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Existence of fixed point theorems of contractive mappings in complete generalized metric spaces

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Abstract
In this paper we prove some fixed point theorems for contractive mappings in complete G – Metric Spaces.

Keywords
Complete Generalized Metric Spaces, Contractive Mapping, Fixed Point Theorems.

AMS Subject Classification
47H10, 54H25.

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Contents

1. Introduction

The theory of fixed point is one of the most influential tools of modern mathematics. Theorem concerning the existence and properties of fixed points are notorious as fixed point theorem. Fixed point theory is a beautiful mixture of analysis, topology & geometry. The concept of generalized metric space was discussed by various authors [3-10]. But robust concept of a generalized metric space was introduced by Z. Mustafa and B. Sims in 2006 as follows,

Definition 1.1. *Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:*

(D1) $G(x,y,z) = 0$ *if* $x = y = z$

(D2) $0 < G(x,x,y)$, *for all* $x,y \in X$ *with* $x \neq y$

(D3) $G(x,x,y) \leq G(x,y,z)$ *for all* $x,y,z \in X$ *with* $z \neq y$

(D4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ *(Symmetry in all three variables)*

(D5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ *for all* $x,y,z,a \in X$ *(rectangle inequality)*

Then the function G is called a generalized metric or more specifically a $G - metric$ on X and the pair (X,G) is called a G - metric space.

Example 1.2. *Let (X,d) be a metric space, then (X,G_s) and (X,G_m) are G - metric space, where*

$$G_s(x,y,z) = d(x,y) + d(y,z) + d(x,z) \forall x,y,z \in X$$

$$G_m(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\} \forall x,y,z \in X$$

Definition 1.3. *Let (X,G) be a G - metric space, let (x_n) be a sequence of points of X , we say that (x_n) is G - convergent to x if $\lim_{n\to\infty} G(x,x_n,x_n) = 0$ that is for any $\varepsilon > 0$, there exists $N \in N$ Such that $G(x,x_n,x_n) < \varepsilon$, for all $n,m \geq N$.*

Proposition 1.4. *Let (X,G) be G -metric space then the following conditions are equivalent*

1. *(x_n) is G -convergent to x*
2. $G(x_n,x_n,x) \to 0, asn \to \infty$
3. $G(x_n,x,x) \to 0, asn \to \infty$

Definition 1.5. *Let (X,G) be a G -metric space, a sequence (x_n) is called G -Cauchy if given $\varepsilon > 0$, there is $N \in N$ such that $G(x_n,x_m,x_l) < \varepsilon$, for all $n,m,l \geq N$ that is if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.*

Proposition 1.6. *In a G -metric space (X,G) , the following are equivalent*

1. *The sequence (x_n) is G-cauchy*

2. *For every $\varepsilon > 0$, there exists $N \in N$ such that*

$$G(x_n, x_m, x_m) < \varepsilon$$

for all $n,m \geq N$

Definition 1.7. *Let (X,G) and (X', G') be G - metric spaces and let $f : (X,G) \to (X', G')$ be a function, then f is said to be G -contiunuous at a point $a \in X$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that $x,y \in X$, $G(a,x,y) < \delta$ implies*

$$G'(f(a), f(x), f(y)) < \varepsilon.$$

A function f is G -continuous on X if and only if it is G -continous at all $a \in X$

Proposition 1.8. *Let $(X,G), (X', G')$ be G -metric spaces, then a function $f : X \to X'$ is G-continous at a point $x \in X$ if and only if it is G-sequently continuous at x , that is whenever (x_n) is G -convergent to $x, f((x_n))$ is G -convergent to $f(x)$.*

Proposition 1.9. *Let (X,G) be a G -metric space, then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Definition 1.10. *A G - metric space (X,G) is said to be G -complete (or a complete G -metric space) if every G -cauchy sequence in (X,G) is G -convergent in (X,G).*

2. Main results

Theorem 2.1. *Let T is mapping from X to X and let (X,G) be a complete G -metric space . T satisfies the succeeding condition for some $k \in N$ and for all $x,y,z \in X$.*

$$\begin{aligned} &G(T(x), T(y), T(z)) \\ &\quad \leq \lambda \max\{G(x,y,z), G(x,x,T(x)), \\ &\qquad G(y,y,T(y)), G(z,z,T(z)) \\ &\qquad G(x,T(x),T(x)), G(y,T(y),T(y),) \\ &\qquad G(z,T(z),T(z)), G(x,T(y),T(y)), \\ &\qquad G(y,T(z),T(z)), G(z,T(x),T(x))\} \end{aligned} \tag{2.1}$$

Where $\lambda \in [0, \frac{1}{2})$. Then u is the unique fixed point of T.

Proof. If the condition (1) satisfied by T , let x_0 in X be an arbitrary point, and (x_n) by $x_n = T^n x_0$ be the sequence.

$$\begin{aligned} &G(T(x_{n-1}), T(x_n), T(x_n)) \\ &\quad \leq \lambda max\{G(x_{n-1},x_n,x_n), G(x_{n-1},x_{n-1},T(x_{n-1})), \\ &\qquad G(x_n,x_n,T(x_n)), G(x_n,x_n,T(x_n)), \\ &\qquad G(x_{n-1},T(x_{n-1}),T(x_{n-1})), G(x_n,T(x_n),T(x_n)), \\ &\qquad G(x_n,T(x_n),T(x_n)), G(x_{n-1},T(x_n),T(x_n)), \\ &\qquad G(x_n,T(x_n),T(x_n)), G(x_n,T(x_{n-1}),T(x_{n-1})),\} \end{aligned}$$

$$\begin{aligned} &G(x_n,x_{n+1},x_{n+1}) \\ &\quad \leq \lambda max\{G(x_{n-1},x_n,x_n), \\ &\qquad G(x_{n-1},x_{n-1},x_n), G(x_n,x_n,x_{n+1}), \\ &\qquad G(x_n,x_n,x_{n+1}), G(x_{n-1},x_n,x_n), \\ &\qquad G(x_n,x_{n+1},x_{n+1}), G(x_n,x_{n+1},x_{n+1}), \\ &\qquad G(x_{n-1},x_{n+1},x_{n+1}), G(x_n,x_{n+1},x_{n+1}), G(x_n,x_n,x_n)\} \end{aligned}$$

$$\begin{aligned} &G(x_n,x_{n+1},x_{n+1}) \\ &\quad \leq \lambda max\{G(x_{n-1},x_n,x_n), G(x_{n-1},x_{n-1},x_n), \\ &\qquad G(x_n,x_n,x_{n+1}, G(x_n,x_{n+1},x_{n+1}), \\ &\qquad G(x_{n-1},x_{n+1},x_{n+1})\} \end{aligned}$$

So,

$$\begin{aligned} &G(x_n,x_{n+1},x_{n+1}) \\ &\quad \leq \lambda max\{G(x_{n-1},x_{n+1},x_{n+1}), G(x_{n-1},x_n,x_n)\} \end{aligned}$$

But, by (D5), we have

$$\begin{aligned} G(x_{n-1},x_{n+1},x_{n+1}) &\leq G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1}) \\ G(x_n,x_{n+1},x_{n+1}) &\leq \lambda max\{G(x_{n-1},x_n,x_n) \\ &\qquad + G(x_n,x_{n+1},x_{n+1}), G(x_{n-1},x_n,x_n)\} \\ G(x_n,x_{n+1},x_{n+1}) &\leq \lambda\{G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1})\} \\ G(x_n,x_{n+1},x_{n+1}) &\leq \lambda G(x_{n-1},x_n,x_n) + \lambda G(x_n,x_{n+1},x_{n+1}) \end{aligned}$$

$$\begin{aligned} &G(x_n,x_{n+1},x_{n+1}) - \lambda G(x_n,x_{n+1},x_{n+1}) \\ &\quad \leq \lambda G(x_{n-1},x_n,x_n)(1-\lambda)G(x_n,x_{n+1},x_{n+1}) \\ &\quad \leq \lambda G(x_{n-1},x_n,x_n)G(x_n,x_{n+1},x_{n+1}) \\ &\quad \leq \frac{\lambda}{1-\lambda} G(x_{n-1},x_n,x_n)G(x_n,x_{n+1},x_{n+1}) \\ &\quad \leq \beta\, G(x_{n-1},x_n,x_n) \end{aligned}$$

Let $\beta = \frac{\lambda}{1-\lambda}$, then $\beta < 1$ and by frequent above progression, we have

$$G(x_n,x_{n+1},x_{n+1}) \leq \beta^n\, G(x_{n-1},x_n,x_n)$$

Then, for all $n,m \in M, n < m$, By using the rectangle inequality

$$\begin{aligned} G(x_n,x_m,x_m) &\leq G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_{n+2},x_{n+2}) + \ldots \\ &\qquad + G(x_{m-1},x_m,x_m) \\ &\leq (\beta^n + \beta^{n+1} + \ldots + \beta^{m-1})G(x_0,x_1,x_1) \\ &\leq \frac{\beta^n}{1-\beta}\, G(x_0,x_1,x_1) \end{aligned}$$

Then, $\lim G(x_n,x_m,\ x_m) = 0$ as $n,m \to \infty$, since

$$\frac{\beta^n}{1-\beta}\, G(x_0,x_1,x_1) = 0$$

as $n,m \to \infty$.
For $n,m,l \in N$ which implies,

$$G(x_n,x_m,x_l) \le G(x_n,x_m,x_m) + G(x_l,x_m,x_m)$$

taking limit as $n,m,l \to \infty$.

We get $G(x_n,x_m,x_l) \to 0$. So, (x_n) is G-Cauchy sequence. By completeness of (X,G), $\exists u \in X$ is G-converges to u. If $T(u) \ne u$. Then,

$$\begin{aligned} &G(x_n,T(u),T(u)) \\ &\quad \le \lambda max\{G(x_{n-1},y,z), G(x_{n-1},x_{n-1},x_n), \\ &\qquad G(u,u,T(u)), G(u,u,T(u)), G(x_{n-1},x_n,x_n), \\ &\qquad G(u,u,T(u)), G(u,u,T(u)), G(x_{n-1},T(u),T(u)), \\ &\qquad G(u,T(u),T(u)), G(u,x_n,x_n)\} \end{aligned}$$

Taking limit $n \to \infty$

$$G(u,T(u),T(u)) \le \lambda G(u,T(u),T(u))$$

Which is a $\Rightarrow\Leftarrow$, because $0 \le \lambda < \frac{1}{2} \Rightarrow u = T(u)$
To show uniqueness, if $u \ne v$ is such that $v = T(v)$. Equation (2.1) implies that

$$\begin{aligned} &G(T(u),T(v),T(v)) \\ &\quad \le \lambda max\{G(u,v,v), G(u,u,T(u)), \\ &\qquad G(v,v,T(v)), G(v,v,T(v)), \\ &\qquad G(u,T(v),T(v)), G(v,T(v),T(v)), \\ &\qquad G(v,T(v),T(v)), G(u,T(v),T(v)), \\ &\qquad G(v,T(v),T(v)), G(v,T(u),T(u))\} \end{aligned}$$

$$\begin{aligned} G(u,v,v) \le &\lambda max\{G(u,v,v), G(u,u,u), \\ &G(v,v,v), G(v,v,v), G(u,v,v), G(v,v,v), \\ &G(v,v,v), G(u,v,v), G(v,v,v), G(v,u,u)\} \\ G(u,v,v) \le &\lambda max\{G(u,v,v), G(v,u,u)\} \end{aligned}$$

Thus, $G(u,v,v) \le \lambda G(v,u,u)$. Once more, by the same quarrel,

$$\begin{aligned} G(v,u,u) &\le \lambda G(u,v,v) \Rightarrow G(u,v,v) \le \lambda(\lambda G(u,v,v)) \\ G(u,v,v) &\le k^2 G(u,v,v) \end{aligned}$$

$\Rightarrow u = v$. Hence proved. □

Theorem 2.2. *Let T is mapping from X to X and let (X,G) be a complete G -metric space. T satisfies the succeeding condition for some $k \in N$ and for all $x,\ y,\ z \in X$.*

$$\begin{aligned} &G(T(x),T(y),T(z)) \\ &\quad \le \lambda max\{G(x,T(y),T(y)), G(y,T(x),T(x)), \\ &\qquad G(y,T(z),T(z)), G(z,T(y),T(y)), \\ &\qquad G(x,T(z),T(z)), G(z,T(x),T(x))\} \end{aligned}$$

Where $\lambda \in [0, \frac{1}{2})$, Then u is the unique fixed point of T.

Proof. If the condition (2) satisfied by T, let x_0 in X be an arbitrary point, and (x_n) by $x_n = T^n(x_0)$ be the sequence.

$$\begin{aligned} &G(T(x_{n-1}),T(x_n),T(x_n)) \\ &\quad \le \lambda max\{G(x_{n-1},T(x_n),T(x_n)), \\ &\qquad G(x_n,T(x_{n-1}),T(x_{n-1})), G(x_n,T(x_n),T(x_n)), \\ &\qquad G(x_n,T(x_n),T(x_n)), G(x_{n-1},T(x_n),T(x_n)), \\ &\qquad G(x_n,T(x_{n-1}),T(x_{n-1}))\} \end{aligned}$$

$$\begin{aligned} G(x_n,x_{n+1},x_{n+1}) \le &\lambda max\{G(x_{n-1},x_{n+1},x_{n+1}), \\ &G(x_n,x_n,x_n), G(x_n,x_{n+1},x_{n+1}), \\ &G(x_n,x_{n+1},x_{n+1}), G(x_{n-1},x_{n+1},x_{n+1}), \\ &G(x_n,x_n,x_n)\} \\ G(x_n,x_{n+1},x_{n+1}) \le &\lambda max\{G(x_{n-1},x_{n+1},x_{n+1}), \\ &G(x_n,x_{n+1},x_{n+1})\} \\ G(x_n,x_{n+1},x_{n+1}) \le &\lambda G(x_{n-1},x_{n+1},x_{n+1}) \end{aligned}$$

But, from(D5), we have

$$\begin{aligned} G(x_{n-1},x_{n+1},x_{n+1}) &\le G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1}) \\ G(x_n,x_{n+1},x_{n+1}) &\le \lambda(G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1})) \\ G(x_n,x_{n+1},x_{n+1}) &\le \lambda G(x_{n-1},x_n,x_n) + \lambda G(x_n,x_{n+1},x_{n+1}) \end{aligned}$$

$$\begin{aligned} G(x_n,x_{n+1},x_{n+1}) - \lambda G(x_n,x_{n+1},x_{n+1}) &\le \lambda G(x_{n-1},x_n,x_n) \\ (1-\lambda)G(x_n,x_{n+1},x_{n+1}) &\le \lambda G(x_{n-1},x_n,x_n) \\ G(x_n,x_{n+1},x_{n+1}) &\le \frac{\lambda}{1-\lambda} G(x_{n-1},x_n,x_n) \\ G(x_n,x_{n+1},x_{n+1}) &\le \beta G(x_{n-1},x_n,x_n) \end{aligned}$$

Let $\beta = \frac{\lambda}{1-\lambda}$ then $\beta < 1$ and repeated above process, we have

$$G(x_n,x_{n+1},x_{n+1}) \le \beta^n\ G(x_0,x_1,x_1)$$

Then $\forall n,m \in N, n < m$. Using rectangle inequality

$$\begin{aligned} G(x_n,x_m,x_m) &\le G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_{n+2},x_{n+2}) \\ &\quad + \ldots + G(x_{m-1},x_m,x_m) \\ &\le (\beta^n + \beta^{n+1} + \ldots + \beta^{m-1}) G(x_0,x_1,x_1) \\ &\le \frac{\beta^n}{1-\beta}\ G(x_0,x_1,x_1) \end{aligned}$$

Then, $\lim G(x_n,x_m,\ x_m) = 0$ as $n,m \to \infty$, since

$$\frac{\beta^n}{1-\beta}\ G(x_0,x_1,x_1) = 0$$

as $n,m \to \infty$. For $n,m,l \in N$, implies that

$$G(x_n,x_m,x_l) \le G(x_n,x_m,x_m) + G(x_l,x_m,x_m)$$

taking limit as $n,m,l \to \infty$. We get $G(x_n,x_m,x_l) \to 0$. So, (x_n) is G -Cauchy sequence.

By completeness of (X,G) , $\exists u \in X$ is G -converges to u . If $T(u) \neq u$. Then,

$$
\begin{aligned}
&G(T(x_n),T(u),T(u))\\
&\quad\leq \lambda max\{G(x_{n-1},T(u),T(u)),G(u,T(x_n),T(x_n)),\\
&\quad\quad G(u,T(u),T(u)),G(u,T(u),T(u)),\\
&\quad\quad G(x_{n-1},T(u),T(u)),G(u,T(x_n),T(x_n))\}
\end{aligned}
$$

Taking the limit as $n \to \infty$,

$$G(u,T(u),T(u)) \leq \lambda G(u,T(u),T(u)).$$

Which is contradiction, because $0 \leq \lambda < \frac{1}{2} \Rightarrow u = T(u)$. To show uniqueness, if $u \neq v$, such that $T(v) = v$ then

$$
\begin{aligned}
G(u,v,v) \leq&\lambda max\{G(u,v,v),G(v,u,u),\\
&G(v,v,v),G(v,v,v),G(u,v,v),G(v,u,u)\}\\
G(u,v,v) \leq&\lambda max\{G(u,v,v),G(v,u,u)\}\\
G(u,v,v) \leq&\lambda G(v,u,u)
\end{aligned}
$$

Again by the same argument, $G(v,u,u) \leq \lambda G(u,v,v)$. Thus,

$$
\begin{aligned}
G(u,v,v) \leq&\lambda(\lambda G(u,v,v))\\
G(u,v,v) \leq&k^2 G(u,v,v)
\end{aligned}
$$

Which implies that $u = v$. Hence proved. $\square$

Corollary 2.3. *Let T is mapping from X to X and let (X,G) be a complete G -metric space. T satisfies the succeeding condition for some $k \in N$ and for all $x,y,z \in X$.*

$$
\begin{aligned}
&G(T^k(x),T^k(y),T^k(z))\\
&\quad\leq \lambda max\{G(x,T^k(y),T^k(y)),\\
&\quad\quad G(y,T^k(x),T^k(x)),G(y,T^k(z),T^k(z)),\\
&\quad\quad G(z,T^k(y),T^k(y)),G(x,T^k(z),T^k(z)),\\
&\quad\quad G(z,T^k(x),T^k(x))\}
\end{aligned}
$$

Where $k \in [0,\frac{1}{2})$. Then T has a unique fixed point u.

Proof. The proof of the above theorem, we take, $T^k(u) = u$. But

$$T(u) = T(T^k(u)) = T^{k+1}(u) = T^k(T(u)),T^k$$

has another fixed point $T(u)$ and by uniqueness $Tu = u$. $\square$

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