Some results for the Jacobi-Dunkl transform in the space $L^2(\mathbb{R}, A_{\alpha, \beta}(t)dt)$

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Abstract

In this paper, using a generalized Jacobi-Dunkl translation operator, we prove an analog of Titchmarsh’s theorem for functions satisfying the Jacobi-Dunkl Lipschitz condition in $L^2(\mathbb{R}, A_{\alpha, \beta}(t)dt)$, $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha \neq -\frac{1}{2}$.

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1 Introduction

Titchmarsh’s [[10], Theorem 85] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have:

**Theorem 1.1.** [[10]] Let $\alpha \in (0, 1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following statements are equivalents:

(a) $\|f(t+h) - f(t)\| = O(h^\alpha)$, as $h \to 0$,
(b) $\int_{|\lambda| \geq r} |\hat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha})$, as $r \to \infty$,

where $\hat{f}$ stands for the Fourier transform of $f$.

In this paper, we prove in analog of Theorem 1.1 for the Jacobi-Dunkl transform for functions satisfying the Jacobi-Dunkl Lipschitz condition in the space $L^2(\mathbb{R}, A_{\alpha, \beta}(t)dt)$. For this purpose, we use the generalized translation operator. Similar results have been established in the context of non compact rank one Riemannian symmetric spaces [[9]].

In section 2 below, we recapitulate from [[1], [2], [3], [5]] some results related to the harmonic analysis associated with Jacobi-Dunkl operator $\Lambda_{\alpha, \beta}$. Section 3 is devoted to the main result after defining the class $\text{Lip}(\delta, 2, \alpha, \beta)$ of functions in $L^2_{\alpha, \beta}(\mathbb{R})$ satisfying the Lipschitz condition correspondent to the generalized Jacobi-Dunkl translation.

2 Notation and Preliminaries

The Jacobi-Dunkl function with parameters $(\alpha, \beta)$, $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha \neq -\frac{1}{2}$, is defined by the formula:

$$\forall x \in \mathbb{R}, \psi_{\lambda}^{\alpha, \beta}(x) = \begin{cases} \frac{\phi_{mu}^{\alpha, \beta}(x)}{\lambda} - \frac{i}{\alpha} \frac{d}{dx} \phi_{mu}^{\alpha, \beta}(x) & \text{if } \lambda \in \mathbb{C} \setminus \{0\}, \\ 1 & \text{if } \lambda = 0, \end{cases}$$

with $\lambda^2 = \mu^2 + \rho^2$, $\rho = \alpha + \beta + 1$ and $\phi_{mu}^{\alpha, \beta}$ is the Jacobi function given by:

$$\phi_{mu}^{\alpha, \beta}(x) = F\left(\frac{\rho + i\mu}{2}, \frac{\rho - i\mu}{2}, \alpha + 1; -\sinh x^2\right),$$

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where $F$ is the Gauss hypergeometric function (see [1],[5] and [7]).

$\psi_{\lambda}^{\alpha,\beta}$ is the unique $C^\infty$-solution on $\mathbb{R}$ of the differentiel-difference equation

$$\begin{cases} 
\Lambda_{\alpha,\beta}U = i\lambda U, & \lambda \in \mathbb{C}, \\
U(0) = 1,
\end{cases}$$

where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by:

$$\Lambda_{\alpha,\beta}U(x) = \frac{dU(x)}{dx} + [(2\alpha + 1) \coth x + (2\beta + 1) \tanh x] \times \frac{U(x) - U(-x)}{2}.$$  

The operator $\Lambda_{\alpha,\beta}$ is a particular case of the operator $D$ given by

$$DU(x) = \frac{dU(x)}{dx} + \frac{A'(x)}{A(x)} \times \left( \frac{U(x) - U(-x)}{2} \right),$$

where $A(x) = |x|^{2\alpha+1}B(x)$, and $B$ a function of class $C^\infty$ on $\mathbb{R}$, even and positive. The operator $\Lambda_{\alpha,\beta}$ corresponds to the function

$$A(x) = A_{\alpha,\beta}(x) = 2^\rho (\sinh |x|)^{2\alpha+1} (\cosh |x|)^{2\beta+1}.$$

Using the relation

$$\frac{d}{dx} \phi_{\mu}^{\alpha,\beta}(x) = -\frac{\lambda}{4(\alpha + 1)} \sinh(2x) \phi_{\mu}^{\alpha+1,\beta+1}(x),$$

the function $\psi_{\lambda}^{\alpha,\beta}$ can be written in the form below (see [2])

$$\psi_{\lambda}^{\alpha,\beta}(x) = \phi_{\mu}^{\alpha,\beta}(x) + i\frac{\lambda}{4(\alpha + 1)} \sinh(2x) \phi_{\mu}^{\alpha+1,\beta+1}(x), \ \ x \in \mathbb{R},$$

where $\lambda^2 = \mu^2 + \rho^2$, $\rho = \alpha + \beta + 1$.

Denote $L_{\alpha,\beta}^2(\mathbb{R}) = L_{\alpha,\beta}^2(\mathbb{R}, A_{\alpha,\beta}(t)dt)$ the space of measurable functions $f$ on $\mathbb{R}$ such that

$$\|f\|_{L_{\alpha,\beta}^2(\mathbb{R})} = \left( \int_{\mathbb{R}} |f(t)|^2 A_{\alpha,\beta}(t)dt \right)^{1/2} < +\infty.$$  

Using the eigenfunctions $\psi_{\lambda}^{\alpha,\beta}$ of the operator $\Lambda_{\alpha,\beta}$ called the Jacobi-Dunkl kernels, we define the Jacobi-Dunkl transform of a function $f \in L_{\alpha,\beta}^2(\mathbb{R})$ by:

$$\mathcal{F}_{\alpha,\beta}f(\lambda) = \int_{\mathbb{R}} f(t) \psi_{\lambda}^{\alpha,\beta}(t) A_{\alpha,\beta}(t)dt, \ \ \lambda \in \mathbb{R},$$

and the inversion formula

$$f(t) = \int_{\mathbb{R}} \mathcal{F}_{\alpha,\beta}f(\lambda) \psi_{-\lambda}^{\alpha,\beta}(t)d\sigma(\lambda),$$

where

$$d\sigma(\lambda) = \frac{|\lambda|}{8\pi \sqrt{\lambda^2 - \rho^2}} C_{\alpha,\beta}(\sqrt{\lambda^2 - \rho^2}) \mathbb{I}_{|\lambda| < \rho}(\lambda)d\lambda.$$

Here,

$$C_{\alpha,\beta}(\mu) = \frac{2^\rho \Gamma(\alpha + 1)\Gamma(i\mu)}{\Gamma\left(\frac{1}{2}(\rho + i\mu)\right)\Gamma\left(\frac{1}{2}(\alpha - \beta + 1 + i\mu)\right)}, \ \ \mu \in \mathbb{C}\setminus(i\mathbb{N}),$$

and $\mathbb{I}_{|\lambda| < \rho}$ is the characteristic function of $\mathbb{R}\setminus [\rho, \rho]$.  

Denote $L_{\alpha,\beta}^2(\mathbb{R}) = L^2(\mathbb{R}, d\sigma(\lambda)).$

The Jacobi-Dunkl transform is a unitary isomorphism from $L_{\alpha,\beta}^2(\mathbb{R})$ onto $L_{\alpha,\beta}^2(\mathbb{R})$, i.e.,

$$\|f\| := \|f\|_{L_{\alpha,\beta}^2(\mathbb{R})} = \|\mathcal{F}_{\alpha,\beta}(f)\|_{L_{\alpha,\beta}^2(\mathbb{R})}, \quad (2.1)$$
The operator of Jacobi-Dunkl translation is defined by:

\[ T_x f(y) = \int_{\mathbb{R}} f(z) dv_{x,y}^{\alpha,\beta}(z), \quad \forall x, y \in \mathbb{R}, \tag{2.2} \]

where \( v_{x,y}^{\alpha,\beta}(z) \), \( x, y \in \mathbb{R} \), are the signed measures given by

\[
  dv_{x,y}^{\alpha,\beta}(z) = \begin{cases} 
  K_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz & \text{if } x, y \in \mathbb{R}^*, \\
  \delta_x & \text{if } y = 0, \\
  \delta_y & \text{if } x = 0,
  \end{cases}
\]

here, \( \delta_x \) is the Dirac measure at \( x \). And,

\[
  K_{\alpha,\beta}(x,y,z) = M_{\alpha,\beta}(\sinh(|x|) \sinh(|y|) \sinh(|z|))^{-2\alpha}I_{x,y} \times \int_{0}^{\pi} \rho_{\theta}(x,y,z)
\]

\[
  I_{x,y} = |x| - |y| - |x| \cup |y| \cup (|x| - |y|, |x| + |y|),
\]

\[
  \rho_{\theta}(x,y,z) = 1 - \alpha_{x,y,z} + \alpha_{z,x,y} + \alpha_{z,y,x},
\]

for all \( z \in \mathbb{R}, \theta \in [0, \pi] \), \( \alpha_{x,y,z} = \left\{ \begin{array}{ll}
  \frac{\cosh x + \cosh y - \cosh z \cos \theta}{\sinh x \sinh y} & \text{if } xy \neq 0, \\
  0 & \text{if } xy = 0,
\end{array} \right. \)

\[
  g_{\theta}(x,y,z) = 1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + 2 \cosh x \cosh y \cosh z \cos \theta,
\]

\[
  t_+ = \left\{ \begin{array}{ll}
  t & \text{if } t > 0, \\
  0 & \text{if } t \leq 0,
\end{array} \right.
\]

and,

\[
  M_{\alpha,\beta} = \left\{ \begin{array}{ll}
  \frac{2^{-\alpha} \Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha - \beta) \Gamma(\beta + 1/2)} & \text{if } \alpha > \beta, \\
  0 & \text{if } \alpha = \beta.
\end{array} \right.
\]

In [2], we have

\[
  F_{\alpha,\beta}(T_{h}f)(\lambda) = \psi_{\lambda}^{\alpha,\beta}(h) F_{\alpha,\beta}(f)(\lambda), \quad \lambda, h \in \mathbb{R}. \tag{2.3}
\]

For \( \alpha \geq \frac{1}{2} \), we introduce the Bessel normalized function of the first kind defined by:

\[
  j_{\alpha}(z) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n! \Gamma(n + \alpha + 1)}, \quad z \in \mathbb{C}.
\]

Moreover, we see that

\[
  \lim_{z \to 0} \frac{j_{\alpha}(z) - 1}{z^2} \neq 0,
\]

by consequence, there exists \( C_1 > 0 \) and \( \eta > 0 \) satisfying

\[
  |z| \leq \eta \Rightarrow |j_{\alpha}(z) - 1| \geq C_1 |z|^2. \tag{2.4}
\]

**Lemma 2.1.** The following inequalities are valid for Jacobi functions \( \psi_{\mu}^{\alpha,\beta}(t) \):

(c) \( |\psi_{\mu}^{\alpha,\beta}(t)| \leq 1 \),

(d) \( |1 - \psi_{\mu}^{\alpha,\beta}(t)| \leq \mu^2 + \rho^2 \).

**Proof.** (See [8], Lemma 3.1, Lemma 3.2).

**Lemma 2.2.** Let \( \alpha \geq \beta \geq \frac{1}{2} \), \( \alpha \neq \frac{1}{2} \). Then for \( |v| \leq \rho \), there exists a positive constant \( C_2 \) such that

\[
  |1 - \psi_{\mu + iv}^{\alpha,\beta}(t)| \geq C_2 |1 - j_{\alpha}(\mu t)|.
\]

**Proof.** (See [4], Lemma 9).
3 Main results

In this section we introduce and prove an analog of Theorem 1.1. Firstly we have to define, for functions in $L^2_{\alpha,\beta}(\mathbb{R})$, the conditions of Cauchy-Lipschitz related to the Jacobi-Dunkl translation operator given in 2.2.

**Definition 3.1.** Let $\delta \in (0, 1)$. A function $f \in L^2_{\alpha,\beta}(\mathbb{R})$ is said to be in the Jacobi-Dunkl-Lipschitz class, denoted by $\text{Lip}(\delta, 2, \alpha, \beta)$, if

$$\|N_h \Lambda_{\alpha,\beta}^m f\| = O(h^\delta), \quad \text{as} \quad h \to 0,$$

where $N_h = T_h + T_{-h} - 2I$, $I$ is the unit operator in the space $L^2_{\alpha,\beta}(\mathbb{R})$ and $m = 0, 1, 2, \ldots$.

**Lemma 3.3.** For $f \in L^2_{\alpha,\beta}(\mathbb{R})$, then

$$\|N_h \Lambda_{\alpha,\beta}^m f\|^2 = 4 \int_{\mathbb{R}} \lambda^{2m} |\psi_\lambda^{a,\beta}(h) - 1|^2 |\mathcal{F}_{\alpha,\beta} f(\lambda)|^2 d\sigma(\lambda).$$

**Proof.** Since $\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta} f)(\lambda) = i\lambda \mathcal{F}_{\alpha,\beta}(f)(\lambda)$, we have

$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta} f)(\lambda) = i^m \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad m = 0, 1, 2, \ldots$$

(3.5)

We use formulas 2.3 and 3.5 we conclude that

$$\mathcal{F}_{\alpha,\beta}(N_h \Lambda_{\alpha,\beta}^m f)(\lambda) = i^m (\psi_\lambda^{a,\beta}(h) + \psi_\lambda^{a,\beta}(-h) - 2) \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

Since

$$\psi_\lambda^{a,\beta}(h) = \phi_\mu^{a,\beta}(h) + i \frac{\lambda}{4(\alpha + 1)} \sinh(2h) \phi_\mu^{a+1,\beta+1}(h),$$

and $\phi_\mu^{a,\beta}$ is even (see [2]), then

$$\mathcal{F}_{\alpha,\beta}(N_h \Lambda_{\alpha,\beta}^m f)(\lambda) = 2i^m (\phi_\mu^{a,\beta}(h) - 1) \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

Now by Parseval’s identity (formula 2.1), we have the result.

**Theorem 3.1.** Let $f \in L^2_{\alpha,\beta}(\mathbb{R})$. Then the following statements are equivalent:

(i) $f \in \text{Lip}(\delta, 2, \alpha, \beta)$,

(ii) $\int_{|\lambda| \geq r} \lambda^{2m} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta}), \quad \text{as} \quad r \to \infty$.

**Proof.** (i) $\Rightarrow$ (ii). Assume that $f \in \text{Lip}(\delta, 2, \alpha, \beta)$, then we have

$$\|N_h \Lambda_{\alpha,\beta}^m f\| = O(h^\delta), \quad \text{as} \quad h \to 0.$$

From Lemma 3.3, we have

$$\|N_h \Lambda_{\alpha,\beta}^m f\|^2 = 4 \int_{\mathbb{R}} \lambda^{2m} |1 - \phi_\mu^{a,\beta}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

By 2.4 and Lemma 2.2, we get:

$$\int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{2m} |1 - \phi_\mu^{a,\beta}(h)|^2 |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \geq C_1^2 C_2^2 \int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} |\mu h|^{2m} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).$$

From $\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}$ we have

$$\left(\frac{\eta}{2h}\right)^2 - \rho^2 \leq \mu^2 \leq \left(\frac{\eta}{h}\right)^2 - \rho^2$$

$$\Rightarrow \mu^2 h^2 \geq \frac{\eta^2}{4} - \rho^2 h^2.$$
Take $h \leq \frac{r}{3p}$, then we have $\mu^2 h^2 \geq C_3 = C_3(\eta)$. So,
\[
\int_{\frac{r}{3p} \leq |\lambda| \leq \frac{r}{p}} \lambda^{2m}|1 - \varphi^{\alpha,\beta}_\mu(h)|^2 |F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \geq C_1 C_2^2 C_3 \int_{\frac{r}{3p} \leq |\lambda| \leq \frac{r}{p}} \lambda^{2m}|F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).
\]
There exists then a positive constant $C$ such that
\[
\int_{\frac{r}{3p} \leq |\lambda| \leq \frac{r}{p}} \lambda^{2m}|F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq C \int_{\mathbb{R}} \lambda^{2m}|1 - \varphi^{\alpha,\beta}_\mu(h)|^2 |F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq Cr^{2\delta}.
\]
For all $0 < h < \frac{r}{3p}$, then we have
\[
\int_{|\lambda| \leq 2r} \lambda^{2m}|F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq Cr^{-2\delta}, \quad r \to \infty.
\]
Furthermore, we obtain
\[
\int_{|\lambda| \geq r} \lambda^{2m}|F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = \sum_{i=0}^{\infty} \int_{2^i r \leq |\lambda| \leq 2^{i+1} r} \lambda^{2m}|F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda)
\leq C \sum_{i=0}^{\infty} (2^i r)^{-2\delta}
\leq Cr^{-2\delta}.
\]
This proves that
\[
\int_{|\lambda| \geq r} \lambda^{2m}|F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta}), \quad r \to \infty.
\]
(ii) $\Rightarrow$ (i). Suppose now that
\[
\int_{|\lambda| \geq r} \lambda^{2m}|F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2\delta}), \quad r \to \infty,
\]
and write
\[
\int_{\mathbb{R}} \lambda^{2m}|1 - \varphi^{\alpha,\beta}_\mu(h)|^2 |F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = \int_{|\lambda| < \frac{r}{p}} \lambda^{2m}|1 - \varphi^{\alpha,\beta}_\mu(h)|^2 |F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda)
+ \int_{|\lambda| \geq \frac{r}{p}} \lambda^{2m}|1 - \varphi^{\alpha,\beta}_\mu(h)|^2 |F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).
\]
Using the inequality (c) of Lemma 2.1, we get
\[
\int_{|\lambda| \geq \frac{r}{p}} \lambda^{2m}|1 - \varphi^{\alpha,\beta}_\mu(h)|^2 |F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) \leq 4 \int_{|\lambda| \geq \frac{r}{p}} \lambda^{2m}|F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).
\]
Then
\[
\int_{|\lambda| \geq \frac{r}{p}} \lambda^{2m}|1 - \varphi^{\alpha,\beta}_\mu(h)|^2 |F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = O(h^{2\delta}), \quad h \to 0. \quad (3.6)
\]
Set
\[
\phi(\lambda) = \int_{-\infty}^{\lambda} \lambda^{2m}|F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda).
\]
An integration by parts gives:
\[
\int_{0}^{x} \lambda^{2m}|F_{\alpha,\beta}(f)(\lambda)|^2 d\sigma(\lambda) = \int_{0}^{x} -\lambda^{2}\phi'(\lambda) d\lambda
= -\lambda^{2}\phi(x) + 2 \int_{0}^{x} \lambda \phi(\lambda) d\lambda
\leq 2 \int_{0}^{x} O(\lambda^{1-2\delta}) d\lambda
= O(x^{2-2\delta}).
\]
From Lemma 2.1, we get
\[
\int_{|\lambda| \leq \frac{1}{h}} \lambda^{2m} |1 - \varphi_{a, b}^\mu(h)|^2 |\mathcal{F}_{a, b}(f)(\lambda)|^2 d\sigma(\lambda) \leq \int_{|\lambda| \leq \frac{1}{h}} \lambda^{2m} |1 - \varphi_{a, b}^\mu(h)||\mathcal{F}_{a, b}(f)(\lambda)|^2 d\sigma(\lambda)
\]
\[
\leq \int_{|\lambda| \leq \frac{1}{h}} \lambda^{2m} (\mu^2 + \rho^2) h^2 |\mathcal{F}_{a, b}(f)(\lambda)|^2 d\sigma(\lambda)
\]
\[
\leq h^2 \int_{|\lambda| \leq \frac{1}{h}} \lambda^{2m} |\mathcal{F}_{a, b}(f)(\lambda)|^2 d\sigma(\lambda)
\]
\[
= O(h^2 h^{-2 + 2\delta}).
\]

Hence,
\[
\int_{|\lambda| \leq \frac{1}{h}} \lambda^{2m} |1 - \varphi_{a, b}^\mu(h)|^2 |\mathcal{F}_{a, b}(f)(\lambda)|^2 d\sigma(\lambda) = O(h^{2\delta}).
\]  
(3.7)

Finally, we conclude from \[3.6\] and \[3.7\] that
\[
\int_{\mathbb{R}} \lambda^{2m} |1 - \varphi_{a, b}^\mu(h)|^2 |\mathcal{F}_{a, b}(f)(\lambda)|^2 d\sigma(\lambda) = \int_{|\lambda| < \frac{1}{h}} + \int_{|\lambda| \geq \frac{1}{h}}
\]
\[
= O(h^{2\delta}) + O(h^{2\delta})
\]
\[
= O(h^{2\delta}).
\]

And this ends the proof.

**Corollary 3.1.** Let \( f \in L^2_{a, b}(\mathbb{R}) \), and let
\[
\|N_h \Lambda_{a, b}^m f\| = O(h^\delta), \quad \text{as} \quad h \to 0,
\]

Then
\[
\int_{|\lambda| \geq r} |\mathcal{F}_{a, b}(f)(\lambda)|^2 d\sigma(\lambda) = O(r^{-2m+2\delta}), \quad \text{as} \quad r \to \infty.
\]

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**References**


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