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New Riemann-Liouville generalizations for some inequalities of Hardy type

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Abstract

In this paper, we present new generalizations for some integral results related to Hardy inequalities. For our results, some recent results of Hardy type and other interesting inequalities of integer order of integration can be deduced as some special cases.

Keywords: Integral inequalities, Riemann-Liouville integral, Hardy ineqality.

2010 MSC: 26D15, 26D10.

1 Introduction

The classical integral inequality of Hardy is the following [4] :

$$\int_0^\infty x^{-p}\left(\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx \tag{1.1}$$

where $p > 1$, $x > 0$, f is a nonnegative integrable function on $[0, \infty[$. The constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. During the past decades, the inequality has been developed and applied in almost unbelievable ways. This inequality plays an important role in analysis and its applications, see [1–3, 6–11, 13, 15, 18] and the references therein.

Let us recall some results that have motivated our work and have been reported in the previous literature. We begin by [12], where N. Levinson established the following generalization:

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f^p(t)dt, \tag{1.2}$$

where $f > 0$ on $[a, b] \subset [0, \infty[$, $p > 1$, and $F(x) = \int_0^x f(t)dt$.

Then, W.T. Sulaiman [17] presented the following similar Hardy inequality:

$$p\int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq (b-a)^p \int_a^b \left(\frac{f(x)}{x}\right)^p dx - \int_a^b \left(1-\frac{a}{x}\right)^p f^p(x)dx. \tag{1.3}$$

Recently, B. Sroysang [16] presented the following generalized result

$$p\int_a^b \frac{F^p(x)}{x^q}dx \leq (b-a)^p \int_a^b \frac{f^p(x)}{x^q}dx - \int_a^b \frac{(x-a)^p}{x^q} f^p(x)dx. \tag{1.4}$$

The important integral results presented very recently in the paper of S. Wu et al. [14] is another motivation for us. For our results, some inequalities of this reference can be deduced as some special cases. We also generalise some results obtained by the authors of [16, 17].

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2 Preliminaries

In this section, we present some preliminaries that will be used to prove the main results [5].

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a,b]$ is defined as*

$$J_a^\alpha [f(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau,\ \alpha > 0,\ a < t \leq b, \tag{2.5}$$

where $\Gamma(\alpha) := \int\limits_0^\infty e^{-u} u^{\alpha-1} du.$

For $\alpha > 0$, $\beta > 0$, we have the following properties:

$$J_a^\alpha J_a^\beta [f(t)] = J_a^{\alpha+\beta} [f(t)] \tag{2.6}$$

and

$$J_a^\alpha J_a^\beta [f(t)] = J_a^\beta J_a^\alpha [f(t)]\,. \tag{2.7}$$

For the expression (2.5), when $f(t) = (t-a)^\mu$, we get:

$$J_a^\alpha (t-a)^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t-a)^{\mu+\alpha}, t \in [a,b]. \tag{2.8}$$

For $t = b$, we put

$$J_a^\alpha [f(b)] = \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{2.9}$$

3 Main Results

Throughout the paper, all functions are assumed to be positive and all the integrals appear in the inequalities are exist and finite.

Theorem 3.1. *Let η be a nonnegative real number, and let $f > 0$ and $g > 0$ on $[a,b] \subseteq [0,\infty[$. If $\frac{x-a+\eta}{g(x)}$ is non-increasing, then for all $p > 1$, $\alpha > 0$, the fractional inequality*

$$\begin{aligned} &\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)} \right)^p dx \\ \leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{(\alpha(p-1)-p+\frac{1}{p})\Gamma^{p-1}(1-\frac{1}{p}+\alpha)} \\ &\times \left[(b-a)^{\alpha(p-1)+\frac{1}{p}-p} \left(J_a^\alpha \left[\left(\frac{f(b)}{g(b)} (b-a+\eta) \right)^p (b-a)^{1-\frac{1}{p}} \right] \right) \right. \\ &\left. -J_a^\alpha \left[\left(\frac{f(b)}{g(b)} (b-a+\eta) \right)^p (b-a)^{1+\alpha(p-1)-p} \right] \right] \end{aligned} \tag{3.10}$$

is valid.

Proof. We have:

$$\begin{aligned} &\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)} \right)^p dx \\ =\ & \int_a^b g^{-p}(x) \left(\int_a^x \frac{1}{\Gamma(\alpha)} (x-t)^{\alpha-1} f(t) (t-a)^{\frac{p-1}{p^2}} (t-a)^{\frac{1-p}{p^2}} dt \right)^p dx. \end{aligned} \tag{3.11}$$

Thanks to the fractional Hölder inequality [4], we obtain:

$$\begin{aligned} &\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)} \right)^p dx \leq \frac{1}{\Gamma^p(\alpha)} \int_a^b g^{-p}(x) \\ &\times \left[\int_a^x \left((x-t)^{\alpha-1} f^p(t) (t-a)^{\frac{p-1}{p}} dt \right)^{\frac{1}{p}} \left(\int_a^x (x-t)^{\alpha-1} (t-a)^{\left(\frac{1-p}{p^2}\right)\left(\frac{p}{p-1}\right)} dt \right)^{1-\frac{1}{p}} \right]^p dx. \end{aligned} \tag{3.12}$$

It yields then that

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{1}{\Gamma^p(\alpha)}\int_a^b g^{-p}(x)\int_a^x \left((x-t)^{\alpha-1} f^p(t)(t-a)^{\frac{p-1}{p}}dt\right) \\
&\times\left(\int_a^x (x-t)^{\alpha-1}(t-a)^{\frac{-1}{p}}dt\right)^{p-1} dx \qquad (3.13)\\
=\ & \frac{1}{\Gamma(\alpha)}\int_a^b g^{-p}(x)\int_a^x \left((x-t)^{\alpha-1} f^p(t)(t-a)^{\frac{p-1}{p}}dt\right)\left(J_a^\alpha (x-a)^{\frac{-1}{p}}\right)^{p-1} dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{\Gamma(\alpha)\Gamma^{p-1}(1-\frac{1}{p}+\alpha)}\int_a^b g^{-p}(x)(x-a)^{\left(\alpha-\frac{1}{p}\right)(p-1)} \qquad (3.14)\\
&\times\int_a^x \left((x-t)^{\alpha-1} f^p(t)(t-a)^{\frac{p-1}{p}}dt\right)dx.
\end{aligned}$$

This is to say that

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{\Gamma(\alpha)\Gamma^{p-1}(1-\frac{1}{p}+\alpha)}\int_a^b \left(\frac{x-a}{g(x)}\right)^p (x-a)^{\alpha(p-1)-1-p+\frac{1}{p}} \\
&\times\int_a^x \left((x-t)^{\alpha-1} f^p(t)(t-a)^{\frac{p-1}{p}}dt\right)dx \qquad (3.15)\\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{\Gamma(\alpha)\Gamma^{p-1}(1-\frac{1}{p}+\alpha)}\int_a^b \left(\frac{x-a+\eta}{g(x)}\right)^p (x-a)^{\alpha(p-1)-1-p+\frac{1}{p}} \\
&\times\int_a^x \left((x-t)^{\alpha-1} f^p(t)(t-a)^{\frac{p-1}{p}}dt\right)dx.
\end{aligned}$$

Since $\frac{x-a+\eta}{g(x)}$ is non-increasing and with the change of integration order, then we can write

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{\Gamma(\alpha)\Gamma^{p-1}(1-\frac{1}{p}+\alpha)}\int_a^b \left(\frac{t-a+\eta}{g(t)}\right)^p (b-t)^{\alpha-1} f^p(t)(t-a)^{\frac{p-1}{p}} \\
&\times\left(\int_t^b (x-a)^{\alpha(p-1)-1-p+\frac{1}{p}}dx\right)dt. \qquad (3.16)
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{\Gamma^{p-1}(\alpha)\Gamma^{p-1}(1-\frac{1}{p}+\alpha)(\alpha(p-1)-p+\frac{1}{p})} \\
&\times\int_a^b \left(\frac{t-a+\eta}{g(t)}\right)^p (b-t)^{\alpha-1} f^p(t)(t-a)^{\frac{p-1}{p}} \qquad (3.17)\\
&\times\left((b-a)^{\alpha(p-1)-p+\frac{1}{p}} - (t-a)^{\alpha(p-1)-p+\frac{1}{p}}\right)dt.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{(\alpha(p-1)-p+\frac{1}{p})\Gamma(\alpha)\Gamma^{p-1}(1-\frac{1}{p}+\alpha)} \\
&\times\Bigg[(b-a)^{\alpha(p-1)-p+\frac{1}{p}} \int_a^b \left(\tfrac{t-a+\eta}{g(t)}\right)^p (b-t)^{\alpha-1} f^p(t)(t-a)^{\frac{p-1}{p}} \\
&\quad -\int_a^b \left(\tfrac{t-a+\eta}{g(t)}\right)^p (b-t)^{\alpha-1} f^p(t)(t-a)^{\alpha(p-1)-p+1}\Bigg].
\end{aligned}
\tag{3.18}
$$

It follows that

$$
\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{g(x)}\right)^p dx \\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{(\alpha(p-1)-p+\frac{1}{p})\Gamma^{p-1}(1-\frac{1}{p}+\alpha)} \\
&\times\Bigg[(b-a)^{\alpha(p-1)-p+\frac{1}{p}} J_a^\alpha \left[\left(\tfrac{f(b)}{g(b)}(b-a+\eta)\right)^p (b-a)^{\frac{p-1}{p}}\right] \\
&\quad -J_a^\alpha \left[\left(\tfrac{f(b)}{g(b)}(b-a+\eta)\right)^p (b-a)^{1+\alpha(p-1)-p}\right]\Bigg].
\end{aligned}
\tag{3.19}
$$

□

Remark 3.1. *Putting $\alpha = 1$ in Theorem 3.1, we obtain Theorem 3.1 of [14].*

Corollary 3.1. *Let f be a nonnegative function on $[a, b]$, $a > 0$ and $0 < \eta < a$. Then for all $p > 1, \alpha > 0$, we have*

$$
\begin{aligned}
&\int_a^b \left(\frac{J_a^\alpha f(x)}{x-a+\eta}\right)^p dx \\
\leq\ & \frac{\Gamma^{p-1}(1-\frac{1}{p})}{(\alpha(p-1)-p+\frac{1}{p})\Gamma^{p-1}(1-\frac{1}{p}+\alpha)} \\
&\left[(b-a)^{\alpha(p-1)+\frac{1}{p}-p} J_a^\alpha \left(f^p(b)(b-a)^{1-\frac{1}{p}}\right) - J_a^\alpha \left(f^p(b)(b-a)^{1+\alpha(p-1)-p}\right)\right].
\end{aligned}
\tag{3.20}
$$

Remark 3.2. *Putting $\alpha = 1$ in Corollary 3.1, we obtain*

$$
\begin{aligned}
&\int_a^b (x-a+\eta)^{-p}\left(\int_a^x f(t)dt\right)^p dx \\
\leq\ & \left(\frac{p}{p-1}\right)^p \left[\int_a^b f^p(t)\left(1-\frac{(t-a)^{1-\frac{1}{p}}}{(b-a)^{1-\frac{1}{p}}}\right)dt\right] \\
\leq\ & \left(\frac{p}{p-1}\right)^p \int_a^b f^p(t)dt.
\end{aligned}
\tag{3.21}
$$

Moreover, it is clear that for $0 < \eta < a$, we have

$$
\int_a^b \left(\frac{J_a^\alpha f(x)}{x}\right)^p dx \leq \int_a^b \left(\frac{J_a^\alpha f(x)}{x-a+\eta}\right)^p dx. \tag{3.22}
$$

Hence, the inequality (3.21) implies Levinson inequality (1.2).

We prove also the following theorem.

Theorem 3.2. *Let $f>0$ and $g>0$ on $[a,b]\subseteq[0,\infty[$ such that g is non decreasing, then for all $p>1$, $q>0$, $\alpha>0$, we have*

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ & \frac{1}{(\alpha p-\alpha+1)\Gamma^{p-1}(\alpha+1)}\\ &\times\left[(b-a)^{\alpha p-\alpha+1} J_a^\alpha\left(\frac{f^p(b)}{g^q(b)}\right)-J_a^\alpha\left(\frac{f^p(b)}{g^q(b)}(b-a)^{\alpha p-\alpha+1}\right)\right].\end{aligned} \tag{3.23}$$

Proof. We have:

$$\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx=\int_a^b g^{-q}(x)\left(\int_a^x \frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}f(t)dt\right)^p dx, \tag{3.24}$$

and then,

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ & \int_a^b g^{-q}(x)\left[(J_a^\alpha f^p(x))^{\frac{1}{p}}(J_a^\alpha 1)^{1-\frac{1}{p}}\right]^p dx.\end{aligned} \tag{3.25}$$

Therefore,

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ & \int_a^b g^{-q}(x)\left[\left(\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f^p(t)dt\right)\left(\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}dt\right)^{p-1}\right]dx.\end{aligned} \tag{3.26}$$

Hence, we can write

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ & \frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)}\int_a^b g^{-q}(x)\left(\int_a^x(x-t)^{\alpha-1}f^p(t)dt\right)(x-a)^{\alpha(p-1)}dx.\end{aligned} \tag{3.27}$$

Since g is non decreasing and with the change of integration order, we obtain

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ & \frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)}\int_a^b g^{-q}(t)(b-t)^{\alpha-1}f^p(t)dt\int_t^b(x-a)^{\alpha(p-1)}dx.\end{aligned} \tag{3.28}$$

Therefore,

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ & \frac{1}{(\alpha p-\alpha+1)\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)}\int_a^b g^{-q}(t)(b-t)^{\alpha-1}f^p(t)\\ &\times\left[(b-a)^{\alpha p-\alpha+1}-(t-a)^{\alpha p-\alpha+1}\right]dt.\end{aligned} \tag{3.29}$$

This implies that

$$\begin{aligned}&\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)}dx\\ \leq\ & \frac{1}{(\alpha p-\alpha+1)\Gamma^{p-1}(\alpha+1)}\\ &\times\left((b-a)^{\alpha p-\alpha+1} J_a^\alpha\left(\frac{f^p(b)}{g^q(b)}\right)-J_a^\alpha\left[\left(\frac{f^p(b)}{g^q(b)}\right)(b-a)^{\alpha p-\alpha+1}\right]\right).\end{aligned} \tag{3.30}$$

$\square$

Remark 3.3. (i) : *Applying Theorem 3.2 for $\alpha = 1$, we obtain the first part of Theorem 3.5 in [14].*
(ii) : *Taking $\alpha = 1$, $g(x) = x$ in Theorem 3.2, we obtain Sroysang inequality (1.4).*
(iii) : *Taking $\alpha = 1$, $g(x) = x$, $p = q$ in Theorem 3.2, we obtain Sulaiman inequality (1.3).*

The third main result is given by the following theorem.

Theorem 3.3. *Let $f \geq 0$ and $g > 0$ on $[a,b] \subseteq [0,\infty[$ such that g is non decreasing. Then, for all $0 < p < 1$, $q > 0$, $\alpha > 0$, we have*

$$\begin{aligned} &\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)} dx \\ \geq\ & \frac{g^{-q}(b)}{(\alpha p - \alpha + 1)\Gamma^{p-1}(\alpha+1)} \left[\frac{(-1)^{\alpha p - \alpha + 1}}{\Gamma(\alpha)} \Gamma(\alpha p + 1) J_b^{\alpha p + 1} f^p(a) - (b-a)^{\alpha p - \alpha + 1} J_b^\alpha f^p(a) \right] \end{aligned} \tag{3.31}$$

Proof. Using the weighted reverse Hölder inequality, we obtain

$$\begin{aligned} &\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)} dx \\ \geq\ & \frac{1}{\Gamma^p(\alpha)} \int_a^b g^{-q}(x) \left[\left(\int_a^x (x-t)^{\alpha-1} f^p(t) dt \right)^{\frac{1}{p}} \left(\int_a^x (x-t)^{\alpha-1} dt \right)^{1-\frac{1}{p}} \right]^p dx \\ =\ & \frac{1}{\Gamma(\alpha)} \int_a^b g^{-q}(x) \left[\left(\int_a^x (x-t)^{\alpha-1} f^p(t) dt \right) (J_a^\alpha 1)^{p-1} \right] dx. \end{aligned} \tag{3.32}$$

Therefore, we have

$$\begin{aligned} &\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)} dx \\ \geq\ & \frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)} \int_a^b g^{-q}(x)(x-a)^{\alpha(p-1)} \left(\int_a^x (x-t)^{\alpha-1} f^p(t) dt \right) dx. \end{aligned} \tag{3.33}$$

Thanks to the non decreasing of g and with the change of the order of integration, it yields that

$$\begin{aligned} &\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)} dx \\ \geq\ & \frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)} \int_a^b g^{-q}(b)(x-a)^{\alpha(p-1)} \left(\int_a^x (x-t)^{\alpha-1} f^p(t) dt \right) dx. \end{aligned} \tag{3.34}$$

Consequently,

$$\begin{aligned} &\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)} dx \\ \geq\ & \frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)} \int_b^a g^{-q}(b)(a-t)^{\alpha-1} f^p(t) \left(\int_b^t (x-a)^{\alpha(p-1)} dx \right) dt \\ =\ & \frac{1}{\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)(\alpha p - \alpha + 1)} \int_b^a g^{-q}(b)(a-t)^{\alpha-1} f^p(t) \\ & \times \left[(t-a)^{\alpha p - \alpha + 1} - (b-a)^{\alpha p - \alpha + 1} \right] dt. \end{aligned} \tag{3.35}$$

Therefore,

$$\begin{aligned} &\int_a^b \frac{(J_a^\alpha f(x))^p}{g^q(x)} dx \\ \geq\ & \frac{1}{(\alpha p - \alpha + 1)\Gamma(\alpha)\Gamma^{p-1}(\alpha+1)} \\ & \left[(b-a)^{\alpha p - \alpha + 1} \int_a^b (a-t)^{\alpha-1} g^{-q}(b) f^p(t) dt \right. \\ & \left. - \int_a^b (a-t)^{\alpha-1} g^{-q}(b) f^p(t)(t-a)^{\alpha p - \alpha + 1} dt \right], \end{aligned} \tag{3.36}$$

and then,

$$\begin{aligned}&\int_a^b \frac{\left(J_a^\alpha f(x)\right)^p}{g^q(x)}dx\\ \geq\quad & \frac{g^{-q}(b)}{(\alpha p-\alpha+1)\Gamma^{p-1}(\alpha+1)}\left[\frac{(-1)^{\alpha p-\alpha+1}}{\Gamma(\alpha)}\Gamma(\alpha p+1)J_b^{\alpha p+1}f^p(a)-(b-a)^{\alpha p-\alpha+1}J_b^\alpha f^p(a)\right].\end{aligned}$$

This ends the proof. □

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