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Riemann-Liouville Fractional Hermite-Hadamard Inequalities for differentiable $\lambda\varphi$-preinvex functions

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Abstract

In this work, we demonstrate Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via once differentiable and twice differentiable defined using $\lambda\varphi$-preinvex functions.

Keywords: Fractional Hermite-Hadamard ineauqualities, φ-preinvex functions, Riemann-Liouville Fractional Integral.

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1 INTRODUCTION

The recently, Fractional calculus and generalizations is handled much. In especially the issue of fractional calculus is done various applications. These areas is physical sciences, economics, engineering, medicine and biological sciences$[1-8]$.

In this work, we give some Hermite-Hadamard type inequalities and the results via classical Riemann-Liouville fractional integrals for $\lambda\varphi-$preinvex functions by considering recent studies about this field.

2 Preliminaries

In this section, we will give some definitions, lemmas and notations which we use later in this work.

Definition 2.1. (*see* [3]) *Let $f \in L\left[a,b\right]$ .The Riemann-Liouville fractional integral $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a > 0$ are defined by*

$$\begin{aligned} J_{a+}^{\alpha}f\left(x\right) &= \tfrac{1}{\Gamma(\alpha)}\textstyle\int_a^x (x-t)^{\alpha-1}f\left(t\right)dt \quad ,0 \leq a < x \leq b \\ J_{b-}^{\alpha}f\left(x\right) &= \tfrac{1}{\Gamma(\alpha)}\textstyle\int_x^b (t-x)^{\alpha-1}f\left(t\right)dt \quad ,0 \leq a < x \leq b \end{aligned} \tag{2.1}$$

Where Γ is the gamma function.

Definition 2.2. (*see* [9]) *The incomplete beta function is defined as follows:*

$$B_x\left(a,b\right) = \textstyle\int_0^x t^{a-1}\left(1-t\right)^{b-1}dt, \tag{2.2}$$

Here $x \in \left[0,1\right], a,b > 0$.

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Definition 2.3. (*see* [10]) *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class $MT(I)$ if f is positive and $\forall x, y \in I$ and $t \in (0,1)$ satisfies the inequality:*

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{2.3}$$

Definition 2.4. (*see* [11]) *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class $m - MT(I)$ if f is positive and $\forall x, y \in I$ and $t \in (0,1)$, with $m \in [0,1]$ satisfies the inequality:*

$$f(tx + m(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{2.4}$$

Definition 2.5. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to a $\lambda - MT-$convex function or said to belong to the class $\lambda - MT(I)$ if f is positive and $\forall x, y \in I$, $\lambda \in \left(0, \frac{1}{2}\right]$ and $t \in (0,1)$ satisfies the inequality:*

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(y). \tag{2.5}$$

Lemma 2.0. (*see* [12]) *Let $f : [a,b] \to \mathbb{R}$ be a once differentiable mapping on (a,b) for $a < b$. If $f' \in L[a,b]$, there is a following equality for fractional integrals*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \\ &\quad = \frac{b-a}{2} \int_0^1 \left[(1-t)^{\alpha} - t^{\alpha} \right] f'(ta + (1-t)b)\, dt. \end{aligned} \tag{2.6}$$

Lemma 2.0. (*see* [13]) *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) for $a < b$. If $f'' \in L[a,b]$, there is following equality for fractional integrals*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \\ &\quad = \frac{(b-a)^2}{2} \int_0^1 \left[\frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] f''(ta + (1-t)b)\, dt. \end{aligned} \tag{2.7}$$

Lemma 2.0. (*see* [14]) *For $t \in [0,1]$,we have*

$$\begin{aligned} (1-t)^m &\leq 2^{1-m} - t^m \quad for\ m \in [0,1], \\ (1-t)^m &\geq 2^{1-m} - t^m \quad for\ m \in [1,\infty). \end{aligned}$$

Let $\mathbb{R}^n$ be Euclidian space and K is said to a nonempty closed in $\mathbb{R}^n$. Let $f : K \to \mathbb{R}$, $\varphi : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be a continuous functions.

Definition 2.6. ([15]) *Let $u \in K$. The set K is said to be $\varphi-$invex at u according to η and φ if*

$$u + te^{i\varphi}\eta(v,u) \in K \tag{2.8}$$

for all $u, v \in K$ and $t \in [0,1]$.

Remark 2.1. *Some special cases of Definition 6 are as follows.*

(1) If $\varphi = 0$, there K is defined an invex set.
(2) If $\eta(v,u) = v - u$, there K is defined a $\varphi-$convex set.
(3) If $\varphi = 0$ and $\eta(v,u) = v - u$, there K is defined a convex set.

Definition 2.7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function.Afunction f on the set $K_{\varphi\eta}$ is said to be $\lambda_{\varphi} -$ preinvex function according to φ and bifunction η and $\forall u, v \in I$, $t \in (0,1)$ and $0 \leq \varphi \leq \frac{\pi}{2}$ then*

$$f\left(u + te^{i\varphi}\eta(v,u)\right) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(u). \tag{2.9}$$

Remark 2.2. *In Definition 7, if $\lambda = \frac{1}{2}$, $\varphi = 0$ and $\eta(v,u) = v - u$. Definition 7 reduces to Definition 3;*

$$f(tv + (1-t)u) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(u).$$

Remark 2.3. *By considering Definition 7, if $\lambda = \frac{1}{2}$, $\varphi = 0$, and $\eta(v,u) = v - u$. for $m \in [0,1]$, we can write;*

$$f\left(mu + te^{i\varphi}\eta(v, mu)\right) = f(tv + m(1-t)u) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(u).$$

Remark 2.4. *In Definition 7, if $\varphi = 0$ and $\eta(v,u) = v - u$. Definition 7 reduces to Definition 5;*

$$f(tv + (1-t)u) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(v) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(u).$$

3 Main Results

Lemma 3.0. *Let $f:[a,b]\to\mathbb{R}$ be a once differentiable mappings on (a,b) with $a<b$, $\eta(b,a)>0$. If $f'\in L\left[a,a+e^{i\varphi}\eta(b,a)\right]$, then the following equality for fractional integral holds:*

$$\begin{aligned}&\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\\&=\frac{e^{i\varphi}\eta(b,a)}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt.\end{aligned}\tag{3.10}$$

Proof. By using Definition 7 and via the partial integration method ,we have following equality.

$$\begin{aligned}&\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt\\&=\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{e^{i\varphi}\eta(b,a)}-\frac{\alpha}{e^{i\varphi}\eta(b,a)}\\&\quad\times\left[\frac{1}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\int_a^{a+e^{i\varphi}\eta(b,a)}(x-a)^{\alpha-1}f(x)\,dx\right.\\&\quad\left.+\frac{1}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\int_a^{a+e^{i\varphi}\eta(b,a)}\left(a+e^{i\varphi}\eta(b,a)-x\right)^{\alpha-1}f(x)\,dx\right]\\&=\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{e^{i\varphi}\eta(b,a)}-\frac{\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha+1}}\\&\quad\times\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right].\end{aligned}\tag{3.11}$$

By multiplying the both sides of (3.2) by $\frac{e^{i\varphi}\eta(b,a)}{2}$, we have:

$$\begin{aligned}&\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\\&=\frac{e^{i\varphi}\eta(b,a)}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt.\end{aligned}$$

The proof is done. □

Remark 3.5. *In Lemma 4, if $\varphi=0$ and $\eta(b,a)=b-a$, Lemma 4 reduces to Lemma 1;*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\\&=\frac{b-a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(ta+(1-t)b)\,dt.\end{aligned}$$

Theorem 3.1. *Let $I\subseteq\mathbb{R}\to\mathbb{R}$ be a open invex set with respect to bifunction $\eta:I\times I\to\mathbb{R}$ where $\eta(b,a)>0$. Let $f:[0,b]\to\mathbb{R}$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ decreasing and $\lambda_{\varphi}-$ preinvex function on I for $\alpha>0$ and $0\le a<b$, then:*

$$\begin{aligned}&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\right|\\&\le\frac{e^{i\varphi}\eta(b,a)}{4}\left[|f'(a)|+\frac{1-\lambda}{\lambda}|f'(b)|\right]\left(B_{\frac12}\left(\frac12,\alpha+\frac12\right)-B_{\frac12}\left(\alpha+\frac12,\frac12\right)\right).\end{aligned}$$

Proof. By using Definition 7 and Lemma 4,we have:

$$\begin{aligned}&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\right|\\&\le\frac{e^{i\varphi}\eta(b,a)}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|dt\\&\le\frac{e^{i\varphi}\eta(b,a)}{2}\left[\int_0^{\frac12}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|dt\right.\\&\quad\left.+\int_{\frac12}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|dt\right]\\&\le\frac{e^{i\varphi}\eta(b,a)}{2}\left[\int_0^{\frac12}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}|f'(b)|\right)dt\right.\\&\quad\left.+\int_{\frac12}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}|f'(a)|+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}|f'(b)|\right)dt\right]\\&\le\frac{e^{i\varphi}\eta(b,a)}{2}\left[|f'(a)|\int_0^{\frac12}\left[(1-t)^{\alpha}-t^{\alpha}\right]\frac{1}{2\sqrt{t(1-t)}}dt\right.\\&\quad\left.+\frac{(1-\lambda)}{\lambda}|f'(b)|\int_0^{\frac12}\left[t^{\alpha}-(1-t)^{\alpha}\right]\frac{1}{2\sqrt{t(1-t)}}dt\right]\\&\le\frac{e^{i\varphi}\eta(b,a)}{4}\left[|f'(a)|+\frac{1-\lambda}{\lambda}|f'(b)|\right]\left(B_{\frac12}\left(\frac12,\alpha+\frac12\right)-B_{\frac12}\left(\alpha+\frac12,\frac12\right)\right)\end{aligned}$$

The proof is done. □

Theorem 3.2. *Let $I=[a,b]\to\mathbb{R}$ be a open invex set with respect to bifunction $\eta: I\times I\to\mathbb{R}$ and $f:[0,b]\to\mathbb{R}$ be a differentiable mapping and $1<q<\infty$. If $|f'|^q$ is measurable and $|f'|^q$ decreasing and $\lambda_\varphi-$ preinvex function on I for $0\le a<b$ and $\eta(b,a)>0$ then:*

$$\begin{aligned}&\left|\tfrac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\tfrac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J^{\alpha}_{a^+}f\left(a+e^{i\varphi}\eta(b,a)\right)+J^{\alpha}_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}f(a)\right]\right|\\&\qquad\le\tfrac{e^{i\varphi}\eta(b,a)}{2}\left[\tfrac{\pi}{4}\left|f'(a)\right|^q+\tfrac{\pi}{4}\left(\tfrac{1-\lambda}{\lambda}\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\left(\tfrac{2-2^{1-\alpha p}}{p\alpha+1}\right)^{\frac{1}{p}}\end{aligned}$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Definition 7, Lemma 4 and Hölder's inequality, we have:

$$\begin{aligned}&\left|\tfrac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\tfrac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J^{\alpha}_{a^+}f\left(a+e^{i\varphi}\eta(b,a)\right)+J^{\alpha}_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}f(a)\right]\right|\\&\le\tfrac{e^{i\varphi}\eta(b,a)}{2}\int_0^1\left|(1-t)^\alpha-t^\alpha\right|\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|dt\\&\le\tfrac{e^{i\varphi}\eta(b,a)}{2}\left(\int_0^1\left|(1-t)^\alpha-t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\\&\le\tfrac{e^{i\varphi}\eta(b,a)}{2}\left(\int_0^1\left|(1-t)^\alpha-t^\alpha\right|^p dt\right)^{\frac{1}{p}}\\&\qquad\times\left(\int_0^1\left(\tfrac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(a)\right|^q+\tfrac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\\&\le\tfrac{e^{i\varphi}\eta(b,a)}{2}\left[\tfrac{\pi}{4}\left|f'(a)\right|^q+\tfrac{\pi}{4}\left(\tfrac{1-\lambda}{\lambda}\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\\&\qquad\times\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha p}-t^{\alpha p}\right]dt+\int_{\frac{1}{2}}^1\left[t^{\alpha p}-(1-t)^{\alpha p}\right]dt\right)^{\frac{1}{p}}\\&\le\tfrac{e^{i\varphi}\eta(b,a)}{2}\left[\left|f'(a)\right|^q+\tfrac{1-\lambda}{\lambda}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\left(\tfrac{\pi}{4}\right)^{\frac{1}{q}}\left(\tfrac{2-2^{1-\alpha p}}{p\alpha+1}\right)^{\frac{1}{p}}.\end{aligned}$$

Here, we $(A_1-A_2)^P\le A_1^p-A_2^p$ for any $A_1>A_2\ge 0$ and $p\ge 1$. The proof is done. □

Theorem 3.3. *Let $I=[0,b]\to\mathbb{R}$ be a open invex set with respect to bifunction $\eta: I\times I\to\mathbb{R}$ and $f:[0,b]\to\mathbb{R}$ be a differentiable mapping and $1<q<\infty$, $f'\in L\left[a+e^{i\varphi}\eta(b,a)\right]$. If $|f'|^q$ is measurable and $|f'|^q$ decreasing and $\lambda_\varphi-$ preinvex function on I for $0\le a<b$ and $\eta(b,a)>0$ then:*

$$\begin{aligned}&\left|\tfrac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\tfrac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J^{\alpha}_{a^+}f\left(a+e^{i\varphi}\eta(b,a)\right)+J^{\alpha}_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}f(a)\right]\right|\\&\le 2^{-\frac{1}{q}}e^{i\varphi}\eta(b,a)\left(\tfrac{1-2^{-\alpha}}{\alpha+1}\right)^{\frac{q-1}{q}}\left[\tfrac{\left|f'(a)\right|^q}{2}\left(B_{\frac{1}{2}}\left(\tfrac{1}{2},\alpha+\tfrac{1}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\tfrac{1}{2},\tfrac{1}{2}\right)\right)\right.\\&\qquad\left.+\left(\tfrac{1-\lambda}{\lambda}\right)\tfrac{\left|f'(b)\right|^q}{2}\left(B_{\frac{1}{2}}\left(\tfrac{1}{2},\alpha+\tfrac{1}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\tfrac{1}{2},\tfrac{1}{2}\right)\right)\right]^{\frac{1}{q}}\end{aligned}$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Definition 7, Lemma 4 and Power Mean inequality, we have:

$$\begin{aligned}&\left|\tfrac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\tfrac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J^{\alpha}_{a^+}f\left(a+e^{i\varphi}\eta(b,a)\right)+J^{\alpha}_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}f(a)\right]\right|\\&\le\tfrac{e^{i\varphi}\eta(b,a)}{2}\int_0^1\left|(1-t)^\alpha-t^\alpha\right|\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|dt\\&\le\tfrac{e^{i\varphi}\eta(b,a)}{2}\left(\int_0^1\left|(1-t)^\alpha-t^\alpha\right|dt\right)^{1-\frac{1}{q}}\\&\qquad\times\left(\int_0^1\left|(1-t)^\alpha-t^\alpha\right|\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\\&\le\tfrac{e^{i\varphi}\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}}\left[(1-t)^\alpha-t^\alpha\right]dt+\int_{\frac{1}{2}}^1\left[t^\alpha-(1-t)^\alpha\right]dt\right)^{1-\frac{1}{q}}\\&\qquad\times\left(\int_0^1\left|(1-t)^\alpha-t^\alpha\right|\left|f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

$$\leq \frac{e^{i\varphi}\eta(b,a)}{2}\left(\frac{2-2^{1-\alpha}}{\alpha+1}\right)^{\frac{q-1}{q}}\left[\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(a)\right|^q+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f'(b)\right|^q\right)dt\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(a)\right|^q+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f'(b)\right|^q\right)dt\right]^{\frac{1}{q}}$$
$$\leq 2^{-\frac{1}{q}}e^{i\varphi}\eta(b,a)\left(\frac{1-2^{-\alpha}}{\alpha+1}\right)^{\frac{q-1}{q}}\left[\frac{\left|f'(a)\right|^q}{2}\left(B_{\frac{1}{2}}\left(\frac{1}{2},\alpha+\frac{1}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\frac{1}{2},\frac{1}{2}\right)\right)\right.$$
$$\left.+\left(\frac{1-\lambda}{\lambda}\right)\frac{\left|f'(b)\right|^q}{2}\left(B_{\frac{1}{2}}\left(\frac{1}{2},\alpha+\frac{1}{2}\right)-B_{\frac{1}{2}}\left(\alpha+\frac{1}{2},\frac{1}{2}\right)\right)\right]^{\frac{1}{q}}.$$

The proof is done. □

Lemma 3.0. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mappings on (a,b) with $a<b$, $\eta(b,a)>0$. If $f''\in L\left[a,a+e^{i\varphi}\eta(b,a)\right]$, then the following equality for fractional integral holds:*

$$\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\right|$$
$$=\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2(\alpha+1)}\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt. \tag{3.12}$$

Proof. By using Definition 7 and Lemma 2, if use twice the partial integration method, we have:

$$\int_0^1\left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right]f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt$$
$$=-\left.\frac{\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)}{(\alpha+1)e^{i\varphi}\eta(b,a)}\right|_0^1$$
$$+\frac{1}{e^{i\varphi}\eta(b,a)}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt \tag{3.13}$$
$$=\frac{1}{e^{i\varphi}\eta(b,a)}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt$$

Motivated by Lemma 4, then:

$$\frac{1}{e^{i\varphi}\eta(b,a)}\left(\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{e^{i\varphi}\eta(b,a)}-\frac{\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha+1}}\right.$$
$$\left.\times\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\right)$$
$$=\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{\left(e^{i\varphi}\eta(b,a)\right)^2}-\frac{\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha+2}}$$
$$\left.\times\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\right).$$

By multipling the both sides of (3.5) by $\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2}$, we have:

$$\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\right|$$
$$=\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{2}\int_0^1\left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right]f''\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt$$

The proof is done. □

Remark 3.6. *In Lemma 5, if $\varphi=0$ and $\eta(b,a)=b-a$. Lemma 5 reduces to Lemma 2;*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]$$
$$=\frac{(b-a)^2}{2}\int_0^1\left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right]f''(ta+(1-t)b)\,dt.$$

Theorem 3.4. *Let $f:[0,b]\to\mathbb{R}$ be a differentiable mapping. If $|f''|$ is measurable and $|f''|$ is decreasing and $\lambda-$preinvex function on $[0,b]$ for $0\leq a<b$, $\eta(b,a)>0$ and $\alpha>0$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^+}^{\alpha}f\left(a+e^{i\varphi}\eta(b,a)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{4(\alpha+1)}\left\{\left|f''(a)\right|\left[\frac{\pi}{2}-B\left(\frac{3}{2},\alpha+\frac{3}{2}\right)-B\left(\alpha+\frac{5}{2},\frac{1}{2}\right)\right]\right.$$
$$\left.+\left(\frac{1-\lambda}{\lambda}\right)\left|f''(b)\right|\left[\frac{\pi}{2}-B\left(\frac{1}{2},\alpha+\frac{5}{2}\right)-B\left(\alpha+\frac{3}{2},\frac{3}{2}\right)\right]\right\}.$$

Proof. By using Definition 7 and Lemma 5, we have:

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}+\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^{+}}^{\alpha}f\left(a+e^{i\varphi}\eta\left(b,a\right)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^{-}}^{\alpha}f\left(a\right)\right]\right|\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2}\int_{0}^{1}\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left|f''\left(a+(1-t)\,e^{i\varphi}\eta\left(b,a\right)\right)\right|dt\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\int_{0}^{1}\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''\left(a\right)\right|+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''\left(b\right)\right|\right)dt\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left\{\frac{|f''(a)|}{2}\left(\int_{0}^{1}t^{\frac{1}{2}}\left(1-t\right)^{\frac{-1}{2}}dt-\int_{0}^{1}t^{\frac{1}{2}}\left(1-t\right)^{\alpha+\frac{1}{2}}dt-\int_{0}^{1}t^{\alpha+\frac{3}{2}}\left(1-t\right)^{\frac{-1}{2}}dt\right)\right.\\
&\left.+\left(\frac{1-\lambda}{\lambda}\right)\frac{|f''(b)|}{2}\left(\int_{0}^{1}t^{\frac{-1}{2}}\left(1-t\right)^{\frac{1}{2}}dt-\int_{0}^{1}t^{\frac{-1}{2}}\left(1-t\right)^{\alpha+\frac{3}{2}}dt-\int_{0}^{1}t^{\alpha+\frac{1}{2}}\left(1-t\right)^{\frac{1}{2}}dt\right)\right\}\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{4(\alpha+1)}\left\{\left|f''\left(a\right)\right|\left[\frac{\pi}{2}-B\left(\frac{3}{2},\alpha+\frac{3}{2}\right)-B\left(\alpha+\frac{5}{2},\frac{1}{2}\right)\right]\right.\\
&\qquad\left.+\left(\frac{1-\lambda}{\lambda}\right)\left|f''\left(b\right)\right|\left[\frac{\pi}{2}-B\left(\frac{1}{2},\alpha+\frac{5}{2}\right)-B\left(\alpha+\frac{3}{2},\frac{3}{2}\right)\right]\right\}.
\end{aligned}$$

The proof is done. □

Theorem 3.5. *Let $f:[0,b]\to\mathbb{R}$ be a differentiable mapping and $1<q<\infty$. If $|f''|^{q}$ is measurable and $|f''|^{q}$ is decreasing and $\lambda_{\varphi}-$ preinvex function on $[0,b]$ for $\eta\left(b,a\right)>0$ and $0\leq a<b$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^{+}}^{\alpha}f\left(a+e^{i\varphi}\eta\left(b,a\right)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^{-}}^{\alpha}f\left(a\right)\right]\right|\\
&\qquad\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(1-2^{1-\alpha}\right)\left(\frac{\pi}{4}\left|f''\left(a\right)\right|^{q}+\frac{\pi}{4}\left(\frac{1-\lambda}{\lambda}\right)\left|f''\left(b\right)\right|^{q}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Definition 7, Lemma 5 and Hölder's inequality we have:

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^{+}}^{\alpha}f\left(a+e^{i\varphi}\eta\left(b,a\right)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^{-}}^{\alpha}f\left(a\right)\right]\right|\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2}\int_{0}^{1}\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left|f''\left(a+(1-t)\,e^{i\varphi}\eta\left(b,a\right)\right)\right|dt\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(\int_{0}^{1}\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|f''\left(a+(1-t)\,e^{i\varphi}\eta\left(b,a\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(\int_{0}^{1}\left[1-2^{-\alpha}\right]^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''\left(a\right)\right|^{q}+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''\left(b\right)\right|^{q}\right)^{q}dt\right)^{\frac{1}{q}}\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(1-2^{-\alpha}\right)\left(\frac{\pi}{4}\left|f''\left(a\right)\right|^{q}+\frac{\pi}{4}\left(\frac{1-\lambda}{\lambda}\right)\left|f''\left(b\right)\right|^{q}\right)^{\frac{1}{q}}.
\end{aligned}$$

The proof is done. □

Theorem 3.6. *Let $f:[0,b]\to\mathbb{R}$ be a differentiable mapping and $1<q<\infty$. If $|f''|^{q}$ is measurable and $|f''|^{q}$ is decreasing and $\lambda_{\varphi}-$ preinvex function on $[0,b]$ for $0\leq a<b$ and $\eta\left(b,a\right)>0$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a^{+}}^{\alpha}f\left(a+e^{i\varphi}\eta\left(b,a\right)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^{-}}^{\alpha}f\left(a\right)\right]\right|\\
&\qquad\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(1-2^{-\alpha}\right)^{\frac{q-1}{q}}\left(\frac{|f''(a)|^{q}}{2}\left[B\left(\frac{3}{2},\alpha+\frac{3}{2}\right)+B\left(\alpha+\frac{5}{2},\frac{1}{2}\right)-\frac{\pi}{2}\right]\right.\\
&\qquad\qquad\left.+\left(\frac{1-\lambda}{\lambda}\right)\frac{|f''(b)|^{q}}{2}\left[B\left(\frac{1}{2},\alpha+\frac{5}{2}\right)+B\left(\alpha+\frac{3}{2},\frac{3}{2}\right)-\frac{\pi}{2}\right]\right)^{\frac{1}{q}}.
\end{aligned}$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Definition 7, Lemma 5 and Power Mean's inequality, we have:

$$
\begin{aligned}
&\left|\frac{f(a)+f\left(a+e^{i\varphi}\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J_{a+}^{\alpha}f\left(a+e^{i\varphi}\eta\left(b,a\right)\right)+J_{\left(a+e^{i\varphi}\eta(b,a)\right)^{-}}^{\alpha}f\left(a\right)\right]\right|\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2}\int_{0}^{1}\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left|f''\left(a+(1-t)\,e^{i\varphi}\eta\left(b,a\right)\right)\right|dt\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(\int_{0}^{1}\left|1-(1-t)^{\alpha+1}-t^{\alpha+1}\right|dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{0}^{1}\left|1-(1-t)^{\alpha+1}-t^{\alpha+1}\right|\left|f''\left(a+(1-t)\,e^{i\varphi}\eta\left(b,a\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(\int_{0}^{1}\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{0}^{1}\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f''\left(a\right)\right|^{q}+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\left|f''\left(b\right)\right|^{q}\right)dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(1-2^{-\alpha}\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\frac{|f''(a)|^{q}}{2}\left(\int_{0}^{1}t^{\frac{1}{2}}\left(1-t\right)^{-\frac{1}{2}}dt-\int_{0}^{1}t^{\frac{1}{2}}\left(1-t\right)^{\alpha+\frac{1}{2}}dt-\int_{0}^{1}t^{\alpha+\frac{3}{2}}\left(1-t\right)^{-\frac{1}{2}}dt\right)\right.\\
&\left.+\left(\frac{1-\lambda}{\lambda}\right)\frac{|f''(b)|^{q}}{2}\left(\int_{0}^{1}t^{-\frac{1}{2}}\left(1-t\right)^{\frac{1}{2}}dt-\int_{0}^{1}t^{-\frac{1}{2}}\left(1-t\right)^{\alpha+\frac{3}{2}}dt-\int_{0}^{1}t^{\alpha+\frac{1}{2}}\left(1-t\right)^{\frac{1}{2}}dt\right)\right)^{\frac{1}{q}}\\
&\leq\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{2(\alpha+1)}\left(1-2^{-\alpha}\right)^{1-\frac{1}{q}}\left(\frac{|f''(a)|^{q}}{2}\left(\frac{\pi}{2}-B\left(\frac{3}{2},\alpha+\frac{3}{2}\right)-B\left(\alpha+\frac{5}{2},\frac{1}{2}\right)\right)\right.\\
&\qquad\left.+\left(\frac{1-\lambda}{\lambda}\right)\frac{|f''(b)|^{q}}{2}\left(\frac{\pi}{2}-B\left(\frac{1}{2},\alpha+\frac{5}{2}\right)-B\left(\alpha+\frac{3}{2},\frac{3}{2}\right)\right)\right)^{\frac{1}{q}}.
\end{aligned}
$$

The proof is done. □

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