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On a delay Volterra-Stieltjes quadratic integral equation

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Abstract

In this work, we study the existence of at least one and exactly one continuous solution $x \in C[0,T]$ for a delay quadratic integral equation of Volterra-Stieltjes type. The continuous dependent of the unique solution will be also proved. The delay Volterra quadratic integral equation of Chandrasekhar's type will be considered.

Keywords

Volterra-Stieltjes type, continuous solution, delay functional integral equation, continuous dependence.

AMS Subject Classification

74H10. 45G10, 47H30.

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Contents

1. Introduction

Integral equations of various types play an important role in many branches of functional analysis and in their applications in physics, economics and other fields. In particular, quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example problems in the theory of radioactive transfer, in the theory of neutron transport and the kinetic theory of gasses lead to quadratic integral equations.

Banas and et al.[3] studied a nonlinear functional integral equation of Volterra type on unbounded interval using the technique associated with measures of noncompactness.

In this paper, we study delay Volterra-Stieltjes quadratic integral equation, which includes many key integral and functional equations that arise in nonlinear analysis and its applications. We shall Schauder fixed point theorem instead of using the technique associated with measures of noncompactness.

Consider the delay quadratic integral equation

$$x(t) = a(t) + \int_0^{\varphi_1(t)} f_1(t,s,x(s))d_s g_1(t,s) \int_0^{\varphi_2(t)} f_2(t,s,x(s))d_s g_2(t,s), \quad t \in [0,T] \tag{1.1}$$

where $g_i : [0,T] \times [0,T] \to R$ are nondecreasing in the second argument and the symbol d_s indicates the integration with respect to s.

For the properties of Volterra-Stieltjes integral equations see [4]-[7].

The aim of this paper is to investigate the solvability of the delay Volterra-Stieltjes quadratic integral equation (1.1). The existence of at least one or exact one solution $x \in C[0,T]$ of the delay quadratic integral equation (1.1) will be proved. The continuous dependence of the unique solution $x \in C[0,T]$ of the delay functions $\varphi_i(t)$ and the functions $g_i(t,s)$ will be studied.

The delay Volterra quadratic integral equation of Chandrasekhar's type [1]

$$x(t) = a(t) + \int_0^{\varphi_1(t)} \frac{t}{t+s}\, k_1(t,s)x(s)ds$$
$$\int_0^{\varphi_2(t)} \frac{t}{t+s}\, k_2(t,s)x(s)ds, \ \ t \in [0,T] \tag{1.2}$$

will be given as example.

2. Existence of at least one solution

Consider the quadratic integral equation (1.1) under the following assumptions:

(i) $\varphi_i : [0,T] \to [0,T], \ i = 1,2, \ \varphi_i(t) \le t$ are continuous and increasing

(ii) $a : [0,T] \to [0,T]$ is continuous and $\sup_t |a(t)| = a$

(iii) $f_i : [0,T] \times [0,T] \times R \to R$ are continuous and there exist the functions b_i and k_i such that

$$|f_i(t,s,x)| \le b_i(t,s) + k_i(t,s)|x|$$

where $b_i, \ k_i : [0,T] \times [0,T] \to R$ are continuous, $b = max\{b_1, b_2\}, \ k = max\{k_1, k_2\}$ and

$$b_i = \sup_{t,s}\{b_i(t,s)\}, \ k_i = \sup_{t,s}\{k_i(t,s)\}$$

(iv) The functions $g_i : [0,T] \times [0,T] \times R \to R, \ i = 1,2$ are continuous with

$$\mu = \max_i\{\sup_t |g_i(t,\varphi_i(t))| + \sup |g_i(t,0)|, \ on \ [0,T]\}$$

(v) For all $t_1, t_2 \in I$ such that $t_1 < t_2$ the functions $s \to g_i(t_2,s) - g_i(t_1,s)$ are nondecreasing on $[0,T]$

(vi) $g_i(0,s) = 0$ for any $s \in [0,T]$

(vii) $2bk\mu^2 < 1$

(viii) There exists a positive root r of the algebraic equation

$$k^2\mu^2 r^2 - (1 - 2bk\mu^2)r + (a + b^2\mu^2) = 0.$$

Remark 2.1. *(see [4]) Observe that the functions $s \to g_i(t,s)$ are nondecreasing on the interval $[0,T]$. In fact, for a fixed $t \in [0,T]$, for $s_1, s_2 \in [0,T]$, with $s_1 < s_2$, from assumptions (v), we obtain*

$$\begin{aligned} & g_i(t,s_2) - g_i(t,s_1) \\ = \ & [g_i(t,s_2) - g_i(0,s_2)] - [g_i(t,s_1) - g_i(0,s_1)] \ge 0. \end{aligned}$$

Lemma 2.2. *(see [4]) Assume that the function g satisfies assumption (vi). Then for arbitrary $s_1, s_2 \in I$, such that $s_1 < s_2$, the function $t \to g(t,s_2) - g(t,s_1)$ is nondecreasing on the interval I.*
In fact, for a fixed $t \in [0,T]$, take $t_1, t_2 \in [0,T]$ such that $t_1 < t_2$. Then, by assumption (v), we get

$$\begin{aligned} & [g(t_2,s_2) - g(t_2,s_1)] - [g(t_1,s_2) - g(t_1,s_1)] \\ = \ & [g(t_2,s_2) - g(t_1,s_2)] - [g(t_2,s_1) - g(t_1,s_1)] \ge 0. \end{aligned}$$

Let $C[0,T]$ be the Banach space of all continuous functions defined on $[0,T]$ with the norm

$$\|x\| = \sup_{t \in [0,T]} |x(t)|.$$

For the existence of at least one solution of the quadratic integral equation (1.1), we have the following theorem.

Theorem 2.3. *Let the assumptions (i)-(viii) be satisfied, then the functional integral equation (1.1) have at least one solution $x \in C[0,T]$.*

Proof. Define the operator

$$\begin{aligned} Fx(t) \ &= \ a(t) + \int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s) \\ & \quad \int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s) \end{aligned}$$

Define the set Q_r by

$$Q_r = \{x \in C[0,T] : ||x|| \le r\}$$

where r is a positive solution of the algebraic equation $a + (b+kr)^2\mu^2 = r$.
It is clear that the set Q is nonempty, bounded, closed, and convex set.
Now, let $x \in Q_r$, then

$$\begin{aligned} |Fx(t)| \ &= \ |a(t) + \int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s) \\ & \quad \int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s)| \\ &\le \ a + \int_0^{\varphi_1(t)} |f_1(t,s,x(s))|\, d_s g_1(t,s) \\ & \quad \int_0^{\varphi_2(t)} |f_2(t,s,x(s))|\, d_s g_2(t,s) \\ &\le \ a + \int_0^{\varphi_1(t)} (b_1(t,s) + k_1(t,s)||x||)\, d_s g_1(t,s)) \\ & \quad \int_0^{\varphi_2(t)} (b_2(t,s) + k_2(t,s)||x||)\, d_s(g_2(t,s)) \\ &\le \ a + \int_0^{\varphi_1(t)} (b + k||x||)\, d_s g_1(t,s)) \\ & \quad \int_0^{\varphi_2(t)} (b + k||x||)\, d_s g_2(t,s)) \\ &\le \ a + (b+kr)(g_1(t,\varphi_1(t)) - g_1(t,0))(b+kr) \\ & \quad (g_2(t,\varphi_2(t)) - g_2(t,0)) \\ &\le \ a + (b+kr)^2\mu^2 = r. \end{aligned}$$

This proves that the operator F maps Q_r into itself and the class of functions $\{Fx\}$ is uniformly bounded on Q_r .
Let $x \in Q_r$ and define

$$\theta(\delta) = \sup_{x \in Q_r} \{|f_i(t_2,s,x(s)) - f_i(t_1,s,x(s))| : \ t_1, t_2 \in [0,T],$$
$$t_1 < t_2, \ |t_2 - t_1| < \delta\}, \ i = 1,2,$$

then from the uniform continuity of the function $f_i : [0,T] \times [0,T] \times [-r,r] \to R$, and assumption (iii), we deduce that $\theta(\delta) \to 0$, as $\delta \to 0$ independent of $x \in Q_r$.
Now, let $t_2,\ t_1 \in [0,T]$, such that $|t_2 - t_1| < \delta$, then we have

$$
\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= |a(t_2) + \int_0^{\varphi_1(t_2)} f_1(t_2,s,x(s))\, d_s g_1(t_2,s) \\
&\quad \int_0^{\varphi_2(t_2)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s)) \\
&- a(t_1) - \int_0^{\varphi_1(t_1)} f_1(t_1,s,x(s))\, d_s g_1(t_1,s) \\
&\quad \int_0^{\varphi_2(t_1)} f_2(t_1,s,x(s))\, d_s g_2(t_1,s)| \\
&\leq |a(t_2) - a(t_1)| \\
&+ |\int_0^{\varphi_1(t_2)} f_1(t_2,s,x(s))\, d_s g_1(t_2,s) \\
&\quad \int_0^{\varphi_2(t_2)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s)) \\
&- \int_0^{\varphi_1(t_1)} f_1(t_1,s,x(s))\, d_s g_1(t_1,s) \\
&\quad \int_0^{\varphi_2(t_2)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s) \\
&+ \int_0^{\varphi_1(t_1)} f_1(t_1,s,x(s))\, d_s g_1(t_1,s) \\
&\quad \int_0^{\varphi_2(t_2)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s)) \\
&- \int_0^{\varphi_1(t_1)} f_1(t_1,s,x(s))\, d_s g_1(t_1,s) \\
&\quad \int_0^{\varphi_2(t_1)} f_2(t_1,s,x(s))\, d_s g_2(t_1,s)| \\
&= |a(t_2) - a(t_1)| \\
&+ |\int_0^{\varphi_2(t_2)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s)) \\
&\quad [\int_0^{\varphi_1(t_2)} f_1(t_2,s,x(s))\, d_s g_1(t_2,s) \\
&- \int_0^{\varphi_1(t_1)} f_1(t_1,s,x(s))\, d_s g_1(t_1,s)] \\
&+ \int_0^{\varphi_1(t_1)} f_1(t_1,s,x(s))\, d_s g_1(t_1,s)) \\
&\quad [\int_0^{\varphi_2(t_2)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s) \\
&- \int_0^{\varphi_2(t_1)} f_2(t_1,s,x(s))\, d_s g_2(t_1,s)]| \\
&\leq |a(t_2) - a(t_1)| \\
&+ |\int_0^{\varphi_2(t_2)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s)) \\
&\quad [\int_0^{\varphi_1(t_1)} f_1(t_2,s,x(s))\, d_s g_1(t_2,s) \\
&+ \int_{\varphi_1(t_1)}^{\varphi_1(t_2)} f_1(t_2,s,x(s))\, d_s g_1(t_2,s) \\
&- \int_0^{\varphi_1(t_1)} f_1(t_1,s,x(s))\, d_s g_1(t_1,s)] \\
&+ \int_0^{\varphi_2(t_1)} f_2(t_1,s,x(s))\, d_s g_2(t_1,s)) \\
&\quad [\int_0^{\varphi_2(t_1)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s) \\
&+ \int_{\varphi_2(t_1)}^{\varphi_2(t_2)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s) \\
&- \int_0^{\varphi_2(t_1)} f_2(t_1,s,x(s))\, d_s g_2(t_1,s)]| \\
&\leq |a(t_2) - a(t_1)| \\
&+ |\int_0^{\varphi_2(t_2)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s)) \\
&\quad [\int_0^{\varphi_1(t_1)} f_1(t_2,s,x(s))\, d_s g_1(t_2,s) \\
&+ \int_{\varphi_1(t_1)}^{\varphi_1(t_2)} f_1(t_2,s,x(s))\, d_s g_1(t_2,s) \\
&- \int_0^{\varphi_1(t_1)} f_1(t_1,s,x(s))\, d_s g_1(t_1,s) \\
&+ \int_0^{\varphi_1(t_1)} f_1(t_2,s,x(s))\, d_s g_1(t_1,s) \\
&- \int_0^{\varphi_1(t_1)} f_1(t_2,s,x(s))\, d_s g_1(t_1,s)] \\
&+ \int_0^{\varphi_2(t_1)} f_2(t_1,s,x(s))\, d_s g_2(t_1,s)) \\
&\quad [\int_0^{\varphi_2(t_1)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s) \\
&+ \int_{\varphi_2(t_1)}^{\varphi_2(t_2)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s) \\
&- \int_0^{\varphi_2(t_1)} f_2(t_1,s,x(s))\, d_s g_2(t_1,s) \\
&+ \int_0^{\varphi_2(t_1)} f_2(t_1,s,x(s))\, d_s g_2(t_2,s) \\
&- \int_0^{\varphi_2(t_1)} f_2(t_1,s,x(s))\, d_s g_2(t_2,s)]| \\
&\leq |a(t_2) - a(t_1)| \\
&+ |\int_0^{\varphi_2(t_2)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s)) \\
&\quad [\int_{\varphi_1(t_1)}^{\varphi_1(t_2)} f_1(t_2,s,x(s))\, d_s g_1(t_2,s) \\
&+ \int_0^{\varphi_1(t_1)} f_1(t_2,s,x(s))\, d_s [g_1(t_2,s) - g_1(t_1,s)] \\
&+ \int_0^{\varphi_1(t_1)} [f_1(t_2,s,x(s)) - f_1(t_2,s,x(s))] d_s g_1(t_1,s)] \\
&+ \int_0^{\varphi_2(t_1)} f_2(t_1,s,x(s))\, d_s g_2(t_1,s)) \\
&\quad [\int_{\varphi_2(t_1)}^{\varphi_2(t_2)} f_2(t_2,s,x(s))\, d_s g_2(t_2,s)
\end{aligned}
$$

$$\begin{aligned}
+ &\quad \int_0^{\varphi_2(t_1)} [f_2(t_2,s,x(s)) - f_2(t_1,s,x(s))]\, d_s g_2(t_2,s) \\
+ &\quad \int_0^{\varphi_2(t_1)} f_2(t_1,s,x(s))\, d_s[g_2(t_2,s) - g_2(t_1,s)]]| \\
\leq &\quad |a(t_2) - a(t_1)| \\
+ &\quad (b+kr)[g_2(t_2,\varphi_2(t_2)) - g_2(t_2,0)] \\
&\quad [(b+kr)([g_1(t_2,\varphi_1(t_2)) - g_1(t_2,\varphi_1(t_1))] \\
+ &\quad (b+kr)[g_1(t_2,\varphi_1(t_1)) - g_1(t_1,\varphi_1(t_1))] \\
- &\quad [g_1(t_2,0) - g_1(t_1,0)]) \\
+ &\quad \theta(\varepsilon)[g_1(t_1,\varphi_1(t_1)) - g_1(t_1,0)] \\
+ &\quad (b+kr)[g_2(t_1,\varphi_2(t_1)) - g_2(t_1,0))] \\
&\quad [(b+kr)[g_2(t_2,\varphi_2(t_2)) - g_2(t_2,\varphi_2(t_1)] \\
+ &\quad \theta(\varepsilon)([g_2(t_2,\varphi_2(t_1)) - g_2(t_2,0))] \\
+ &\quad (b+kr)[g_2(t_2,\varphi_2(t_1)) - g_2(t_1,\varphi_2(t_1)) \\
- &\quad [g_2(t_2,0) - g_2(t_1,0)]]
\end{aligned}$$

The above inequality means that the class of functions $\{Fx\}$ is equicontinuous on Q_r.
Then from Arzela-Ascoli theorem (see [9]) the operator F is compact on Q_r.
Let $x_n \subset Q_r$, such that $x_n \to x_0$ in Q_r, then

$$\begin{aligned}
Fx_n(t) &= a(t) + \int_0^{\varphi_1(t)} f_1(t,s,x_n(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x_n(s))\, d_s g_2(t,s) \\
\lim_{n\to\infty} Fx_n(t) &= a(t) + \lim_{n\to\infty}\Big(\int_0^{\varphi_1(t)} f_1(t,s,x_n(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x_n(s))\, d_s g_2(t,s)\Big)
\end{aligned}$$

Applying Lebesgue dominated convergence theorem(see [9]), then

$$\begin{aligned}
&= a(t) + \int_0^{\varphi_1(t)} f_1(t,s,\lim_{n\to\infty} x_n(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,\lim_{n\to\infty} x_n(s))\, d_s g_2(t,s)) \\
&= a(t) + \int_0^{\varphi_1(t)} f_1(t,s,x_0(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x_0(s))\, d_s g_2(t,s)) = Fx_0(t).
\end{aligned}$$

This means that the operator F is continuous on Q_r.
Since all conditions of Schauder fixed point theorem (see [8]) are satisfied, then the operator F has at least one fixed point $x \in Q_r$, and the quadratic integral equation (1.1) have at least one solution $x \in C[0,T]$.
This completes the proof. ■

3. Uniqueness of the solution

To study the uniqueness of the solution of the functional integral equation (1.1) we replace the assumption (iii) by:

$(iii)^*$ $f_i : I \times [0,T] \times R \to R$, i=1,2 are continuous and satisfies the Lipschitz condition,

$$|f_i(t,s,x) - f_i(t,s,y)| \leq k|x-y|.$$

From the assumption $(iii)^*$ we have

$$|f_i(t,s,x(s))| - |f_i(t,s,0)| \leq |f_i(t,s,x(s)) - f_i(t,s,0)| \leq k|x|$$

$$|f_i(t,s,x(s))| \leq k|x| + |f_i(t,s,0)|,$$

then

$$|f_i(t,s,x(s))| \leq k|x| + b,$$

where $b = \sup_t |f_i(t,s,0)|$.
For the uniqueness of the solution of the functional integral equation (1.1) we have the following theorem.

Theorem 3.1. *Let the assumptions (i)-(ii)-$(iii)^*$-(iv)-(v)-(vi)-(vii)-(vii) be satisfied, if $2k\mu^2(b+kr) < 1$, then the solution $x \in C[0,T]$ of the functional integral equation (1.1) is unique .*

Proof. Let x_1, x_2 be two solutions of the integral equation (1.1), then

$$\begin{aligned}
|x_1(t) - x_2(t)| &= |a(t) + \int_0^{\varphi_1(t)} f_1(s,x_1(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x_1(s))\, d_s g_2(t,s)) \\
&- a(t) - \int_0^{\varphi_1(t)} f_1(t,s,x_2(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x_2(s))\, d_s g_2(t,s))| \\
&= |\int_0^{\varphi_1(t)} f_1(t,s,x_1(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x_1(s))\, d_s g_2(t,s) \\
&- \int_0^{\varphi_1(t)} f_1(t,s,x_2(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x_2(s))\, d_s g_2(t,s))| \\
&= |\int_0^{\varphi_1(t)} f_1(t,s,x_1(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x_1(s))\, d_s g_2(t,s) \\
&- \int_0^{\varphi_1(t)} f_1(t,s,x_2(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x_2(s))\, d_s g_2(t,s))
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{\varphi_1(t)} f_1(t,s,x_1(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x_2(s))\, d_s g_2(t,s) \\
&- \int_0^{\varphi_1(t)} f_1(t,s,x_1(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x_2(s))\, d_s g_2(t,s)| \\
&\leq |\int_0^{\varphi_1(t)} f_1(t,s,x_1(s))\, d_s g_1(t,s) \\
&\quad [\int_0^{\varphi_2(t)} (f_2(t,s,x_1(s)) - f_2(t,s,x_2(s)))\, d_s g_2(t,s)] \\
&+ |\int_0^{\varphi_2(t)} f_2(t,s,x_2(s))\, d_s g_2(t,s) \\
&\quad [\int_0^{\varphi_1(t)} (f_1(t,s,x_1(s)) - f_1(t,s,x_2(s)))\, d_s g_2(t,s)] \\
&\leq (b+kr)\mu^2 k||x_1 - x_2|| + (b+kr)\mu^2 k||x_1 - x_2||,
\end{aligned}
$$

then

$$
||x_1 - x_2||(1 - 2k\mu^2(b+kr)) \quad \leq \quad 0.
$$

This means that $x_1 = x_2$ and the solution of the functional integral equation (1.1) is unique. ■

4. Continuous dependence of the solution

In this section we are going to study the continuous dependence of the unique solution $x \in C[0,T]$ of the functional integral equation (1.1) on the delay functions and the functions $g_i(t,s)$.

4.1 Continuous dependence on the delay functions $\varphi_i(t)$

Definition 4.1. *The solutions of the functional integral equation (1.1) is depends continuously on the delay functions $\varphi_i(t)$ if $\forall\ \varepsilon > 0$, $\exists \delta > 0$, such that if x, x^* are solutions of equation (1) related to functions φ_i and φ_i^*, respectively, then*

$$
|\varphi_i(t) - \varphi_i^*(t)| \leq \delta \ \Rightarrow\ \|x - x^*\| \leq \varepsilon.
$$

Theorem 4.2. *Let the assumptions of Theorem 3 be satisfied, then the solution of the functional integral equation (1.1) dependence continuously on the delay functions $\varphi_i(t)$.*

Proof. Let $\delta > 0$ be given such that $|\varphi_i(t) - \varphi_i^*(t)| \leq \delta$, $\forall t \ \geq\ 0$, then

$$
\begin{aligned}
|x(t) - x^*(t)| \ &\leq\ |a(t) + \int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s)) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s)) \\
&- a(t) - \int_0^{\varphi_1^*(t)} f_1(t,s,x^*(s))\, d_s g_1(t,s)) \\
&\quad \int_0^{\varphi_2^*(t)} f_2(t,s,x^*(s))\, d_s g_2(t,s))| \\
&\leq |\int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s) \\
&- \int_0^{\varphi_1^*(t)} f_1(t,s,x^*(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2^*(t)} f_2(t,s,x^*(s))\, d_s g_2(t,s)) \\
&+ \int_0^{\varphi_1^*(t)} f_1(t,s,x^*(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s) \\
&- \int_0^{\varphi_1^*(t)} f_1(t,s,x^*(s))\, d_s g_1(t,s) \\
&\quad \int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s)| \\
&\leq |\int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s) \\
&\quad [\int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s)) \\
&- \int_0^{\varphi_1^*(t)} f_1(t,s,x^*(s))\, d_s g_1(t,s))] \\
&+ \int_0^{\varphi_1^*(t)} f_1(t,s,x^*(s))\, d_s g_1(t,s) \\
&\quad [\int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s) \\
&- \int_0^{\varphi_2^*(t)} f_2(t,s,x^*(s))\, d_s g_2(t,s)]| \\
&\leq |\int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s) \\
&\quad [\int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s)) \\
&- \int_0^{\varphi_1(t)} f_1(t,s,x^*(s))\, d_s g_1(t,s)) \\
&+ \int_0^{\varphi_1(t)} f_1(t,s,x^*(s))\, d_s g_1(t,s)) \\
&- \int_0^{\varphi_1^*(t)} f_1(t,s,x^*(s))\, d_s g_1(t,s))] \\
&+ \int_0^{\varphi_1^*(t)} f_1(t,s,x^*(s))\, d_s g_1(t,s) \\
&\quad [\int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s)
\end{aligned}
$$

$$
\begin{aligned}
- & \int_0^{\varphi_2(t)} f_2(t,s,x^*(s))\, d_s g_2(t,s) \\
+ & \int_0^{\varphi_2(t)} f_2(t,s,x^*(s))\, d_s g_2(t,s) \\
- & \int_0^{\varphi_2^*(t)} f_2(t,s,x^*(s))\, d_s g_2(t,s)]| \\
\leq & \int_0^{\varphi_2(t)} |f_2(t,s,x(s))|\, d_s g_2(t,s) \\
& [\int_0^{\varphi_1(t)} k|x(s)-x^*(s)|\, d_s g_1(t,s)] \\
+ & \int_{\varphi_1^*(t)}^{\varphi_1(t)} |f_1(t,s,x^*(s))|\, d_s g_1(t,s)] \\
+ & \int_0^{\varphi_1^*(t)} |f_1(t,s,x^*(s))|\, d_s g_1(t,s) \\
& [\int_0^{\varphi_2(t)} k|x(s)-x^*(s)|\, d_s g_2(t,s))] \\
+ & \int_{\varphi_2^*(t)}^{\varphi_2(t)} |f_2(t,s,x^*(s))|\, d_s g_2(t,s)] \\
\leq & (b+kr)[g_2(t,\varphi_2(t))-g_2(t,0)] \\
& \{k\|x-x^*\|(g_1(t,\varphi_1(t))-g_1(t,0)) \\
+ & (b+kr)[g_1(t,\varphi_1(t))-g_1(t,\varphi_1^*(t))]\} \\
+ & (b+kr)[g_1(t,\varphi_1^*(t))-g_1(t,0)] \\
& \{k\|x-x^*\|(g_2(t,\varphi_2(t))-g_2(t,0)) \\
+ & (b+kr)[g_2(t,\varphi_2(t))-g_2(t,\varphi_2^*(t))]\} \\
\leq & (b+kr)\mu\{k\|x-x^*\|\mu+((b+kr) \\
& [g_1(t,\varphi_1^*(t))-g_1(t,0)]\} \\
+ & (b+kr)\mu\{k\|x-x^*\|\mu+(b+kr) \\
& [g_2(t,\varphi_2(t))-g_2(t,\varphi_2^*(t))]\}.
\end{aligned}
$$

This implies that

$$
\|x-x^*\| \leq \big((b+kr)([g_1(t,\varphi_1(t))-g_1(t,\varphi_1^*(t))] \\
+(b+kr)[g_2(t,\varphi_2(t))-g_2(t,\varphi_2^*(t))]\big)\big)/(1-2k(b+kr)\mu^2)
$$

But from the continuity of g_i we have

$$
|\varphi_i(t)-\varphi_i^*(t)| \leq \delta \Rightarrow |g_i(t,\varphi_i(t))-g_i(t,\varphi_i^*(t))| < \varepsilon_1,
$$

then

$$
\begin{aligned}
\leq & \frac{(b+kr)\varepsilon_1+(b+kr)\varepsilon_1}{1-2k(b+kr)\mu^2} \\
\leq & \frac{2(b+kr)\varepsilon_1}{1-2k(b+kr)\mu^2} = \varepsilon.
\end{aligned}
$$

This completes the proof. ■

4.2 Continuous dependence on the functions $g_i(t,s)$

Definition 4.3. *The solutions of the quadratic functional integral equation (1.1), is dependence continuously on the functions $g_i(t,s)$, i=1,2 if $\forall\ \varepsilon>0,\ \exists\ \delta>0$, such that if x,x^* are solutions of equation (1) related to functions $g(t,s)_i$ and $g(t,s)_i^*$, respectively,*

$$
|g_i(t,s)-g_i^*(t,s)| \leq \delta \;\Rightarrow\; \|x-x^*\| \leq \varepsilon.
$$

Theorem 4.4. *Let the assumptions of Theorem 3 be satisfied, then the solution of the delay quadratic functional integral equation (1.1) depends continuously on the functions $g_i(t,s)$.*

Proof.

$$
\begin{aligned}
|x(t)-x^*(t)| = & |a(t)+\int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s)) \\
& \int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s)) \\
- & a(t)-\int_0^{\varphi_1(t)} f_1(t,s,x^*(s))\, d_s g_1{}^*(t,s)) \\
& \int_0^{\varphi_2(t)} f_2(t,s,x^*(s))\, d_s g_2{}^*(t,s))| \\
\leq & |\int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s)) \\
& \int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s)) \\
- & \int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s)) \\
& \int_0^{\varphi_2(t)} f_2(t,s,x^*(s))\, d_s g_2{}^*(t,s))| \\
+ & \int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s)) \\
& \int_0^{\varphi_2(t)} f_2(t,s,x(s)^*)\, d_s g_2^*(t,s)) \\
- & \int_0^{\varphi_1(t)} f_1(t,s,x^*(s))\, d_s g_1{}^*(t,s)) \\
& \int_0^{\varphi_2(t)} f_2(t,s,x^*(s))\, d_s g_2{}^*(t,s))| \\
\leq & |\int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s)) \\
[& \int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s)) \\
- & \int_0^{\varphi_2(t)} f_2(t,s,x(s)^*)\, d_s g_2^*(t,s))] \\
+ & \int_0^{\varphi_2(t)} f_2(t,s,x(s)^*)\, d_s g_2^*(t,s)) \\
[& \int_0^{\varphi_1(t)} f_1(t,s,x^*(s))\, d_s g_1{}^*(t,s)) \\
- & \int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s))]| \\
\leq & |\int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s)) \\
[& \int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2(t,s)) \\
- & \int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2^*(t,s))
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s g_2^*(t,s)) \\
&- \int_0^{\varphi_2(t)} f_2(t,s,x^*(s))\, d_s g_2^*(t,s))] \\
&+ \int_0^{\varphi_2(t)} f_2(t,s,x^*(s))\, d_s g_2^*(t,s)) \\
&[\int_0^{\varphi_1(t)} f_1(t,s,x^*(s))\, d_s g_1{}^*(t,s)) \\
&- \int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1^*(t,s)) \\
&+ \int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1{}^*(t,s)) \\
&- \int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s))]|
\end{aligned}
$$

$$
\begin{aligned}
&\leq |\int_0^{\varphi_1(t)} f_1(t,s,x(s))\, d_s g_1(t,s)) \\
&\quad [\int_0^{\varphi_2(t)} f_2(t,s,x(s))\, d_s[g_2(t,s)-g_2^*(t,s)] \\
&+ \int_0^{\varphi_2(t)} [f_2(t,s,x(s))-f_2(t,s,x^*(s))]\, d_s g_2^*(t,s))] \\
&+ \int_0^{\varphi_2(t)} f_2(t,s,x^*(s))\, d_s g_2^*(t,s)) \\
&\quad [\int_0^{\varphi_1(t)} [f_1(t,s,x^*(s))-f_1(t,s,x(s))]\, d_s g_1{}^*(t,s)] \\
&+ \int_0^{\varphi_1(t)} f_1(t,s,x(s)) d_s[g_1^*(t,s)-g_1(t,s)]]| \\
&\leq (b+kr)[g_1(t,\varphi_1(t))-g_1(t,0)] \\
&\quad \{(b+kr)|[g_2(t,\varphi_2(t))-g_2^*(t,\varphi_2(t))] \\
&- [g_2(t,0)-g_2^*(t,0)]| \\
&+ k|x-x^*|[g_2^*(t,\varphi_1(t))-g_2^*(t,0)]\} \\
&+ (b+kr)[g_2^*(t,\varphi_1(t))-g_2^*(t,0)] \\
&\quad \{k|x-x^*|[g_1(t,\varphi_1(t))-g_1(t,0)] \\
&+ (b+kr)|[g_1^*(t,\varphi_1(t)))-g_1(t,\varphi_1(t))] \\
&- [g_1^*(t,0))-g_1(t,0))|\} \\
&\leq 2(b+kr)\mu[(b+kr)2\delta+k\|x-x^*\|\mu] \\
&+ (b+kr)\mu[k\|x-x^*\|\mu+(b+kr)2\delta] \\
&\leq 2(b+kr)^2\delta\mu+(b+kr)k\mu^2\|x-x^*\| \\
&+ (b+kr)k\mu^2\|x-x^*\|+(b+kr)^2 2\mu\delta,
\end{aligned}
$$

then

$$
\|x-x^*\|[1-2(b+kr)\mu^2 k] \quad \leq \quad 4(b+kr)^2\mu\delta
$$

and

$$
\|x-x^*\| \quad \leq \quad \frac{4(b+kr)^2\mu\delta}{1-2k(b+kr)\mu^2} = \varepsilon.
$$

This completes the proof. ■

5. Example

Let $f_i(t,s,x(s)) = k_i(t,s)|x(s)|$ and let the functions $g_i(t,s)$ be given by

$$
g_i(t,s) = \begin{cases} t \ln\frac{t+s}{t}, & t \in (0,T] \\ 0, & t = 0, \end{cases}
$$

then

$$
dg_i(t,s) = \frac{t}{t+s} ds
$$

and the assumptions (iii)-(vi) are satisfied (see [7]). Then our result can be applied to the delay Volterra quadratic integral equation of Chandrasekhar's type (1.2) and the unique solution of (1.2) depends continuously on the delay functions $\varphi_i(t)$.

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