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The b-chromatic number of some degree splitting graphs

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Abstract

A b-coloring of a graph G is a variant of proper coloring in which each color class contains a vertex that has a neighbor in all the other color classes. We investigate some results on b-coloring in the context of degree splitting graph of P_n, $B_{n,n}$, S_n and G_n.

Keywords: graph coloring, b-coloring, b-vertex, degree splitting graph.

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1 Introduction

In this paper we deal with finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. The notations and terminology here are used in the sense of Clark and Holton [1]. A *proper k-coloring* of a graph G is a function $c : V(G) \to \{1, 2, ..., k\}$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. The color class c_i is the subset of vertices of G that is assigned to color i. The chromatic number $\chi(G)$ is the minimum number k for which G admits proper k-coloring.

A proper k-coloring c of a graph G is a *b-coloring* if for every color class c_i, there is a vertex with color i which has at least one neighbor in every other color classes. Such vertex is called a *b-vertex*. The *b-chromatic number* of a graph G, denoted by $\varphi(G)$, is the largest integer k for which G admits a b-coloring.

The concept of b-coloring was introduced by Irving and Manlove [2]. If G has a b-coloring by k colors for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$ then G is called *b-continuous*. The *b-spectrum* $S_b(G)$ of a graph G is the set of integers k such that G has a b-coloring by k colors.

The concept of b-coloring is explored by many researchers. The bounds for the b-chromatic number of a graph is investigated by Kouider and Mahéo [3] while b-chromatic number for Peterson graph and power of a cycle is discussed by Chandrakumar and Nicholas [6]. The b-continuity of chordal graphs is discussed by Faik [7].

Definition 1.1. *([2], [4])The m-degree of a graph G, denoted by $m(G)$, is the largest integer m such that G has m vertices of degree at least $m - 1$.*

Proposition 1.2. *([1]) For any graph G, $\chi(G) \geq 3$ if and only if G has an odd cycle.*

Proposition 1.3. *([2]) If G admits a b-coloring with m colors, then G must have at least m vertices with degree at least $m - 1$.*

Proposition 1.4. *([3]) $\chi(G) \leq \varphi(G) \leq m(G)$.*

It is obvious that if $\chi(G) = k$, then every coloring of a graph G by k colors is a b-coloring of G.

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Proposition 1.5. *([5]) If P_n, C_n, K_n, $K_{m,n}$ and $W_n : C_n + K_1$ are respectively path, cycle, complete graph, complete bipartite graph and wheel graph, then*

1. $\chi(C_{2n}) = 2$, $\chi(C_{2n+1}) = 3$.
2. $\chi(W_n) = 3$, *if n is odd and* $\chi(W_n) = 4$, *if n is even.*
3. $\chi(K_{m,n}) = 2$.
4. $\varphi(P_n) = 2$, *if* $1 < n < 5$ *and* $\varphi(P_n) = 3$, *if* $n \geq 5$.
5. $\varphi(C_n) = 2$, *if* $n = 4$ *and* $\varphi(C_n) = 3$, *if* $n \neq 4$.
6. $\varphi(W_n) = 3$, *if* $n = 4$ *and* $\varphi(W_n) = 4$, *if* $n \neq 4$.
7. $\chi(K_n) = \varphi(K_n) = n$.

2 Main Results

Definition 2.1. *Let $G = (V(G), E(G))$ be a graph with $V(G) = S_1 \cup S_2 \cup .... \cup S_t \cup T$ where each S_i is a set of all vertices of the same degree with at least two elements and $T = V(G) \setminus \bigcup_{i=1}^{t} S_i$. The degree splitting graph of G, denoted by $DS(G)$, is obtained from G by adding vertices $w_1, w_2, ..., w_t$ and joining w_i to each vertex of S_i for $1 \leq i \leq t$.*

Lemma 2.2. $\chi(DS(P_n)) = \begin{cases} 2, & n = 3 \\ 3, & n \neq 3. \end{cases}$

Proof. The path P_n has two pendant vertices and the remaining $n-1$ vertices are of degree two. Thus $V(P_n) = \{v_i\,; 1 \leq i \leq n\} = S_1 \cup S_2$ where $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_i\,; 2 \leq i \leq n-1\}$. For obtaining $DS(P_n)$ from P_n, add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus $V(DS(P_n)) = V(P_n) \cup \{w_1, w_2\}$ and $E(DS(P_n)) = E(P_n) \cup \{w_1 v_i$ where $v_i \in S_1; i = 1, n\} \cup \{w_2 v_j$ where $v_j \in S_2; 2 \leq j \leq n-1\}$. $|V(DS(P_n))| = n+2$ and $|E(DS(P_n))| = 2n-1$.

When $n = 3$, the graph $DS(P_3)$ is isomorphic to C_4. Then by Proposition 1.5, $\chi(DS(P_3)) = 2$. But when $n \neq 3$, $DS(P_n)$ contains a cycle C_3. Then by Proposition 1.2, $\chi(DS(P_n)) \geq 3$. If we assign the proper coloring as $c(w_1) = c(w_2) = 1$, $c(v_{2k+1}) = 2$, $c(v_{2k}) = 3$; $k \in$ N then $\chi(DS(P_n)) = 3$. □

Theorem 2.3. $\varphi(DS(P_n)) = \begin{cases} 2, & n = 3 \\ 3, & n = 2, 4 \\ 4, & n \geq 5. \end{cases}$

Proof. The graphs $DS(P_2)$ and $DS(P_3)$ are isomorphic to C_3 and C_4 respectively. Then by Proposition 1.5, $\varphi(DS(P_2)) = 3$ and $\varphi(DS(P_3)) = 2$.

In the graph $DS(P_4)$ there are four vertices of degree 2. Then the m-degree, $m(DS(P_4))$ =3. Then by Proposition 1.4, $\varphi(DS(P_4)) \leq 3$. Moreover $DS(P_4)$ induces a path of length greater than four, $\varphi(DS(P_4)) \geq 3$. Hence $\varphi(DS(P_4)) = 3$.

For $n \geq 5$, the graph $DS(P_n)$ has at least four vertices of degree at least 3. Then the m-degree, $m(DS(P_n)) = 4$. Then by Proposition 1.4, $\varphi(DS(P_n)) \leq 4$. Moreover $DS(P_n)$ induces a path of length greater than four, $\varphi(DS(P_n)) \geq 3$. We suppose that $DS(P_n)$ has a b-coloring using four colors. By assigning the proper coloring as $c(w_1) = c(w_2) = 1$, $c(v_{3k-2}) = 2$, $c(v_{3k-1}) = 3$, $c(v_{3k}) = 4$; $k \in$ N then the vertices w_2, v_4, v_2 and v_3 are the b-vertices for the color classes 1, 2, 3 and 4 respectively. Thus $\varphi(DS(P_n)) = 4$. Hence the result. □

Definition 2.4. *The bistar $B_{n,n}$ is a graph obtained by joining the center(apex) vertices of two copies of $K_{1,n}$ by an edge.*

Lemma 2.5. *For all n, $\chi(DS(B_{n,n})) = 3$.*

Proof. In $B_{n,n}$, $V(B_{n,n}) = \{u, v, u_i, v_i;\ 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uu_i, vv_i;\ 1 \leq i \leq n\} \cup \{uv\}$. The graph bistar $B_{n,n}$ contains two types of vertices - pendant vertices and vertices of degree $n+1$. Thus $V(B_{n,n}) = S_1 \cup S_2$ where $S_1 = \{u_i, v_i;\ 1 \leq i \leq n\}$ and $S_2 = \{u, v\}$. For obtaining $DS(B_{n,n})$ from $B_{n,n}$, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus $V(DS(B_{n,n})) = V(B_{n,n}) \cup \{w_1, w_2\}$ and $E(DS(B_{n,n})) = E(B_{n,n}) \cup \{u_iw_1, v_iw_1, uw_2, vw_2\}$. Hence $|V(DS(B_{n,n}))| = 2n+4$ and $|E(DS(B_{n,n}))| = 4n+3$.

As the graph $DS(B_{n,n})$ contains a C_3, $\chi(DS(B_{n,n})) \geq 3$. If we assign the proper coloring as $c(w_2) = 1$, $c(u) = 2$, $c(v) = 3$, $c(w_1) = 2$, $c(u_i) = c(v_i) = 1$, for $i = 1, 2, .., n$, then $\chi(DS(B_{n,n})) = 3$ for all n. □

Theorem 2.6. *For all n, $\varphi(DS(B_{n,n})) = 3$.*

Proof. By Lemma 2.5, $\varphi(DS(B_{n,n})) \geq \chi(DS(B_{n,n})) = 3$. The graph $DS(B_{n,n})$ has at least three vertices of degree at least two. Then $m(DS(B_{n,n})) = 3$ and hence by Proposition 1.4, $\varphi(DS(B_{n,n})) \leq 3$. Thus $\varphi(DS(B_{n,n})) = 3$ for all n. □

Definition 2.7. *A shell S_n is the graph obtained by taking $n-3$ concurrent chords in cycle C_n. That is, $S_n = P_{n-1} + K_1$.*

Lemma 2.8. $\chi(DS(S_n)) = \begin{cases} 4, & n = 3 \\ 3, & n \neq 3. \end{cases}$

Proof. In the shell graph S_n, $V(S_n) = \{u, v_1, v_2, ..., v_{n-1}\}$ where u is the apex vertex and $E(S_n) = \{uv_i\ ;\ 1 \leq i \leq n-1\} \cup \{v_iv_{i+1}; 1 \leq i \leq n-2\}$. Clearly $|V(S_n)| = n$ and $|E(S_n)| = 2n-3$.
There are three types of vertices

(i) vertices of degree 2,

(ii) vertices of degree 3,

(iii) a vertex of degree $n-1$.

Thus $V(S_n) = \{u, v_1, v_2, ..., v_{n-1}\} = S_1 \cup S_2 \cup T$ where $S_1 = \{v_1, v_{n-1}\}$, $S_2 = \{v_i\ ;\ 2 \leq i \leq n-2\}$ and $T = \{u\} = V(S_n) \setminus \bigcup_{i=1}^{2} S_i$. For obtaining $DS(S_n)$ from S_n, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus $V(DS(S_n)) = V(S_n) \cup \{w_1, w_2\}$ and $E(DS(S_n)) = E(S_n) \cup \{v_1w_1, v_{n-1}w_1\} \cup \{v_iw_2; 2 \leq i \leq n-2\}$.

When $n = 3$, the graph $DS(S_3)$ is isomorphic to K_4. Then by Proposition 1.5, $\chi(DS(S_3)) = 4$. But when $n \neq 3$, $DS(S_n)$ contains a C_3, then by Proposition 1.2, $\chi(DS(S_n)) \geq 3$. If we assign the colors as $c(w_1) = c(w_2) = c(u) = 1$, $c(v_{2k+1}) = 2$, $c(v_{2k}) = 3$; $k \in \mathrm{N}$, then $\chi(DS(S_n)) = 3$. □

Theorem 2.9. $\varphi(DS(S_n)) = \begin{cases} 3, & n = 4 \\ 4, & n \neq 4. \end{cases}$

Proof. When $n = 3$, the graph $DS(S_3)$ is isomorphic to K_4, by Proposition 1.5, $\varphi(DS(S_3)) = 4$.

When $n = 4$, the graph $DS(S_4)$ has four vertices of degree at least three. Then $m(DS(S_4)) = 4$. Then by Proposition 1.4, $\varphi(DS(S_4)) \leq 4$. Suppose that $DS(S_4)$ does have a b-chromatic 4-coloring. By assigning the proper coloring as $c(u) = 1$, $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 4$ which in turn forces to assign $c(w_1)$ is either by the color 1 or 3 and $c(w_2)$ is either by the color 2 or 4. This proper coloring gives the b-vertices for the color classes 1 and 3 but not for 2 and 4. Similarly all other proper coloring using 4 colors will generate b-vertices at most for two color classes only. Hence $\varphi(DS(S_4)) \neq 4$. Thus $\varphi(DS(S_4)) \leq 3$. Also by Lemma 2.8, $\varphi(DS(S_4)) \geq 3$. Hence $\varphi(DS(S_4)) = 3$.

When $n = 5$ and 6, the graph $DS(S_n)$ has the m-degree four. Thus $\varphi(DS(S_n)) \leq 4$. Suppose that $DS(S_n)$ does have a b-chromatic 4-coloring. By assigning the proper coloring as $c(u) = 1$, $c(v_1) = c(v_4) = 2$, $c(v_2) = 3$, $c(v_3) = c(w_1) = 4$ which gives the b-vertices u, v_1, v_2, v_3 for the color classes 1, 2, 3, and 4 respectively. Thus $\varphi(DS(S_n)) = 4$.

When $n \geq 7$, the graph $DS(S_n)$ has the m-degree five. Thus $\varphi(DS(S_n)) \leq 5$. Suppose that $DS(S_n)$ does have a b-chromatic 5-coloring. By assigning the proper coloring as $c(v_2) = 1$, $c(v_1) = 2$, $c(u) = 4$, $c(v_3) = 5$, $c(w_2) = 3$, $c(v_4) = 2$ which in turn forces to assign $c(v_5) = 1$. This proper coloring gives the b-vertices for the color classes 1, 2 and 5 but not for 3 and 4. Similarly all other proper coloring with 5 colors will generate b-vertices at most for three color classes only. Hence $\varphi(DS(S_n)) \neq 5$. Thus $\varphi(DS(S_n)) \leq 4$. If we assign the colors as $c(w_1) = c(w_2) = 1$, $c(v_{3k-2}) = 2$, $c(v_{3k-1}) = 3$, $c(v_{3k}) = 4$; $k \in \mathrm{N}$ gives the b-vertices u, v_2, v_3, v_4 for the color classes 1, 3, 4 and 2 respectively. Thus $\varphi(DS(S_n)) = 4$. □

Definition 2.10. *The gear Graph, G_n, is obtained from the wheel by subdividing each of its rim edge.*

That is, let $W_n = C_n + K_1$ be the wheel graph with apex vertex v and the rim vertices $v_1, v_2, ..., v_n$. To obtain the gear graph G_n, subdivide each rim edge of wheel W_n by the vertices $u_1, u_2, ..., u_n$ where each u_i subdivides the edge v_iv_{i+1} for $i = 1, 2, .., n-1$ and u_n subdivides the edge v_1v_n. Then $|V(G_n| = 2n + 1$ and $|E(G_n| = 3n$.

Lemma 2.11. $\chi(DS(G_n)) = \begin{cases} 3, & n = 3 \\ 2, & n \neq 3. \end{cases}$

Proof. The gear graph G_n has three types of vertices

(i) vertices of degree 2,

(ii) vertices of degree 3

(iii) a vertex of degree n.

Thus $V(G_n) = \{v_i, u_i, v\} = S_1 \cup S_2 \cup T$ where $S_1 = \{v_i\}$, $S_2 = \{u_i\}$, $T = \{v\} = V(G_n) \setminus \bigcup_{i=1}^{2} S_i$. For obtaining $DS(G_n)$ from G_n, we add two vertices w_1 and w_2 corresponding to S_1 and S_2 respectively. Thus $V(DS(G_n)) = V(G_n) \cup \{w_1, w_2\}$ and $E(DS(G_n)) = E(G_n) \cup \{v_iw_1, u_iw_2\}$.

When $n = 3$, $DS(G_3)$ contains a K_3 (formed by the vertices v, w_1 and v_2), $\chi(DS(G_3)) \geq 3$. If we assign the colors as $c(v) = 1$, $c(w_1) = 2$, $c(w_2) = 3$, $c(u_i) = 2$, $c(v_i) = 3$ for $i = 1, 2, ..., n$ gives the proper coloring using 3 colors. Thus $\chi(DS(G_3)) = 3$. But when $n \neq 3$, $DS(G_n)$ contains no odd cycles and it is a bipartite graph. Hence by Proposition 1.5, $\chi(DS(G_n)) = 2$. □

Theorem 2.12. $\varphi(DS(G_n)) = \begin{cases} 5, & n = 3 \\ 4, & n \neq 3. \end{cases}$

Proof. When $n = 3$, the graph $DS(G_3)$ contains five vertices of degree 4. Consequently $m(DS(G_3) = 5$. Then by Proposition 1.4, $\varphi(DS(G_3)) \leq 5$. Suppose that $DS(G_3)$ does have a b-chromatic 5-coloring. By assigning the proper coloring as $c(u_1) = 1$, $c(u_2) = 3$, $c(u_3) = 2$, $c(v_1) = 3$, $c(v_2) = 2$, $c(v_3) = 1$, $c(v) = 4$, $c(w_2) = 4$, $c(w_1) = 5$ then the vertices v_3, v_2, v_1, v, and w_1 are the b-vertices for the color classes 1, 2, 3, 4 and 5 respectively. Thus $\varphi(DS(G_3)) = 5$.

When $n \neq 3$, the graph $DS(G_n)$ contains at least five vertices of degree 4. Then $m(DS(G_n) = 5$. Then by Proposition 1.4, $\varphi(DS(G_n)) \leq 5$. Suppose that $DS(G_n)$ does have a b-chromatic 5-coloring. By assigning the proper coloring as $c(v) = 1$, $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 4$, $c(v_4) = 5$ gives the b- vertex v for the color class 1. Again assume that $c(u_1) = 4$ and $c(u_n) = 3$ which in turn forces to assign $c(w_1) = 5$ which is not possible as the adjacent vertices w_1 and v_4 will receive the same color. Thus v_1 is not a b- vertex for the color class 2. Similarly we can prove that no v_i's are b-vertices when five colors are used for b-coloring. Hence $\varphi(DS(G_n)) \neq 5$. But if we assign the colors as $c(v) = 1$, $c(v_{3k-2}) = 2$, $c(v_{3k-1}) = 3$, $c(v_{3k}) = 4$; $k \in \mathrm{N}$ which gives the b- vertices v, v_1, v_2 and v_3 for the color classes 1, 2, 3 and 4 respectively. Thus $\varphi(DS(G_n)) = 4$. Hence the result.

We have the following obvious result stating the b-spectrum of $DS(G_n)$ as any proper coloring with $\chi(G)$ colors is a b-coloring. □

Corollary 2.13. $S_b(DS(G_n)) = \begin{cases} \{3,4,5\}, & n = 3 \\ \{2,3,4\}, & n \neq 3 \end{cases}$ *and $DS(G_n)$ is b-continuous.*

Proof. When $n = 3$, by assigning the colors as $c(v) = 1$, $c(v_1) = 2$, $c(v_2) = 3$, $c(v_3) = 4$, $c(w_1) = c(w_2) = 4$ and $c(u_i) = 1$ for $i = 1, 2$ and 3, the graph $DS(G_3)$ has the b-chromatic 4-coloring. But when $n \neq 3$, by assigning the colors as $c(v) = c(w_1) = c(w_2) = 1$, $c(v_i) = 2$, $c(u_i) = 3$ for $i = 1, 2, .., n$, $DS(G_n)$ has the b-chromatic 3-coloring. Thus by Lemma 2.11 and Theorem 2.12, $DS(G_n)$ is b-continuous and the b-spectrum $S_b(DS(G_n)) = \begin{cases} \{3,4,5\}, & n = 3 \\ \{2,3,4\}, & n \neq 3. \end{cases}$ □

3 Concluding Remarks

The study of b-coloring is important due to its applications in many real life problems like scheduling problem, channel assignment problem, routing networks etc. Here we investigate b-chromatic number and related parameters for the degree splitting graph of some graphs. We show that the degree splitting graph of G_n is b-continuous. The degree splitting graph of P_n, $B_{n,n}$ and S_n are obviously b-continuous as any proper coloring with $\chi(G)$ colors is a b-coloring.

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