Malaya Journal of Matematik, Vol. 9, No. 1, 1-8, 2021

https://doi.org/10.26637/MJM0901/0001

Haar wavelet method for solving the system of linear Volterra integral equations with variable coefficients

A. Padmanabha Reddy[1]*, S. H. Manjula[2], C. Sateesha[3]

Abstract
This paper deals solutions for system of linear Volterra integral equations with variable coefficients using the Haar wavelet method. The powerful properties of Haar wavelets are used to reduce the system of Volterra integral equations to a system of algebraic equations. Few problems are considered to examine the efficiency and applicability of the method. A collocation technique is utilized to find the approximate solution. Accuracy of the method is exemplified by the graph and table results.

Keywords
Haar wavelets; System of algebraic equations; Integral equations; Collocation method.

AMS Subject Classification
65T60, 15A06, 45A05, 65L60.

[1] *Department of Studies in Mathematics, V. S. K. University, Ballari, Karnataka, India.*
[2] *Department of Mathematics, KLE's J.T.College, Gadag, Karnataka, India.*
[3] *Department of Mathematics, R.T.E.Society's Rural Engineering College Hulkoti, Gadag, Karnataka, India.*
***Corresponding author**: [1] apreddy@vskub.ac.in, [2] manjulaharageri@gmail.com,[3] csatish9980@gmail.com
Article History: Received **11** October **2020**; Accepted **12** December **2020**

Contents

1. Introduction

Volterra integral equations arise in many applications of science and technology. They are population dynamics, spread of epidemics and semiconductor devices, potential theory, dirichlet problems, electrostatics, mathematical modeling of radioactive equilibrium, the particle transport problems of astrophysics and reactor theory and radioactive heat transfer problems etc. The system of linear Volterra integral equations was solved by many researchers with different numerical methods. Berenguer et al. [1] have solved with the aid of bi-orthogonal systems in Banach spaces, Niyazi et al. [2] have used Bessel polynomials method,, Roodaki et. al.[3] have employed delta basis functions(DBFs), Balakumar et al. [4] have applied the block-pulse functions method, Li-Hong et al. [5] have applied reproducing kernel method.

From the beginning of 1990's wavelet method have been applied for solving integral equations. Alfred Haar a Hungarian mathematician introduced the pair of piecewise constant functions called wavelets. Many types of wavelets exist but Haar wavelet is the simplest among them. Haar wavelets are compactly supported and orthogonal functions. Haar wavelets have many valued properties, so they can be used in the solution of differential equations, integral equations and integro-differential equations, signal and image processing. Haar wavelet collocation method is used for the solution of seventh and eighth order boundary value problems by Reddy et. al.[6, 7]. Haar wavelets are used to solve linear and nonlinear integral equations by Lepik [8], Maleknejad et al.[9], Babolian et. al.[10],. Farshid [11], Aziz et. al.[12]. By the inspiration of these articles we proposed this work.

General form of system of linear Volterra integral equations with variable coefficients are defined over $[a,b]$ as

$$\sum_{s=1}^{n} c_{q,s}(x)y_s(x) - \int_a^x \{\sum_{s=1}^{n} k_{q,s}(x,t)y_s(t)\}dt = f_q(x), \quad q = 1,2,...,n, \tag{1.1}$$

$c_{q,s}(x)$ and $f_q(x)$ are given functions of $x \in [a,b]$, $k_{q,s}(x,t)$ is kernal of integral equation, $y_s(x)$ is unknown function.

This paper is organized as follows. In section 2, Haar wavelets and their integrals is explained. In section 3, solving the problem by the proposed method is presented. In section 4, Experiment of method on few test problems is demonstrated. In section 5, we discussed the results. Finally in section 6, we gave the conclusion.

2. Haar wavelets and their integrals

Let us assume that integration domain $[a,b]$ is divided into 2^{J+1} subintervals of equal length $\Delta x = \frac{(b-a)}{2^{J+1}}$. Here $J \in N$ is maximal level of resolution. Next two parameters, translation and dilation are denoted as $j = 0,1,2,...J$ and $k = 0,1,2,...,2^j - 1$ respectively. Haar family is defined as

$$h_i(x) = \begin{cases} 1, & \text{for} \quad x \in [\alpha, \beta), \\ -1, & \text{for} \quad x \in [\beta, \gamma), \\ 0, & \quad \text{otherwise}, \end{cases} \tag{2.1}$$

where $i = m+k+1$, $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$, $\gamma = \frac{k+1}{m}$, where $m = 2^j$.
Eq. (2.1) is valid for $i > 2$. For $i = 1$ we have Haar scaling function which is also called father wavelet

$$h_1(x) = \begin{cases} 1, & \text{for} \quad x \in [a,b), \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

For $i = 2$ we have mother wavelet

$$h_2(x) = \begin{cases} 1, & \text{for} \quad x \in [a, \frac{a+b}{2}), \\ -1, & \text{for} \quad x \in [\frac{a+b}{2}, b), \\ 0, & \quad \text{otherwise}. \end{cases} \tag{2.3}$$

Integrals of Haar wavelets are as follows:

$$p_{1,i}(x) = \int_0^x h_i(x')dx' = \begin{cases} x - \alpha, & \text{for} \quad x \in [\alpha, \beta), \\ \beta - x, & \text{for} \quad x \in [\beta, \gamma), \\ 0, & \quad \text{otherwise}, \end{cases} \tag{2.4}$$

Haar wavelets with ν times of integration is given by

$$p_{\nu,i}(x) = \int_0^x p_{\nu-1}(x')dx', \qquad \nu = 2,3,...,n, n \in N. \tag{2.5}$$

Haar matrix H and integrated Haar matrices p_1 for $J = 2$ are given as

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$p_1 = \begin{bmatrix} 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.5625 & 0.6875 & 0.8125 & 0.9375 \\ 0.0625 & 0.1875 & 0.3125 & 0.4375 & 0.3125 & 0.1875 & 0.0625 & \\ 0.0625 & 0.1875 & 0.1875 & 0.0625 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0625 & 0.1875 & 0.1875 & 0.0625 \\ 0.0625 & 0.0625 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0625 & 0.0625 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0625 & 0.0625 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0625 & 0.0625 \end{bmatrix}$$

Any function which is having finite energy on $[a,b]$ and square integrable i.e. $f \in L^2[a,b]$ can be expressed as infinite sum of Haar wavelets:

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x), \tag{2.6}$$

where a_i are called Haar coefficients. if f is either piecewise constant or wish to approximate by piecewise constant during each subinterval then series can be terminated to finite terms as

$$f(x) = \sum_{i=1}^{2^{J+1}} a_i h_i(x). \tag{2.7}$$

3. Method of Solution

The application of Haar wavelet method to the system of linear Volterra integral equations with variable coefficients defined over $[0,1]$ had the following steps:

(1): General form of system of linear Volterra integral equations with variable coefficients Eq.(1.1) can be rewrite as

$$C(x)Y(x) - \int_0^x \{K(x,t)Y(t)\}dt = F(x) \tag{3.1}$$

where,

$$C(x) = \begin{bmatrix} c_{1,1}(x) & c_{1,2}(x) & \ldots & c_{1,n}(x) \\ c_{2,1}(x) & c_{2,2}(x) & \ldots & c_{2,n}(x) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ c_{n,1}(x) & c_{n,2}(x) & \ldots & c_{n,n}(x) \end{bmatrix}, Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ . \\ . \\ . \\ y_n(x) \end{bmatrix}$$

$$, K(x,t) = \begin{bmatrix} k_{1,1}(x,t) & k_{1,2}(x,t) & \ldots & k_{1,n}(x,t) \\ k_{2,1}(x,t) & k_{2,2}(x,t) & \ldots & k_{2,n}(x,t) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ k_{n,1}(x,t) & k_{n,2}(x,t) & \ldots & k_{n,n}(x,t) \end{bmatrix},$$

$$F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ . \\ . \\ . \\ f_n(x) \end{bmatrix}.$$

(2): Approximate the unknown functions $y_s(x)$ in terms of Haar functions

$$y_s(x) = \sum_{i=1}^{2^{J+1}} a_{s,i} h_i(x), \qquad s = 1,2,...,n, \qquad J \in N. \tag{3.2}$$

(3): Substitute Eq.(3.2) into (3.1) we get

$$\begin{bmatrix} c_{1,1}(x) & c_{1,2}(x) & \dots & c_{1,n}(x) \\ c_{2,1}(x) & c_{2,2}(x) & \dots & c_{2,n}(x) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ c_{n,1}(x) & c_{n,2}(x) & \dots & c_{n,n}(x) \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{2^{J+1}} a_{i,1}h_i(x) \\ \sum_{i=1}^{2^{J+1}} a_{i,2}h_i(x) \\ . \\ . \\ . \\ \sum_{i=1}^{2^{J+1}} a_{i,n}h_i(x) \end{bmatrix} -$$

$$\int_0^x \begin{bmatrix} k_{1,1}(x,t) & k_{1,2}(x,t) & \cdots & k_{1,n}(x,t) \\ k_{2,1}(x,t) & k_{2,2}(x,t) & \dots & k_{2,n}(x,t) \\ \vdots & \vdots & & \vdots \\ k_{n,1}(x,t) & k_{n,2}(x,t) & \dots & k_{n,n}(x,t) \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{2^{J+1}} a_{i,1}h_i(t) \\ \sum_{i=1}^{2^{J+1}} a_{i,2}h_i(t) \\ \vdots \sum_{i=1}^{2^{J+1}} a_{i,n}h_i(t) \end{bmatrix} dt = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}, \tag{3.3}$$

system (3.3) can be written as

$$\sum_{i=1}^{2^{J+1}} a_{s,i}[h_i(x)c_{q,s}(x) - \int_0^x k_{q,s,i}(x,t)h_i(t)dt] = f_q(x), \quad q,s = 1,2,...,n, \tag{3.4}$$

$$\sum_{i=1}^{2^{J+1}} a_{s,i}D_{q,s,i}(x) = f_q(x), \tag{3.5}$$

where $D_{q,s,i}(x) = h_i(x)c_{q,s}(x) - \int_0^x k_{q,s}(x,t)h_i(t)dt$.

(4): Discritize the equation (3.5) at collocation points $x_l = \frac{(\tilde{x}_{l-1}+\tilde{x}_l)}{2}$, $l = 1,2,...,2^{J+1}$, where $\tilde{x}_r$ is the grid point given by $\tilde{x}_r = a + r\triangle x$, $r = 0,1,...,2^{J+1}$, we get linear system

$$\sum_{i=1}^{2^{J+1}} a_{s,i}D_{q,s,i}(x_l) = f_q(x_l), l = 1,2,...,2^{J+1}, \quad q,s = 1,2,...,n. \tag{3.6}$$

Block matrix of the above system is:

$$\begin{bmatrix} D_{1,1} & D_{1,2} & \dots & D_{1,n} \\ D_{2,1} & D_{2,2} & \dots & D_{2,n} \\ . & . & & . \\ . & . & & . \\ D_{n,1}(x) & D_{n,2}(x) & \dots & D_{n,n}(x) \end{bmatrix}_{(n2^{J+1}\times n2^{J+1})} \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix}_{(n2^{J+1}\times 1)} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}_{(n2^{J+1}\times 1)} \tag{3.7}$$

$$D_{(n2^{J+1})\times(n2^{J+1})}a_{(n2^{J+1})\times 1} = F_{(n2^{J+1})\times 1}, \quad i = 1,2,...,2^{J+1}, \quad l = 1,2,...,2^{J+1}, \tag{3.8}$$

here $D_{q,s}(i,l) = D_{q,s,i}(x_l)$, $i = 1,2,...,2^{J+1}$, $l = 1,2,...,2^{J+1}$.

(5): Calculate the Haar wavelet coefficients $a_{s,i}$ and obtain the Haar solutions for unknown functions y_s for $s = 1,2,..,n$.

4. Numerical Examples

In this section, we applied the proposed method on six problems to check the applicability and accuracy of the method. We compared the obtained results with existing numerical methods in the literature. We computed the maximum absolute errors $\left(e_s^J = max|y_s(x_l)\text{app} - y_s(x_l)\text{exact}|, s = 1,2,...,n,\ l = 1,...,2^{J+1}\right)$ using MATLAB software.

Example 1 : Consider the system of linear Volterra integral equations with variable coefficients[5]

$$\begin{cases} 2xy_1(x) + xy_2(x) - \int_0^x 3ty_1(t)dt - \int_0^x (2x+1)y_2(t)dt \\ = f_1(x) \\ xy_1(x) - 2xy_2(x) - \int_0^x 2(x+t)y_1(t)dt - \int_0^x 2(x+t)ty_2(t)dt \\ = f_2(x), \end{cases} \tag{4.1}$$

with $F(x) = \left[2x, x - \frac{5x^3}{3} + \frac{7x^4}{6}\right]^T$.

Exact solution of this system is $Y(x) = [x+1, -x]^T$. In Figure1, the approximate and exact solution of Eq.(4.1) for $J = 4$ is compared. In Figure 2 and Figure 3 absolute errors obtained to $y_1(x)$ and $y_2(x)$ of Eq.(4.1) for $J = 3,5$ and 7 are presented. In Table 1 maximum absolute errors for various values of resolution are inserted.

Example 2 :Consider the system of Volterra integral equations with variable coefficients [2] is considered

$$\begin{cases} y_1(x) + xy_2(x) - \int_0^x x^2cos(t)y_1(t)dt + \int_0^x x^2sin(t) \\ y_2(t)dt = f_1(x) \\ -2xy_1(x) + y_2(x) - \int_0^x sin(x)cos(t)y_1(t)dt + \int_0^x sin(x) \\ sin(t)y_2(t)dt = f_2(x), \end{cases} \tag{4.2}$$

with $F(x) = [sin(x) + xcos(x), cos(x) - 2xsin(x)]^T$. Its analytic solution is $Y(x) = [sin(x), cos(x)]^T$. Approximate and Haar solution of Eq.(4.2) for $J = 5$ is shown in the Figure 4. In Table 2 maximum absolute errors for various values of resolution are tabulated.

Example 3 :Consider the system of linear Volterra integral equations[13,4]

$$\begin{aligned} y_1(x)-\int_0^x (t^2-x)y_1(t)dt-\int_0^x (t^2-x)y_2(t)dt &= f_1(x) \\ y_2(x)-\int_0^x ty_1(t)dt-\int_0^x ty_2(t)dt & \\ &= f_2(x), \end{aligned} \tag{4.3}$$

with $F(x)=\left[x+\frac{x^3}{2}+\frac{x^4}{12}-\frac{x^5}{5}, x^2-\frac{x^3}{3}-\frac{x^4}{4}\right]^T$. Exact solution of above problem is $Y(x)=[x,x^2]^T$. The comparison of approximate and exact solution of Eq.(4.3) for $J=3$ is represented in Figure 5. Absolute errors obtained to $y_1(x)$ and $y_2(x)$ of Eq. (4.3) for $J=3,5$ and 7 are demonstrated in Figure 6 and Figure 7. The maximum absolute errors for various values of resolution are inserted in Table 3 and compared to the block pulse function method (BPFM)[4].

Example 4 :Solve the system of linear Volterra integral equations[3]

$$\begin{aligned} y_1(x)+\int_0^x x^2ty_1(t)dt-\int_0^x xty_2(t)dt &= f_1(x) \\ y_2(x)-2\int_0^x (x+t)y_1(t)dt+3\int_0^x xty_2(t)dt &= f_2(x), \end{aligned} \tag{4.4}$$

with $F(x)=\left[x+\frac{x^5}{12}, x^2-\frac{5x^3}{3}+\frac{3x^5}{4}\right]^T$. The analytical solution of this problem is $Y(x)=\left[x,x^2\right]^T$. In Figure 8 the comparison of approximate and exact solution of Eq.(4.4) for $J=5$ is presented. Absolute errors obtained to $y_1(x)$ and $y_2(x)$ of Eq. (4.4) for $J=3,5$ and 7 are shown in Figure 9 and Figure 10. The maximum absolute errors for various values of resolution compared with the delta basis functions method is shown in Table 4.

Example 5 :Consider the system of Volterra integral equations with variable coefficients [13]

$$\begin{aligned} y_1(x)-\int_0^x y_1(t)dt-\int_0^x 2y_2(t)dt-\int_0^x y_3(t)dt &= f_1(x) \\ y_2(x)-\int_0^x ty_1(t)dt-\int_0^x (2x-2t)y_2(t)dt &= f_2(x) \\ y_3(x)-\int_0^x y_1(t)dt &= f_3(x), \end{aligned} \tag{4.5}$$

with $F(x)=\left[3x-cos(2x)-x^3, \frac{3sin(2x)}{2}-xe^x, x^2-e^x\right]^T$.
Exact solution of this problem is $y(x)=[e^{2x}, sin(2x), x^2-1]^T$. Approximate and exact solution of Eq. (4.5) for $J=3$ is compared in Figure 11. maximum absolute errors for various values of resolution are inserted in Table 5.

Example 6 :Consider the system of linear Volterra integral equations [13]

$$\begin{aligned} y_1(x) &- \int_0^x y_2(t)dt-\int_0^x y_3(t)dt = f_1(x) \\ y_2(x) &- \int_0^x (x-1)y_1(t)dt-\int_0^x ty_2(t)dt+\int_0^x xy_4(t)dt \\ = f_2(x) & \\ y_3(x) &- \int_0^x (x-t)y_1(t)dt+\int_0^x 3t^2y_4(t)dt = f_3(x) \\ y_4(x) &- \int_0^x (2x-3t)y_1(t)dt = f_4(x). \end{aligned} \tag{4.6}$$

with
$F(x)=\left[-x^3-x, \frac{x^5}{4}-\frac{x^4}{4}-\frac{x^3}{2}-3x^2-1, \frac{x^6}{2}-\frac{31x^3}{6}+2x^2+3, x^5-5\right]^T$.
The analytical solution of the system is $y(x)=[x,x^2-1,2x^2+3,x^3-5]^T$. In Figure 12 approximate and analytical solution of Eq.(4.6) for $J=4$ is compared to each other. In Table 6 maximum absolute errors obtained from Eq.(4.6) for various values of resolutions are shown.

5. Results and Discussions

We analyzed the obtained results in the form of figures and tables. In Examples $1-6$, the approximate and exact solutions obtained at collocation points for $J=4,5,3,5,3$ and 4are compared in Figures 1,4,5,8,11 and 12 respectively. This comparison pointed the accuracy of approximate solutions. Absolute errors with step size 0.1 to each unknown of Examples 1,3 and 4 for $J=3,5$ and 7 are obtained and shown in Figures $2-3$, Figures $6-7$, and Figures $9-10$. These figures showed the relation between absolute errors and resolution values i.e. absolute errors curves approaches to $X-$axis (where the absolute errors are zero) as the resolution value increases. The maximum absolute errors obtained by proposed method are inserted in tables with compared to BPFs and DBFs method. These comparisons have shown accuracy and superiority of the Haar wavelet method.

6. Conclusion

In this paper, we applied the Haar wavelet collocation method for solving system of Volterra integral equations with variable coefficients. Six examples are considered to test the applicability of the method. We proved the accuracy and efficiency of the present method comparing with other numerical methods such as BPEs and DBFs methods.

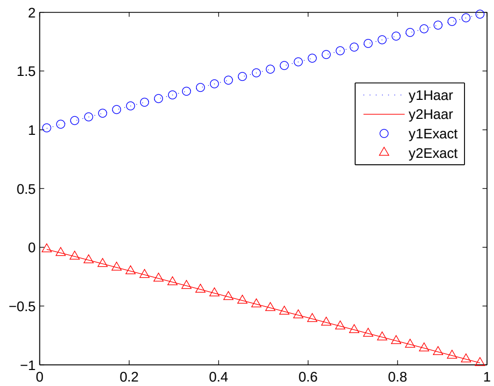

Figure 1. Comparison of absolute errors obtained to y_1 of Ex. 1 for J=3, 5 and 7.

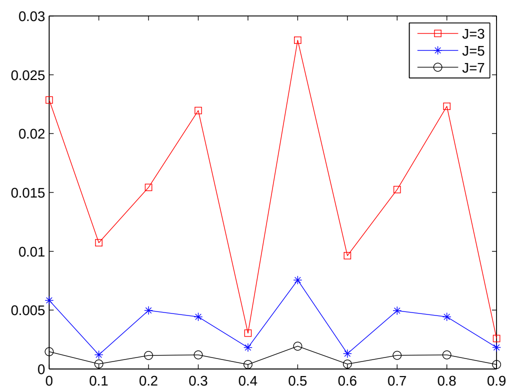

Figure 2. Comparison of absolute errors obtained to y_1 of Ex. 1 for J=3, 5 and 7.

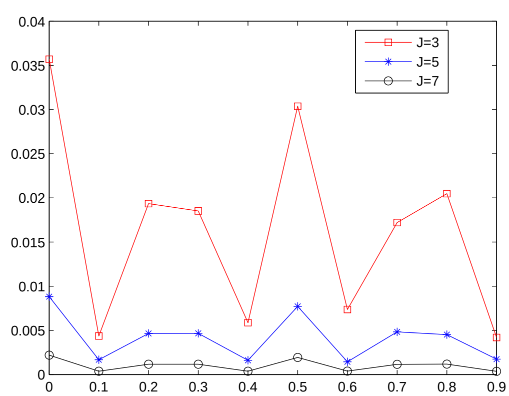

Figure 3. Comparison of absolute errors obtained to y_2 of Ex. 1 for J=3, 5 and 7.

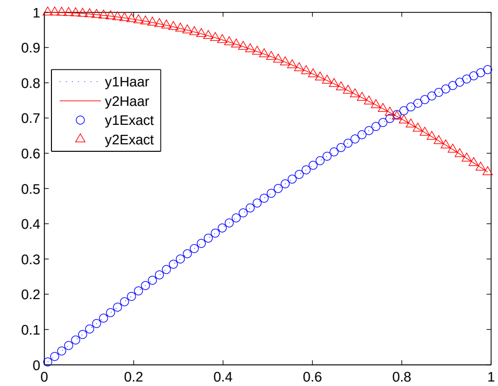

Figure 4. Comparison of approximate and exact solution of Ex. 2 for $J = 5$.

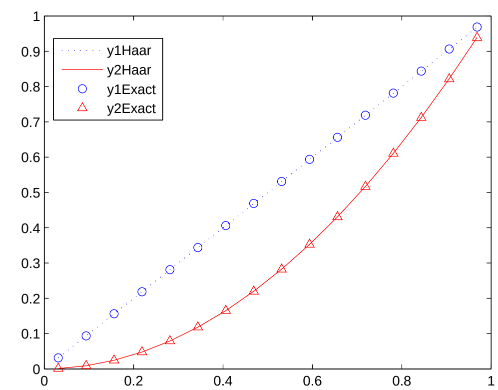

Figure 5. Comparison of approximate and exact solution of Ex. 3 for $J = 3$.

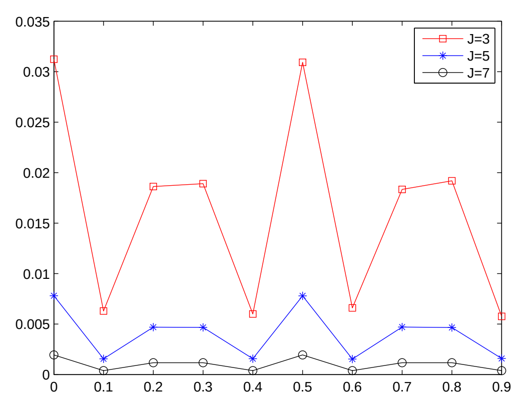

Figure 6. Comparison of absolute errors obtained to $y_1(x)$ of Ex. 3 for $J = 3, 5$ and 7.

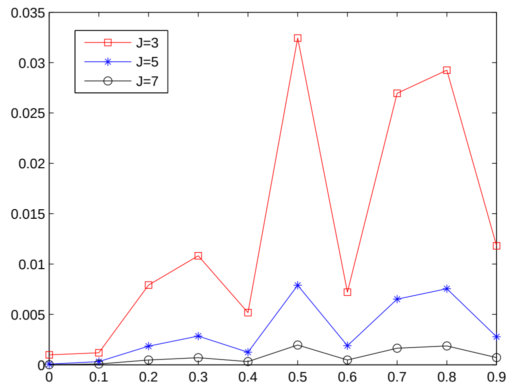

Figure 7. Comparison of absolute errors obtained to $y_2(x)$ of Ex. 3 for $J = 3,5$ and 7..

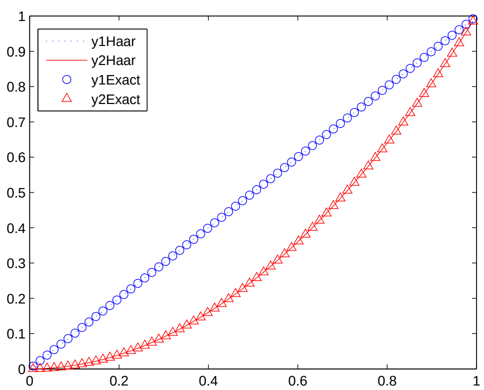

Figure 8. Comparison of approximate and exact solution of Ex. 4 for $J = 5$.

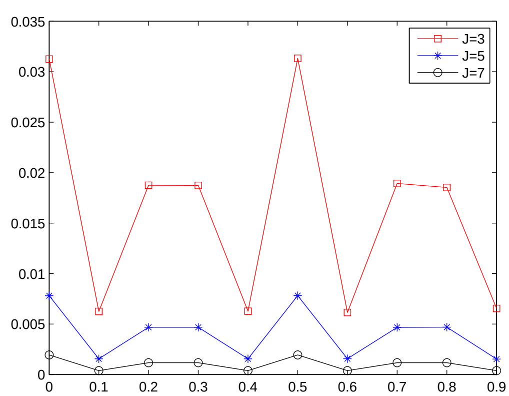

Figure 9. Comparison of absolute errors obtained to $y_1(x)$ of Ex. 4 for $J = 3,5$ and 7.

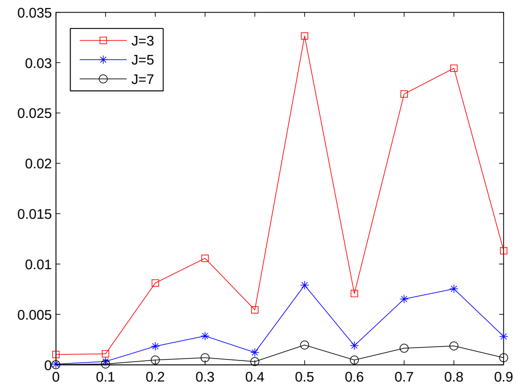

Figure 10. Comparison of absolute errors obtained to $y_2(x)$ of Ex. 4 for $J = 3,5$ and 7..

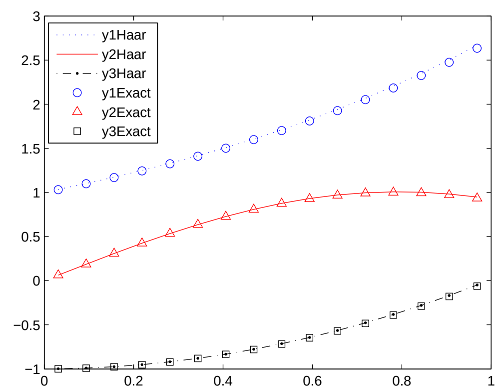

Figure 11. Comparison of approximate and exact solution of Ex. 5 for $J = 3$.

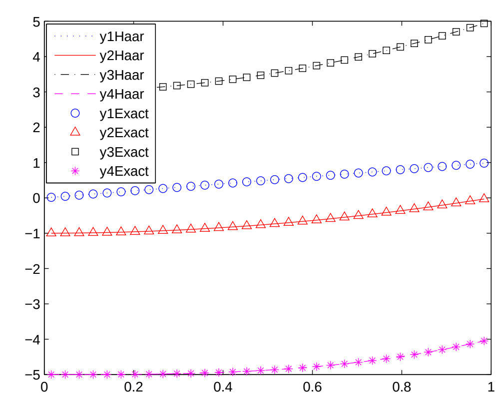

Figure 12. Comparison of approximate and exact solution of Ex. 6 for $J = 4$.

Table 1. Maximum absolute Errors Obtained by HWCM for Ex. 1

2^{J+1}	e_1^J	e_2^J
4	4.9920E-02	2.2735E-02
8	1.8063E-02	9.8495E-03
16	8.3921E-03	4.4424E-03
32	4.0473E-03	2.0902E-03
64	1.9879E-03	1.0111E-03

Table 2. Maximum absolute Errors Obtained by HWCM for Ex. 2.

2^{J+1}	e_1^J	e_2^J
4	2.3912E-04	5.3595E-03
8	6.9063E-05	1.3756E-03
16	1.8401E-05	3.4720E-04
32	4.7402E-06	8.7148E-05
64	1.2023E-06	2.1823E-05

Table 3. Maximum absolute Errors Obtained by HWCM for Ex. 3

HWCM			BPFM		[4]
2^{J+1}	e_1^J	e_2^J	M	e_1^J	e_2^J
32	1.2417E-04	1.5327E-04	32	1.5667E-02	1.6419E-02
64	3.1106E-05	3.8929E-05	64	7.8226E-03	7.9170E-03
128	7.7839E-06	9.8093E-06	128	3.9087E-03	4.2478E-03
256	1.9468E-06	2.4620E-06	256	1.9537E-03	1.9596E-03

Table 4. Maximum absolute Errors Obtained by HWCM for Ex. 4

HWCM			DBFM		[3]
2^{J+1}	e_1^J	e_2^J	M	e_1^J	e_2^J
4	4.4007E-03	6.3413E-03	4	4.0842E-02	7.6737E-02
8	1.2720E-03	1.6616E-03	8	2.0529E-02	3.6356E-02
16	3.3612E-04	4.1834E-04	16	1.0225E-02	1.8777E-02
32	8.6060E-05	1.0456E-04	32	5.1226E-03	9.3665E-03
64	2.1753E-05	2.6134E-05	64	2.5608E-03	4.6931E-03

Table 5. Maximum absolute Errors Obtained by HWCM for Ex. 5

2^{J+1}	e_1^J	e_2^J	e_3^J
4	4.6131E-01	1.8109E-03	1.6945E-01
8	6.9063E-05	1.3756E-03	4.4840E-02
16	1.8401E-05	3.4720E-04	1.1883E-02
32	4.7402E-06	8.7148E-05	3.0815E-03
64	1.2023E-06	2.1823E-05	7.8608E-04

Table 6. Maximum absolute Errors Obtained by HWCM for Ex. 6

2^{J+1}	e_1^J	e_2^J	e_3^J	e_4^J
4	1.8376E-02	1.2799E-02	1.6272E-02	7.3133E-03
8	4.6725E-03	3.6111E-03	5.8342E-03	1.9372E-03
16	1.1860E-03	9.6654E-04	1.7121E-03	5.0253E-04
32	2.9726E-04	2.5051E-04	4.6198E-04	1.2831E-04
64	7.4334E-05	6.3802E-05	1.1988E-04	3.2444E-05

References

[1] M. I. Berenguer, D. Gamez, A.I. Garralda-Guillem, M. Ruiz Galan and M. C.Serrano Perez, Biorthogonal systems for solving Volterra integral equation systems of the second kind, *J. Comp. Appl. Math.*, 235(2011), 1875-1883.

[2] S. Niyazi, Y. Suayip and G. Mustafa, A collocation approach for solving systems of linear Volterra integral equations with variable coefficients, *Comp. Math. Appl.*, 62(2011),755-769.

[3] M. Roodaki and H. Almasieh , Delta basis functions and their applications to system of integral equations, *Comp. Math. Appl.*, 63(2012), 100-109.

[4] V. Balakumar and K. Murugesan, Biorthogonal systems for solving Volterra integral equation systems of the second kind, Numerical solution of systems of linear Volterra integral equations using block-pulse functions, *Malaya J. Mat*., 1(2013), 77-84.

[5] Li-Hong, S. Ji-Hong and W. Yue, The reproducing kernel method for solving the system of the linear Volterra integral equations with variable coefficients, *J. Comp. Appl. Math.*, 236(2013), 2398-2405.

[6] A. Padmanabha Reddy, S. H. Manjula C. Sateesha and N. M. Bujurke, Haar wavelet approach for the solution of seventh order ordinary differential equations, *Math. Model. Eng. Probl.*, 3(2016), 108-114.

[7] A. Padmanabha Reddy,S. H. Manjula, C. Sateesha, A numerical approach to solve eighth order boundary value problems by Haar wavelet collocation method, *J. Math. Mod.*, 5(2017), 61-75.

[8] U. Lepik, Solving differential and integral equations by the Haar wavelet method, *Int. J. Math. Comp.*, 198(2008), 326-332.

[9] K. Maleknjad and B. Mirzaee, Using rationalized Haar wavelet for solving linear integral equations, *Appl. Math. Comp.*, 160(2005), 579-589.

[10] E. Babolian and A. Shahsavaran, Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets, *J. Comp. Appl. Math.*, 225(2009), 87-95.

[11] M. Farshid, Numerical computational solution of linear Volterra Integral equations system via rationalized Haar functions, *J. King Sand Uni.*, 22(2010), 265-268.

[12] I. Aziz and Siraj-ul-Islam, New algorithms for the numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets, *J. Comp. Appl. Math.*, 239(2013), 333-345.

[13] H. S. Huseyin and Y. Salih, Approximate solutions of linear Volterra integral equation systems with variable coefficients, *Appl. Math. Model.*, 34(2010), 3451-3464.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆