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Improvement of conformable fractional Hermite-Hadamard type inequality for convex functions and some new conformable fractional midpoint type inequalities

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Abstract

In this paper, it is proved that conformable fractional Hermite-Hadamard inequality and conformable fractional Hermite-Hadamard-Fejer inequality is just the results of Hermite-Hadamard-Fejér inequality. After this, a new conformable fractional Hermite-Hadamard inequality which is not a result of Hermite-Hadamard-Fejér inequality and better than given in [8] by Set et al. is obtained. Also, new equality is proved and some new conformable fractional midpoint type inequalities are given. Our results have some relations with the results given in [5, 6].

Keywords

Convex functions, Hermite-Hadamard inequalities, Conformable fractional integrals, Midpoint type inequalities.

AMS Subject Classification

26A51, 26A33, 26D10.

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Contents

1. Introduction

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [3, 4].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequality or its weighted versions, the so-called Hermite-Hadamard-Fejér inequality.

In [2], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1.1. *Let $f: [a,b] \to \mathbb{R}$ be convex function. Then, the inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx &\leq \int_a^b f(x)g(x)dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \end{aligned} \tag{1.2}$$

holds, where $g: [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$ (i.e. $g(x) = g(a+b-x)$ for all $x \in [a,b]$).

In [5], Kırmacı used the following equality to obtain midpoint type inequalities and some applications:

Lemma 1.2. *Let $a,b \in I$ with $a<b$ and $f : I^{\circ} \to \mathbb{R}$ is a differentiable mapping (I° tne interior of I). If $f' \in L[a,b]$, then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\\&= (b-a)\left[\int_0^{1/2} t f'(ta+(1-t)b)\,dt\right.\\&\left.+\int_{1/2}^{1}(t-1)f'(ta+(1-t)b)\,dt\right]. \qquad (1.3)\end{aligned}$$

Following definitions of the left and right side Riemann-Liouville fractional are well known in the literature.

Definition 1.3. *Let $a,b \in \mathbb{R}$ with $a<b$ and $f \in L[a,b]$. The left and right Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt,\ x>a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt,\ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-t}t^{\alpha-1}dt$ (see [7, page 69] and [12, page 4]).

The beta function and incomplete beta function defined as follows:

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = \int_0^1 t^{u-1}(1-t)^{v-1}\,dt, \quad u,v>0,$$

$$B_w(u,v) = \int_0^w t^{u-1}(1-t)^{v-1}\,dt \quad u,v>0 \text{ and } 0\le w\le 1.$$

Following definitions of the left and right side conformable fractional integrals given in [1] (see also [8]):

Definition 1.4. *Let $\alpha \in (n,n+1]$, $n = 0,1,2,...$, $\beta = \alpha - n$, $a,b \in \mathbb{R}$ with $a<b$ and $f \in L[a,b]$.The left and right conformable fractional integrals $I_{\alpha}^{a} f$ and ${}^{b}I_{\alpha} f$ of order $\alpha >$ 0 are defined by*

$$I_{\alpha}^{a} f(t) = \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx,\ t>a$$

and

$${}^{b}I_{\alpha} f(t) = \frac{1}{n!}\int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx,\ t<b$$

respectively.

It is easily seen that if one takes $\alpha = n+1$ in the Definition 1.4 (for the left and right conformable fractional integrals), one has the Definition 1.3 (the left and right Riemann-Liouville fractional integrals) for $\alpha \in \mathbb{N}$.

In [8], Set et al. proved following conformable fractional Hermite-Hadamrd type inequality:

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \le a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for conformable fractional integrals hold:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\\&\le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\left[\ I_{\alpha}^{a} f(b)\ +\ {}^{b}I_{\alpha} f(a)\ \right]\\&\le \frac{f(a)+f(b)}{2} \qquad (1.4)\end{aligned}$$

with $\alpha \in (n,n+1]$.

Remark 1.6. *In Theorem 1.3, it is not necessary supposing a,b are positive real numbers. From the Definition 1.4, it is clear that a,b are any real numbers such as $a<b$.*

In [9], Set and Mumcu proved the following conformable fractional Hermite-Hadamard-Fejér type inequality:

Theorem 1.7. *Let $f : [a,b] \to \mathbb{R}$ be a convex function with $a<b$ and $f \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[\ I_{\alpha}^{a} g(b)\ +\ {}^{b}I_{\alpha} g(a)\ \right]\\&\le \left[\ I_{\alpha}^{a}(fg)(b)\ +\ {}^{b}I_{\alpha}(fg)(a)\ \right]\\&\le \frac{f(a)+f(b)}{2}\left[\ I_{\alpha}^{a} g(b)\ +\ {}^{b}I_{\alpha} g(a)\ \right] \qquad (1.5)\end{aligned}$$

with $\alpha > 0$.

In [10], Turhan et al. proved the following left conformable fractional Hermite-Hadamard type inequality and next equality:

Theorem 1.8. *Let $a,b \in \mathbb{R}$ with $a<b$ and $f : [a,b] \to \mathbb{R}$ be a convex function. If $f \in L[a,b]$, then the following inequality for the left conformable fractional integral holds:*

$$\begin{aligned}&f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) \le \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}\ I_{\alpha}^{a} f(b)\\&\le \frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1} \qquad (1.6)\end{aligned}$$

with $n = 0,1,2...$ and $\alpha \in (n,n+1]$.

Lemma 1.9. *Let $a,b \in \mathbb{R}$ with $a<b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $f' \in L[a,b]$, then the following equality for the left conformable fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}\ I_{\alpha}^{a} f(b)\ - f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) \qquad (1.7)$$

$$= (b-a)\left[\begin{array}{c}\int_0^{\frac{n+1}{\alpha+1}} \frac{B_t(n+1,\alpha-n)}{B(n+1,\alpha-n)} f'(ta+(1-t)b)\,dt\\ +\int_{\frac{n+1}{\alpha+1}}^{1}\left(\frac{B_t(n+1,\alpha-n)}{B(n+1,\alpha-n)}-1\right) f'(ta+(1-t)b)\,dt\end{array}\right]$$

with $n = 0,1,2...$ and $\alpha \in (n,n+1]$.

In [11], Turhan et al. proved the following right conformable fractional Hermite-Hadamard type inequality and next equality:

Theorem 1.10. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a convex function. If $f\in L[a,b]$, then the following inequality for the right conformable fractional integral holds:*

$$\begin{aligned} & f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right) \\ & \le \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} \quad {}^{b}I_{\alpha}f(a) \\ & \le \frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1} \end{aligned} \tag{1.8}$$

with $n=0,1,2...$ and $\alpha\in(n,n+1]$.

Lemma 1.11. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $f'\in L[a,b]$, then the following equality for the right conformable fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} \quad {}^{b}I_{\alpha}f(a) \quad -f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right) \tag{1.9}$$

$$=(b-a)\left[\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}}-\frac{B_t(n+1,\alpha-n)}{B(n+1,\alpha-n)}f'(tb+(1-t)a)\,dt \\ +\int_{\frac{n+1}{\alpha+1}}^{1}\left(1-\frac{B_t(n+1,\alpha-n)}{B(n+1,\alpha-n)}\right)f'(tb+(1-t)a)\,dt \end{array}\right]$$

with $n=0,1,2...$ and $\alpha\in(n,n+1]$.

In our studies we noticed that conformable fractional Hermite-Hadamard type inequality given in Theorem 1.5 and conformable fractional Hermite-Hadamard-Fejér type inequality given in Theorem 1.7 are just result of Hermite-Hadamard-Fejér inequality (given in Theorem 1.1), with a special selection of the weighted function. This show how strong the Hermite-Hadamard-Fejér inequality is. However, we will prove new fractional Hermite-Hadamard type inequality which is not a result of Theorem 1.1. Also, we will have new conformable fractional midpoint type inequalities.

2. Results of Hermite-Hadamard-Fejér inequality

Proposition 2.1. *Theorem 1.5 is a result of Theorem 1.1.*

Proof. In Theorem 1.1, let we choose $g(x)=(x-a)^{n}(b-x)^{\alpha-n-1}+(b-x)^{n}(x-a)^{\alpha-n-1}$ for $n=0,1,2...$ and $\alpha\in(n,n+1]$, $a,b\in\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ (It is clear $g(x)$ nonnegative, integrable and symmetric to $\frac{a+b}{2}$). Computing the following integrals, we have

$$\begin{aligned} & \int_a^b g(x)dx \\ & =\int_a^b (x-a)^{n}(b-x)^{\alpha-n-1}+(b-x)^{n}(x-a)^{\alpha-n-1}dx \end{aligned} \tag{2.1}$$

$$=2(b-a)^{\alpha}B(n+1,\alpha-n)=2(b-a)^{\alpha}n!\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)},$$

$$\begin{aligned} & \int_a^b f(x)g(x)dx=\int_a^b\left[\begin{array}{c}(x-a)^{n}(b-x)^{\alpha-n-1}\\ +(b-x)^{n}(x-a)^{\alpha-n-1}\end{array}\right]f(x)dx \\ & =\int_a^b (x-a)^{n}(b-x)^{\alpha-n-1}f(x)dx \\ & +\int_a^b (b-x)^{n}(x-a)^{\alpha-n-1}f(x)dx \\ & =n!\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]. \end{aligned} \tag{2.2}$$

Combining (1.2), (2.1) and (2.2) we have (1.4). This completes the proof. □

Proposition 2.2. *Theorem 1.7 is a result of Theorem 1.1.*

Proof. In Theorem 1.1, let we choose $w(x)=\left[(x-a)^{n}(b-x)^{\alpha-n-1}+(b-x)^{n}(x-a)^{\alpha-n-1}\right]g(x)$ for $n=0,1,2...$ and $\alpha\in(n,n+1]$, $a,b\in\mathbb{R}$ and $g(x)$ nonnegative, integrable and symmetric to $\frac{a+b}{2}$ (It is clear $w(x)$ nonnegative, integrable and symmetric to $\frac{a+b}{2}$). Computing the following integrals, we have

$$\begin{aligned} & \int_a^b w(x)dx=\int_a^b\left[\begin{array}{c}(x-a)^{n}(b-x)^{\alpha-n-1}\\ +(b-x)^{n}(x-a)^{\alpha-n-1}\end{array}\right]g(x)dx \\ & =\int_a^b\left[(x-a)^{n}(b-x)^{\alpha-n-1}\right]g(x)dx \\ & +\int_a^b\left[(b-x)^{n}(x-a)^{\alpha-n-1}\right]g(x)dx \\ & =n!\left[\ I_{\alpha}^{a}g(b)\ +\ {}^{b}I_{\alpha}g(a)\ \right], \end{aligned} \tag{2.3}$$

$$\begin{aligned} & \int_a^b f(x)w(x)dx \\ & =\int_a^b\left[\begin{array}{c}(x-a)^{n}(b-x)^{\alpha-n-1}\\ +(b-x)^{n}(x-a)^{\alpha-n-1}\end{array}\right]f(x)g(x)dx \\ & =\int_a^b\left[(x-a)^{n}(b-x)^{\alpha-n-1}\right]f(x)g(x)dx \\ & +\int_a^b\left[(b-x)^{n}(x-a)^{\alpha-n-1}\right]f(x)g(x)dx \\ & =n!\left[\ I_{\alpha}^{a}(fg)(b)\ +\ {}^{b}I_{\alpha}(fg)(a)\ \right]. \end{aligned} \tag{2.4}$$

Combining (1.2), (2.3) and (2.4) we have (1.5). This completes the proof. □

Remark 2.3. *Theorem 1.8 and Theorem 1.10 are not results of Theorem 1.1.*

3. Improvement of Fractional Hermite-Hadamard Type Inequality

We will use Theorem 1.8 and Theorem 1.10 to have new conformable fractional Hermite-Hadamard type inequality better than (1.4).

Theorem 3.1. *Let $a,b \in \mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a convex function. If $f \in L[a,b]$, then the following inequality for fractional integral holds:*

$$\begin{aligned}
&\frac{f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)+f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)}{2}\\
&\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\left[\ I_{\alpha}^{a}f(b)\ +\ {}^{b}I_{\alpha}f(a)\ \right]\\
&\leq \frac{f(a)+f(b)}{2} \qquad (3.1)
\end{aligned}$$

with $n=0,1,2...$ and $\alpha \in (n,n+1]$.

Proof. If (1.6) and (1.8) gather side by side and dividing into 2, it is hold the desired result. ☐

Remark 3.2. *Since, f is a convex function on $[a,b]$, it is clear $f\left(\frac{a+b}{2}\right) \leq \frac{f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)+f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)}{2}$ for $n=0,1,2...$ and $\alpha \in (n,n+1]$. It means that*

1. *Theorem 3.1 is better than Theorem 1.5,*
2. *In Theorem 3.1 if one takes $\alpha=n+1$, one has [6, Theorem 6],*
3. *In Theorem 3.1 if one takes $\alpha=n+1$, after that if one takes $\alpha=1$ one has (1.1),*
4. *Theorem 3.1 is not a result of Theorem 1.1.*

4. New Conformable Fractional Midpoint Type Inequalities

We will now prove an equality to have new fractional midpoint type inequalities.

Lemma 4.1. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a convex function. If $f\in L[a,b]$, then the following inequality for conformable fractional integrals holds:*

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\left[\begin{array}{c} I_{\alpha}^{a}f(b)\\ +\ {}^{b}I_{\alpha}f(a)\end{array}\right]\\
&-\frac{f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)+f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)}{2}\\
&=\frac{b-a}{2B\left(\begin{array}{c} n+1,\\ \alpha-n\end{array}\right)}\\
&\left[\begin{array}{c}
\int_0^{\frac{n+1}{\alpha+1}} B_t\left(\begin{array}{c} n+1,\\ \alpha-n\end{array}\right) f'(A_t)\,dt\\
+\int_{\frac{n+1}{\alpha+1}}^{1}\left(\begin{array}{c} B_t\left(\begin{array}{c} n+1,\\ \alpha-n\end{array}\right)\\ -B\left(\begin{array}{c} n+1,\\ \alpha-n\end{array}\right)\end{array}\right) f'(A_t)\,dt\\
+\int_0^{\frac{n+1}{\alpha+1}} -B_t\left(\begin{array}{c} n+1,\\ \alpha-n\end{array}\right) f'(B_t)\,dt\\
+\int_{\frac{n+1}{\alpha+1}}^{1}\left(\begin{array}{c} B\left(\begin{array}{c} n+1,\\ \alpha-n\end{array}\right)\\ -B_t\left(\begin{array}{c} n+1,\\ \alpha-n\end{array}\right)\end{array}\right) f'(B_t)\,dt
\end{array}\right] \qquad (4.1)
\end{aligned}$$

with $A_t=ta+(1-t)b$, $B_t=tb+(1-t)a$, $n=0,1,2...$ and $\alpha\in(n,n+1]$.

Proof. If (1.7) and (1.9) gather side by side and dividing into 2, it is hold the desired result. ☐

Corollary 4.2. *In Lemma 4.1,*

1. *If one takes $\alpha=n+1$, one has [6, Lemma 4],*
2. *If one takes $\alpha=n+1$, after that if one takes $\alpha=1$ one has Lemma 1.2.*

Theorem 4.3. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|$ is convex on $[a,b]$, then the following conformable fractional midpoint type inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\left[\ I_{\alpha}^{a}f(b)\ +\ {}^{b}I_{\alpha}f(a)\ \right]\right.\\
&\left.-\frac{f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)+f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)}{2}\right|\\
&\leq \frac{b-a}{2B(n+1,\alpha-n)}\left[\begin{array}{c}|f'(a)|\,T_1(\alpha,n)\\ +|f'(b)|\,T_2(\alpha,n)\end{array}\right], \qquad (4.2)
\end{aligned}$$

where

$$\begin{aligned}
T_1(\alpha,n) &= \left[\begin{array}{c}\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,dt\\ +\int_{\frac{n+1}{\alpha+1}}^{1}|B_t(n+1,\alpha-n)-B(n+1,\alpha-n)|\,dt\end{array}\right],\\
T_2(\alpha,n) &= \left[\begin{array}{c}\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,dt\\ +\int_{\frac{n+1}{\alpha+1}}^{1}|B_t(n+1,\alpha-n)-B(n+1,\alpha-n)|\,dt\end{array}\right],
\end{aligned}$$

with $n=0,1,2...$ and $\alpha\in(n,n+1]$.

Proof. Using Lemma 4.1 and the convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\left[\ I_{\alpha}^{a}f(b)\ +\ {}^{b}I_{\alpha}f(a)\ \right]\right.\\
&\left.-\frac{f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)+f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)}{2}\right|
\end{aligned}$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \, |f'(ta+(1-t)b)| \, dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array} \right| |f'(ta+(1-t)b)| \, dt \\ +\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \, |f'(tb+(1-t)a)| \, dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B(n+1,\alpha-n)- \\ B_t(n+1,\alpha-n) \end{array} \right| |f'(tb+(1-t)a)| \, dt \end{array} \right]$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \left[\begin{array}{c} t|f'(a)| \\ +(1-t)|f'(b)| \end{array} \right] dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array} \right| \left[\begin{array}{c} t|f'(a)| \\ +(1-t)|f'(b)| \end{array} \right] dt \\ +\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \left[\begin{array}{c} t|f'(b)|+ \\ (1-t)|f'(a)| \end{array} \right] dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B(n+1,\alpha-n) \\ -B_t(n+1,\alpha-n) \end{array} \right| \left[\begin{array}{c} t|f'(b)| \\ +(1-t)|f'(a)| \end{array} \right] dt \end{array} \right]$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{l} |f'(a)| \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \, t \, dt \\ +|f'(b)| \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| (1-t) \, dt \\ +|f'(a)| \int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array} \right| t \, dt \\ +|f'(b)| \int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array} \right| (1-t) \, dt \\ +|f'(b)| \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \, t \, dt \\ +|f'(a)| \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| (1-t) \, dt \\ +|f'(b)| \int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B(n+1,\alpha-n) \\ -B_t(n+1,\alpha-n) \end{array} \right| t \, dt \\ +|f'(a)| \int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B(n+1,\alpha-n) \\ -B_t(n+1,\alpha-n) \end{array} \right| (1-t) \, dt \end{array} \right]$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{c} |f'(a)| \left[\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \, dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array} \right| dt \end{array} \right] \\ +|f'(b)| \begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \, dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array} \right| dt \end{array} \end{array} \right].$$

This completes the proof. □

Corollary 4.4. *In Theorem 4.3,*

1. *If one takes $\alpha = n+1$, one has [6, Theorem 7]*
2. *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$ one has [5, Theorem 2.2].*

Theorem 4.5. *Let $a,b \in \mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|^q$ is convex on $[a,b]$ for $q \geq 1$, then the following conformable fractional midpoint type inequality holds:*

$$\left| \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha \Gamma(\alpha-n)} \left[I_\alpha^a f(b) + {}^bI_\alpha f(a) \right] - \frac{f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) + f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)}{2} \right| \tag{4.3}$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{l} T_3^{1-\frac{1}{q}}(\alpha,n) \left(\begin{array}{c} |f'(a)|^q T_4(\alpha,n) \\ +|f'(b)|^q T_5(\alpha,n) \end{array} \right)^{\frac{1}{q}} \\ +T_6^{1-\frac{1}{q}}(\alpha,n) \left(\begin{array}{c} |f'(a)|^q T_7(\alpha,n) \\ +|f'(b)|^q T_8(\alpha,n) \end{array} \right)^{\frac{1}{q}} \\ +T_3^{1-\frac{1}{q}}(\alpha,n) \left(\begin{array}{c} |f'(b)|^q T_4(\alpha,n) \\ +|f'(a)|^q T_5(\alpha,n) \end{array} \right)^{\frac{1}{q}} \\ +T_6^{1-\frac{1}{q}}(\alpha,n) \left(\begin{array}{c} |f'(b)|^q T_7(\alpha,n) \\ +|f'(a)|^q T_8(\alpha,n) \end{array} \right)^{\frac{1}{q}} \end{array} \right],$$

where

$$\begin{aligned} T_3(\alpha,n) &= \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \, dt, \\ T_4(\alpha,n) &= \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \, t \, dt, \\ T_5(\alpha,n) &= \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| (1-t) \, dt, \\ T_6(\alpha,n) &= \int_{\frac{n+1}{\alpha+1}}^1 |B_t(n+1,\alpha-n) - B(n+1,\alpha-n)| \, dt, \\ T_7(\alpha,n) &= \int_{\frac{n+1}{\alpha+1}}^1 |B_t(n+1,\alpha-n) - B(n+1,\alpha-n)| \, t \, dt, \\ T_8(\alpha,n) &= \int_{\frac{n+1}{\alpha+1}}^1 |B_t(n+1,\alpha-n) - B(n+1,\alpha-n)| (1-t) \, dt, \end{aligned}$$

with $n = 0,1,2...$ and $\alpha \in (n,n+1]$.

Proof. Using Lemma 4.1, power mean inequality and the convexity of $|f'|^q$, we have

$$\left| \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha \Gamma(\alpha-n)} \left[I_\alpha^a f(b) + {}^bI_\alpha f(a) \right] - \frac{f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right) + f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)}{2} \right|$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \, |f'(ta+(1-t)b)| \, dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array} \right| |f'(ta+(1-t)b)| \, dt \\ +\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \, |f'(tb+(1-t)a)| \, dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{c} B(n+1,\alpha-n)- \\ B_t(n+1,\alpha-n) \end{array} \right| |f'(tb+(1-t)a)| \, dt \end{array} \right]$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{l} \left(\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}} \\ \times\left(\begin{array}{l}\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \\ |f'(ta+(1-t)b)|^q\,dt\end{array}\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l} B_t(n+1,\alpha-n)- \\ B(n+1,\alpha-n)\end{array}\right| dt\right)^{1-\frac{1}{q}} \\ \times\left(\begin{array}{l}\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l}B_t(n+1,\alpha-n)-\\ B(n+1,\alpha-n)\end{array}\right| \\ |f'(ta+(1-t)b)|^q\,dt\end{array}\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}} \\ \times\left(\begin{array}{l}\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \\ |f'(tb+(1-t)a)|^q\,dt\end{array}\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l} B(n+1,\alpha-n) \\ -B_t(n+1,\alpha-n)\end{array}\right| dt\right)^{1-\frac{1}{q}} \\ \times\left(\begin{array}{l}\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l}B(n+1,\alpha-n)\\ -B_t(n+1,\alpha-n)\end{array}\right| \\ |f'(tb+(1-t)a)|^q\,dt\end{array}\right)^{\frac{1}{q}} \end{array}\right]$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{l} \left(\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}} \\ \times\left(\begin{array}{l}\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \\ \left[\begin{array}{l} t\,|f'(a)|^q+ \\ (1-t)\,|f'(b)|^q\end{array}\right] dt\end{array}\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l} B_t(n+1,\alpha-n)- \\ B(n+1,\alpha-n)\end{array}\right| dt\right)^{1-\frac{1}{q}} \\ \times\left(\begin{array}{l}\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l}B_t(n+1,\alpha-n)\\ -B(n+1,\alpha-n)\end{array}\right| \\ \left[\begin{array}{l} t\,|f'(a)|^q+ \\ (1-t)\,|f'(b)|^q\end{array}\right] dt\end{array}\right)^{\frac{1}{q}} \end{array}\right.$$

$$\left.\begin{array}{l} +\left(\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}} \\ \times\left(\begin{array}{l}\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \\ \left[\begin{array}{l} t\,|f'(b)|^q \\ +(1-t)\,|f'(a)|^q\end{array}\right] dt\end{array}\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l} B(n+1,\alpha-n) \\ -B_t(n+1,\alpha-n)\end{array}\right| dt\right)^{1-\frac{1}{q}} \\ \times\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l}B(n+1,\alpha-n)\\ -B_t(n+1,\alpha-n)\end{array}\right| \left[\begin{array}{l} t\,|f'(b)|^q+ \\ (1-t)\,|f'(a)|^q\end{array}\right] dt\right)^{\frac{1}{q}} \end{array}\right]$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{l} \left(\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}} \\ \times\left(\begin{array}{l}|f'(a)|^q\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,t\,dt \\ +|f'(b)|^q\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,(1-t)\,dt\end{array}\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n)\end{array}\right| dt\right)^{1-\frac{1}{q}} \\ \times\left(\begin{array}{l}|f'(a)|^q\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l}B_t(n+1,\alpha-n)\\ -B(n+1,\alpha-n)\end{array}\right| t\,dt \\ +|f'(b)|^q\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l}B_t(n+1,\alpha-n)\\ -B(n+1,\alpha-n)\end{array}\right| (1-t)\,dt\end{array}\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}} \\ \times\left(\begin{array}{l}|f'(b)|^q\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,t\,dt \\ +|f'(a)|^q\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|\,(1-t)\,dt\end{array}\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l} B(n+1,\alpha-n) \\ -B_t(n+1,\alpha-n)\end{array}\right| dt\right)^{1-\frac{1}{q}} \\ \times\left(\begin{array}{l}|f'(b)|^q\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l}B(n+1,\alpha-n)\\ -B_t(n+1,\alpha-n)\end{array}\right| t\,dt \\ +|f'(a)|^q\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l}B(n+1,\alpha-n)\\ -B_t(n+1,\alpha-n)\end{array}\right| (1-t)\,dt\end{array}\right)^{\frac{1}{q}} \end{array}\right].$$

This completes the proof. □

Corollary 4.6. *In Theorem 4.5,*

1. *If one takes $\alpha = n+1$, one has [6, Theorem 8]*
2. *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$ one has [6, Corollary 3].*

Theorem 4.7. *Let $a,b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $|f'|^q$ is convex on $[a,b]$ for $q > 1$, then the following conformable fractional midpoint type inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\left[\ I^a_{\alpha}f(b)\ +\ {}^bI_{\alpha}f(a)\ \right] - \frac{f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)+f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)}{2}\right| \tag{4.4}$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{l} T_9^{\frac{1}{p}}(\alpha,n)\left(\begin{array}{l} \frac{(n+1)^2}{2(\alpha+1)^2}\left|f'(a)\right|^q + \\ \frac{2(\alpha+1)(n+1)-(n+1)^2}{2(\alpha+1)^2}\left|f'(b)\right|^q \end{array} \right)^{\frac{1}{q}} \\ +T_{10}^{\frac{1}{p}}(\alpha,n)\left(\begin{array}{l} \frac{(\alpha+1)^2-(n+1)^2}{2(\alpha+1)^2}\left|f'(a)\right|^q + \\ \frac{(\alpha+1)^2-2(\alpha+1)(n+1)+(n+1)^2}{2(\alpha+1)^2}\left|f'(b)\right|^q \end{array} \right)^{\frac{1}{q}} \\ +T_9^{\frac{1}{p}}(\alpha,n)\left(\begin{array}{l} \frac{(n+1)^2}{2(\alpha+1)^2}\left|f'(b)\right|^q + \\ \frac{2(\alpha+1)(n+1)-(n+1)^2}{2(\alpha+1)^2}\left|f'(a)\right|^q \end{array} \right)^{\frac{1}{q}} \\ +T_{10}^{\frac{1}{p}}(\alpha,n)\left(\begin{array}{l} \frac{(\alpha+1)^2-(n+1)^2}{2(\alpha+1)^2}\left|f'(b)\right|^q + \\ \frac{(\alpha+1)^2-2(\alpha+1)(n+1)+(n+1)^2}{2(\alpha+1)^2}\left|f'(a)\right|^q \end{array} \right)^{\frac{1}{q}} \end{array} \right],$$

where

$$\begin{aligned} T_9(\alpha,n) &= \int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right|^p dt, \\ T_{10}(\alpha,n) &= \int_{\frac{n+1}{\alpha+1}}^1 \left|B_t(n+1,\alpha-n)-B(n+1,\alpha-n)\right|^p dt, \end{aligned}$$

with $\frac{1}{p}+\frac{1}{q}=1$, $n=0,1,2...$ *and* $\alpha\in(n,n+1]$.

Proof. Using Lemma 4.1, Holder inequality and the convexity of $|f'|^q$, we have

$$\left| \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha \Gamma(\alpha-n)} \left[\ I_\alpha^a f(b) \ + \ {}^bI_\alpha f(a)\ \right] - \frac{f\left(\frac{(n+1)a+(\alpha-n)b}{\alpha+1}\right)+f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)}{2} \right|$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{l} \int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right| \left|f'(ta+(1-t)b)\right| dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{l} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array} \right| \left|f'(ta+(1-t)b)\right| dt \\ +\int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right| \left|f'(tb+(1-t)a)\right| dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{l} B(n+1,\alpha-n) \\ -B_t(n+1,\alpha-n) \end{array} \right| \left|f'(tb+(1-t)a)\right| dt \end{array} \right]$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{l} \left(\int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\int_0^{\frac{n+1}{\alpha+1}} \left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|B_t(n+1,\alpha-n)-B(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\int_0^{\frac{n+1}{\alpha+1}} \left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \end{array} \right]$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{c} \left(\int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\int_0^{\frac{n+1}{\alpha+1}} \left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right] dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|B_t(n+1,\alpha-n)-B(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\int_{\frac{n+1}{\alpha+1}}^1 \left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right] dt\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\int_0^{\frac{n+1}{\alpha+1}} \left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right] dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\int_{\frac{n+1}{\alpha+1}}^1 \left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]\right)^{\frac{1}{q}} \end{array} \right]$$

$$\leq \frac{b-a}{2B(n+1,\alpha-n)} \left[\begin{array}{l} \left(\int_0^{\frac{n+1}{\alpha+1}} \left|B_t(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\begin{array}{l} \frac{(n+1)^2}{2(\alpha+1)^2}\left|f'(a)\right|^q + \\ \frac{2(\alpha+1)(n+1)-(n+1)^2}{2(\alpha+1)^2}\left|f'(b)\right|^q \end{array} \right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left| \begin{array}{l} B_t(n+1,\alpha-n) \\ -B(n+1,\alpha-n) \end{array} \right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\begin{array}{l} \frac{(\alpha+1)^2-(n+1)^2}{2(\alpha+1)^2}\left|f'(a)\right|^q + \\ \frac{(\alpha+1)^2-2(\alpha+1)(n+1)+(n+1)^2}{2(\alpha+1)^2}\left|f'(b)\right|^q \end{array} \right)^{\frac{1}{q}} \end{array} \right]$$

$$\left[\begin{array}{l} +\left(\int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)|^p\,dt\right)^{\frac{1}{p}} \\ \times\left(\begin{array}{l} \frac{(n+1)^2}{2(\alpha+1)^2}|f'(b)|^q+ \\ \frac{2(\alpha+1)(n+1)-(n+1)^2}{2(\alpha+1)^2}|f'(a)|^q \end{array}\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1 \left|\begin{array}{l} B(n+1,\alpha-n) \\ -B_t(n+1,\alpha-n) \end{array}\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\begin{array}{l} \frac{(\alpha+1)^2-(n+1)^2}{2(\alpha+1)^2}|f'(b)|^q+ \\ \frac{(\alpha+1)^2-2(\alpha+1)(n+1)+(n+1)^2}{2(\alpha+1)^2}|f'(a)|^q \end{array}\right)^{\frac{1}{q}} \end{array}\right]$$

This completes the proof. □

Corollary 4.8. *In Theorem 4.7,*

1. *If one takes $\alpha = n+1$, one has [6, Theorem 8]*
2. *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$ one has [5, Theorem 2.3].*

5. Competing Interests

The authors declare that they have no competing interests.

6. Conclusion

We have created a new approach in this article. This proved that conformable fractional Hermite-Hadamard and conformable fractional Hermite-Hadamard-Fejér inequalities are only the result of HF inequality. We achieved the new conformable fractional midpoint type inequality. We correlated our results with different studies in the literature. This method of different convex species.

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