_{n \in \mathbb{N}} |\ln |n| - A(1)| = \infty.
$$
We have to prove (4.5) is true.

Case (i): If $x_1, x_2, \ldots, x_n > 0$ in (4.5) then,

$$f (-x_1 + x_2 + \ldots + x_n) + \ldots + f (x_1 + x_2 + \ldots - x_n) - (n - 2) (f (x_1) + \ldots + f (x_n))$$

$$= |(-x_1 + x_2 + \ldots + x_n) ln |x_1 + x_2 + \ldots + x_n| + \ldots + (x_1 + x_2 + \ldots - x_n) ln |x_1 + x_2 + \ldots - x_n|$$

$$- (n - 2) (x_1 ln |x_1| + \ldots + x_n ln |x_n|)|.$$

Set $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$ it follows that

$$f (-x_1 + x_2 + \ldots + x_n) + \ldots + f (x_1 + x_2 + \ldots - x_n) - (n - 2) (f (x_1) + \ldots + f (x_n))$$

$$= |(-x_1 + x_2 + \ldots + x_n) ln |x_1 + x_2 + \ldots + x_n| + \ldots + (x_1 + x_2 + \ldots - x_n) ln |x_1 + x_2 + \ldots - x_n|$$

$$- (n - 2) (x_1 ln |x_1| + \ldots + x_n ln |x_n|)|$$

Then $f$ satisfies the functional inequality

$$\lambda \left( |y_1|^{\frac{1}{n}} |y_2|^{\frac{1}{n}} \ldots |y_{n-1}|^{\frac{1}{n}} |y_n|^{-\frac{1}{n}} \right) = \lambda \left( |x_1|^{\frac{1}{n}} |x_2|^{\frac{1}{n}} \ldots |x_{n-1}|^{\frac{1}{n}} |x_n|^{-\frac{1}{n}} \right)$$

Similarly, one can verify the following cases. Case (ii): If $x_1 > 0$ and $x_2, \ldots, x_n < 0$, Case (iii): $x_1 < 0$ and $x_2, \ldots, x_n > 0$ and Case (iv): $x_1, x_2, \ldots, x_n < 0$ Case (v): If $x_1 = x_2 = \ldots = x_n = 0.$

Now we will provide an example to illustrate that the functional equation (1.1) is not stable for $s = \frac{1}{n}$ in Condition (iv) of corollary 2.1.

Example 4.3. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$\phi(x) = \begin{cases} \mu x, & \text{if } |x| < \frac{1}{n} \\ \frac{\mu}{n}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi((n-2)^k x)}{(n-2)^k} \quad \text{for all } x \in \mathbb{R}.$$ 

Then $f$ satisfies the functional inequality

$$|Df(x_1, \ldots, x_n)| \leq \frac{(n-1)(n-2)}{n-3} \mu \left( |x_1|^{\frac{1}{n}} \ldots |x_n|^{\frac{1}{n}} + |x_1| + \ldots + |x_n| \right)$$

(4.7)

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Then there do not exists a mapping $A : \mathbb{R} \rightarrow \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - A(x)| \leq \beta |x| \quad \text{for all } x \in \mathbb{R}.$$ 

(4.8)

Proof. Now

$$|f(x)| \leq \sum_{n=0}^{\infty} \frac{|\phi((n-2)^k x)|}{|(n-2)^k|} = \sum_{k=0}^{\infty} \frac{\mu}{(n-2)^k} = \left( \frac{n-2}{n-3} \right) \mu.$$ 

Therefore we see that $f$ is bounded. We are going to prove that $f$ satisfies (4.7). If $x_1 = x_2 = \ldots = x_n = 0$ then (4.7) is trivial. If $|x_1|^{\frac{1}{n}} |x_2|^{\frac{1}{n}} \ldots |x_n|^{\frac{1}{n}} + |x_1| + |x_2| + \ldots + |x_n| \geq \frac{1}{n}$ then the left hand side of (4.7) is less than $\left( \frac{(n-1)(n-2)}{n-3} \right) \mu$. Now suppose that $0 < |x_1|^{\frac{1}{n}} |x_2|^{\frac{1}{n}} \ldots |x_n|^{\frac{1}{n}} + |x_1| + |x_2| + \ldots + |x_n| < \frac{1}{n}$.

The rest of the Proof is similar to that of Example 4.1.
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