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Approximate controllability of nonlocal impulsive fractional neutral stochastic integro-differential equations with state-dependent delay in Hilbert spaces

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Abstract

In this manuscript, we study the approximate controllability results for nonlocal impulsive fractional neutral stochastic integro-differential equations with state-dependent delay conditions in Hilbert spaces under the assumptions that the corresponding linear system is approximately controllable. The results are obtained by using fractional calculus, semigroup theory, stochastic analysis and fixed point theorem. An example is provided to show the application of our result.

Keywords: Fractional differential equations, approximate controllability, stochastic differential system, nonlocal condition, state-dependent delay, fixed point theorem, semigroup theory.

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1 Introduction

In this manuscript, we set up the approximate controllability of mild solutions for nonlocal impulsive fractional neutral stochastic integro-differential systems (abbreviated, NIFNSIDS) with state-dependent delay (abbreviated, SDD) in Hilbert spaces through the utilization of the fixed point theorem thanks to Schauder [30]. We discuss the neutral integro-differential equations of fractional-order with SDD of the model

$$
\begin{aligned}
{}^{C}D_t^{\alpha}[u(t)-\mathscr{G}(t,u_{\varrho(t,u_t)})] &= \mathscr{A}u(t)+Bv(t)+\mathscr{F}\left(t,u_{\varrho(t,u_t)},\int_0^t e_1(t,s,u_{\varrho(s,u_s)})ds\right)\\
&\quad+\Sigma\left(t,u_{\varrho(t,u_t)},\int_0^t e_2(t,s,u_{\varrho(s,u_s)})ds\right)\frac{dw(t)}{dt},\quad t\neq t_k,\quad k=1,2,\cdots,n, \qquad (1.1)\\
\Delta u(t_k) &= \mathcal{I}_k(u(t_k^-)),\quad k=1,2,\cdots,n, \qquad (1.2)\\
u(0)+h(u) &= \varphi\in\mathcal{B}, \qquad (1.3)
\end{aligned}
$$

where ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative of order α, $\alpha\in(0,1)$, the state variable u takes values in a Hilbert space $\mathcal{H}$; $\Delta u(t_k) = u(t_k^+)-u(t_k^-)$, $k=1,2,\ldots,n$ are impulsive function, which the solution is jump at impulsive point t_k, $0<t_1<t_2<\cdots<t_n<T$; $\mathscr{A}: D(\mathscr{A})\subset\mathcal{H}\to\mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operator $\{\mathbb{T}(t): t\geq 0\}$. That is to say, $\|\mathbb{T}(t)\|\leq\mathcal{M}$ for some constant $\mathcal{M}\geq 1$ and every $t\geq 0$; the control function v is given in $\mathcal{L}^2(\mathscr{I},U)$, U is a Hilbert space, B is a bounded linear operator from U into $\mathcal{H}$. The time history u_t: $(-\infty,0]\to\mathcal{H}$, $u_t(\theta)=u(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ described axiomatically in section 2 and ϱ :

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$\mathscr{I} \times \mathcal{B} \to (-\infty, T]$ is a continuous function. Let $\mathcal{K}$ be a another Hilbert space, suppose $\{W(t)\}_{t\geq 0}$ is a given $\mathcal{K}$-valued Brownian motion or wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Denote $\mathcal{PC}(\mathscr{I}, \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{H})) = \{u(t)$ is continuous everywhere except for some t_k at which $u(t_k^-)$ and $u(t_k^+)$ exist and $u(t_k^-) = u(t_k)\}$ be the Banach space of piece-wise continuous function from $\mathscr{I}$ into $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{H})$ with the norm $\|u\|_{\mathcal{PC}} = \sup_{t\in\mathscr{I}} |u(t)| < \infty$, $\mathcal{PC}(\mathscr{I}, \mathcal{L}^2)$ is the closed subspace of $\mathcal{PC}(\mathscr{I}, \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{H}))$ consisting of a measurable and $\mathcal{F}_t$-adapted $\mathcal{H}$-valued process $u(\cdot) \in \mathcal{PC}(\mathscr{I}, \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{H}))$ with the norm defined $\|u\|^2 = \sup\{\mathbb{E}\|u(t)\|^2,\ t \in \mathscr{I}\}$. The functions $\mathscr{G}, \mathscr{F}, \Sigma, e_i, i = 1, 2; \mathcal{I}_k$ and h are suitable functions to be specified later.

The emergence of fractional calculus arise new questions in fundamental physics, which provides great challenging interest for the mathematicians and physicists in the theory of fractional calculus. The fractional differential equations (abbreviated, FDEs) have been considered to be the valuable tool, which can describe dynamical behavior of real life phenomena more accurately. For instance, the nonlinear oscillation of earthquake can be well modeled with fractional derivatives. We can find the numerous applications of FDEs in control theory, nonlinear oscillation of earthquake, the fluid-dynamic traffic model, aerodynamics and in almost every field of science and engineering. For more points of interest on this concept, we allude the reader to Pazy [27]. There has been a lot of enthusiasm toward the solutions of fractional differential equations in systematic and mathematical thoughts. For fundamental certainties about fractional systems, one can make reference to the books [6, 13, 38], and the papers [11, 15], and the references cited therein.

FDEs with delay features happen in several areas such as medical and physical with SDD or non-constant delay. These days, existence and controllability results of mild solutions for such problems became very attractive and several researchers working on it. As of late, few number of papers have been published on the fractional order problems with SDD [1, 2, 9, 20, 30, 35] and references therein. Especially, in [1], the authors analyzed the existence results for fractional integro-differential equations whereas Benchohra et al. [2] examined the existence of mild solutions for fractional integro-differential equations in Banach spaces.

An important feature of real-world dynamic processes that has attracted considerable interest by scientists is the effect of abrupt changes. Hereby, "abrupt" is meant in the sense of a multi-scale problem, i.e. the state of a system changes only slowly for a long time interval, and then undergoes a drastic change within a very short time interval. For example, a football may be flying through the air for several seconds before it changes its flight direction within milliseconds during a collision with a goal post. For the mathematical description of this system, the specification of two sets of equations is appropriate: one for the flight phase, and one for the collision phase.

Several mathematical models can be developed for the football example. In a simplified setting, the motion of the football could be described by the position and velocity of its center of mass, and the encounter with the goal post could be treated as an inelastic collision (i.e. by an immediate change of the football's velocity).

For the description of the collision of the ball with the goal post leads to differential equations in which the velocity experiences, at the time of the collision, a so-called impulse. There is really a noteworthy improvement in impulsive concept, particularly in the region of impulsive differential frameworks having fixed times; for the additional purposes of enthusiasm on this concept and on its uses, see for example the treatise by Lakshmikantham et al. [22], Ivanka M. Stamova [34], Bainov et al. [4], Benchohra et al. [7] and the papers [3, 8, 10, 15], and the references cited therein.

In addition, the investigation of stochastic differential comparisons has pulled in awesome enthusiasm because of its applications in portraying numerous issues in material science, biology, chemistry, mechanics, etc. As a matter of fact, the accurate analysis or assessment subjected to a realistic environment has to take into account the potential randomness in the system properties, such as fluctuations in the stock market or noise in a communication network. All these problems in mathematics are modeled and depicted by stochastic differential equations or stochastic integro-differential equations with delay and impulses.

On the other hand, controllability is one of the important fundamental concepts in mathematical control theory and plays an important role in both deterministic and stochastic control system. In many dynamical systems, the control does not affect the complete state of the dynamical system but only a part on it. Further, very often in real industrial processes it is possible to observe only a certain part of the complete state of the dynamical system. This, the dynamical systems must be treated by the weaker concept of controllability, namely approximate controllability.

The existence, controllability and other qualitative and quantitative attributes of stochastic FDEs are the most progressing area of pursuit, for instance, see [5, 12, 16, 17, 25, 39–44]. In particular, Toufik Guendouzi et al. [16, 17] reviewed existence and approximate controllability of different types of fractional stochastic differential and integro-differential systems with SDD in Hilbert spaces under different suitable fixed point theorems. Lately, Zhang et al. [44] derived a new set of sufficient conditions for approximate controllability of impulsive fractional stochastic differential equations with state-dependent delay in Hilbert spaces with the help of fractional calculus and stochastic analysis. Moreover, Yan et al. [39, 40] investigated for approximate controllability of impulsive partial neutral stochastic functional integro-differential inclusion with infinite delay. Recently, Sakthivel et al. [29, 31] reviewed the approximate controllability of fractional neutral stochastic differential inclusions with nonlocal conditions and infinite delay by utilizing the Krasnoselskii's fixed point theorem. Very recently, Vijayakumar et al. [24, 36, 37] derived the controllability and approximate controllability results for abstract neutral integro-differential inclusions with infinite delay in Hilbert spaces.

The best of our knowledge, it appears that little is thought about approximate controllability results for IFNSIDS with non-local and SDD conditions in Hilbert spaces. The point of this manuscript is to analyze this fascinating model (1.1)- (1.3).

The rest of this paper is organized as follows. In Section 2 is focused on call to mind of some crucial perspectives that will be utilized in this work to accomplish our primary results. In Section 3, we declare and present the existence results about by proposes of Schauder fixed point theorem. In Section 4, an example is given to illustrate our results.

2 Preliminaries

Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ and $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$ denote two real separable Hilbert spaces. For our convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathcal{H}, \mathcal{K}$ and $(\cdot,\cdot)$ to denote the inner product without any confusion. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space furnished with a normal filtration $\mathcal{F}_t, t \in \mathscr{I}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathcal{P}$-null sets), and $\mathbb{E}(\cdot)$ denotes the expectation with respect to the measure $\mathcal{P}$. An $\mathcal{H}$-valued random variable is an $\mathcal{F}$ measurable function $u(t)\colon \Omega \to \mathcal{H}$, and a collection of random variable $\mathcal{W} = \{u(t,\omega)\colon \Omega \to \mathcal{H}|_{t\in T}\}$ is called a stochastic process. We suppress the dependence on $\omega \in \Omega$ and write $u(t)$ instead of $u(t,\omega)$ and $u(t)\colon \mathscr{I} \to \mathcal{H}$ in the place of $\mathcal{W}$. Assume that $\{\beta_n\}_{n\geq 1}$ be a sequence of real valued independent Brownian motions, defined by $W(t)$ = $\sum\limits_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)\chi_n$, $t \geq 0$, where $\{\chi_n\}_{n\geq 1}$ is complete orthonormal system in $\mathcal{K}$ and $\lambda_n \geq 0$ $(n = 1, 2, \dots)$ are non-negative real numbers. Let $Q \in \mathcal{L}(\mathcal{K},\mathcal{K})$ be an operator satisfying $Q\chi_n = \lambda_n\chi_n$ with $tr(Q) = \sum\limits_{n=1}^{\infty} \lambda_n < \infty$. Then, the above $\mathcal{K}$-valued stochastic process $W(t)$ is a Q-wiener process. Let us assume $\mathcal{F}_t = \sigma(W(s) : 0 \leq s \leq t)$ is the σ-algebra generated by W and $\mathcal{F}_T = \mathcal{F}$.

Let $\mathcal{L}(\mathcal{K},\mathcal{H})$ denote the space of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$ equipped with the usual operator norm $\|\cdot\|$. For $\varphi \in \mathcal{L}(\mathcal{K},\mathcal{H})$ and define

$$\|\varphi\|_Q^2 = tr(\varphi Q \varphi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\varphi\chi_n\|^2.$$

If $\|\varphi\|_Q^2 < \infty$, then φ is called a Q-Hilbert-Schmidt operator. Let $\mathcal{L}_Q(\mathcal{K},\mathcal{H})$ denote the space of all Q-Hilbert-Schmidt operators φ. The completion $\mathcal{L}_Q(\mathcal{K},\mathcal{H})$ of $\mathcal{L}(\mathcal{K},\mathcal{H})$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\varphi\|_Q^2 = <\varphi,\varphi>$ is a Hilbert space with the above norm topology.

Without loss of generality, we assume that $0 \in \wp(\mathscr{A})$, the resolvent set of $\mathscr{A}$. Then for $0 < \eta \leq 1$, it is possible to define the fractional power $\mathscr{A}^\eta$ as a closed linear operator on its domain $D(\mathscr{A}^\eta)$, being dense in $\mathcal{H}$, and we denote by $\mathcal{H}_\eta$ the Banach space of $D(\mathscr{A}^\eta)$ endowed with the norm $\|u\|_\eta = \|\mathscr{A}^\eta u\|$, which is equivalent to the graph norm of $\mathscr{A}^\eta$.

Lemma 2.1. *[27] Suppose that the preceding conditions are satisfied.*

(i) *Let* $0 < \eta \leq 1$, *then* $\mathcal{H}_\eta$ *is a Banach space.*

(ii) *If* $0 < \nu \leq \eta$, *then the embedding* $\mathcal{H}_\nu \subset \mathcal{H}_\eta$ *is compact whenever the resolvent operator of* $\mathscr{A}$ *is compact.*

(iii) *For every* $\eta \in (0,1]$, *there exists a positive constant* $\mathcal{C}_\eta$ *such that*

$$\|\mathscr{A}^\eta \mathbb{T}(t)\| \leq \frac{\mathcal{C}_\eta}{t^\eta}, \quad t > 0.$$

It needs to be outlined that, once the delay is infinite, then we should talk about the theoretical phase space $\mathcal{B}$ in a beneficial way.

We assume that the phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a semi-normed linear space of $\mathcal{F}_0$-measurable functions mapping $(-\infty, 0]$ into $\mathcal{H}$ and fulfilling the subsequent elementary adages as a result of Hale and Kato (see the case in point in [15, 18, 19]).

If $u : (-\infty, T] \to \mathcal{H}, T > 0$, is continuous on $\mathscr{I}$ and $u_0 \in \mathcal{B}$, then for every $t \in \mathscr{I}$ the accompanying conditions hold:

(P_1) u_t is in $\mathcal{B}$;

(P_2) $\|u(t)\| \leq H\|u_t\|_{\mathcal{B}}$;

(P_3) $\|u_t\|_{\mathcal{B}} \leq \mathscr{E}_1(t)\sup\{\|u(s)\| : 0 \leq s \leq t\} + \mathscr{E}_2(t)\|u_0\|_{\mathcal{B}}$, where $H > 0$ is a constant and $\mathscr{E}_1(\cdot) : [0, +\infty) \to [0, +\infty)$ is continuous, $\mathscr{E}_2(\cdot) : [0, +\infty) \to [0, +\infty)$ is locally bounded and $\mathscr{E}_1, \mathscr{E}_2$ are independent of $u(\cdot)$.

(P_4) The function $t \to \varphi_t$ is well described and continuous from the set

$$\mathcal{R}(\varrho^-) = \{\varrho(s, \psi) : (s, \psi) \in \mathscr{I} \times \mathcal{B}\},$$

into $\mathcal{B}$ and there is a continuous and bounded function $J^\varphi : \mathcal{R}(\varrho^-) \to (0, \infty)$ to ensure that $\mathbb{E}\|\varphi_t\|^2_{\mathcal{B}} \leq J^\varphi(t)\mathbb{E}\|\varphi\|^2_{\mathcal{B}}$ for every $t \in \mathcal{R}(\varrho^-)$.

(P_5) The space $\mathcal{B}$ is complete.

Let $u : (-\infty, T] \to \mathcal{H}$ be an $\mathcal{F}_t$-adapted measurable process such that we have the $\mathscr{F}_0$-adapted process $u_0 = \varphi(t) \in \mathcal{L}^2(\Omega, \mathcal{B})$, then

$$\mathbb{E}\|u_t\|^2_{\mathcal{B}} \leq \mathscr{E}_1^{*2} \sup_{0 \leq s \leq T} \{\mathbb{E}\|u(s)\|^2\} + \mathscr{E}_2^{*2}\mathbb{E}\|\varphi\|^2_{\mathcal{B}},$$

where $\mathscr{E}_1^* = \sup_{s \in \mathscr{I}} \mathscr{E}_1(s)$ and $\mathscr{E}_2^* = \sup_{s \in \mathscr{I}} \mathscr{E}_2(s)$.

Lemma 2.2. *[14] Let u:* $(-\infty, T] \to \mathcal{H}$ *be a function in a way that* $u_0 = \varphi$ *and* $u \in \mathcal{PC}(\mathscr{I}, \mathcal{L}^2)$ *and if* (P_4) *hold, then*

$$\mathbb{E}\|u_s\|^2_{\mathcal{B}} \leq \mathscr{E}_1^{*2} \sup\{\mathbb{E}\|u(\theta)\|^2_{\mathcal{H}} : \theta \in [0, \max\{0, s\}]\} + (\mathscr{E}_2^* + J^\varphi)^2 \mathbb{E}\|u_0\|^2_{\mathcal{B}}, \quad s \in \mathcal{R}(\varrho^-) \cup \mathscr{I},$$

where $J^\varphi = \sup_{t \in \mathcal{R}(\varrho^-)} J^\varphi(t)$.

Recognize the space

$$\mathscr{B}_T = \left\{u : (-\infty, T] \to \mathcal{H} \text{ such that } u_0 \in \mathcal{B} \text{ and the constraint } u|_{\mathscr{I}} \in \mathcal{PC}(\mathscr{I}, \mathcal{L}^2)\right\}.$$

The function $\|\cdot\|_{\mathscr{B}_T}$ to be a seminorm in $\mathscr{B}_T$, it is described by

$$\|u\|_{\mathscr{B}_T} = \|\varphi\|_{\mathcal{B}} + \sup\left\{\left(\mathbb{E}\|u(s)\|^2\right)^{\frac{1}{2}} : s \in [0, T]\right\}, \ u \in \mathscr{B}_T.$$

Now, we provide some fundamental definitions and results of the fractional calculus theory that happen to be utilized additionally within this manuscript.

Definition 2.1. *[21] The fractional integral of order γ with the lower limit zero for a function f is determined as*

$$I_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0, \quad \gamma > 0,$$

offered the right part is point-wise described on $[0, +\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. *[21] The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f \in \mathcal{L}^1(\mathscr{I}, \mathcal{H})$ is characterized as*

$$D_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{1-n+\gamma}} ds, \quad t > 0, \quad n-1 < \gamma < n.$$

Definition 2.3. *[21, 28] The Caputo derivative of order γ for a function $f \in \mathcal{L}^1(\mathscr{I}, \mathcal{H})$ could be consisting as*

$${}^C D_t^{\gamma} f(t) = D_t^{\gamma}(f(t) - f(0)), \quad t > 0, \quad 0 < \gamma < 1.$$

Definition 2.4. *[45, Definition 4.59] The Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \; \alpha, \beta > 0, \; z \in \widetilde{C},$$

where $\widetilde{C}$ denotes the complex plane. When $\beta = 1$, fix $E_{\alpha}(z) = E_{\alpha,1}(z)$.

Definition 2.5. *[45] The Mainardi's function has the form*

$$\phi_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n - \alpha + 1)}, \; 0 < \alpha < 1, \; z \in \widetilde{C}.$$

Presently, we are in a position to characterize the mild solution for the system (1.1)-(1.3). For this, first we assume that the approximate controllability of its linear fractional differtial system

$${}^C D_t^{\alpha} x(t) = \mathscr{A} x(t) + Bv(t) + \mathscr{F}(t) \frac{dw(t)}{dt}, \tag{2.1}$$

$$x(0) = x_0, \tag{2.2}$$

where ${}^C D_t^{\alpha}$ and $\mathscr{A}$ are defined in (1.1)-(1.3). Now, we first consider the classical solutions to the problem (2.1)-(2.2). Then, based on the expression of such solutions, we define the mild solutions of the problem (2.1)-(2.2). At last, the relations between the analytic semigroup $\{\mathbb{T}(t)\}_{t \geq 0}$ and some solution operators is obtained.

For our convenient at this position to introduce the controllability operator associated with (2.1)-(2.2), thus

$$\Gamma_0^T = \int_0^T \mathcal{S}_{\alpha}(T-s) BB^* \mathcal{S}_{\alpha}^*(T-s) ds,$$

where B^* and $\mathcal{S}_{\alpha}^*$ are the adjoint of B and $\mathcal{S}_{\alpha}$ respectively. It is straightforward that the operator Γ_0^T is a linear bounded operator.

Let $u(T; u_0, v)$ be the state value of (1.1)-(1.3) at teriminal time T corresponding to the control v and the intial value u_0. Introduce the set $\mathscr{R}(T, u_0) = \{u(T; u_0, v) : v \in \mathcal{L}^2(\mathscr{I}, U)\}$, which is called the reachable set of the system (1.1)-(1.3) at terminal time T, its closure in $\mathcal{H}$ is denoted by $\overline{\mathscr{R}(T, u_0)}$.

Definition 2.6. *[44] The system (1.1)-(1.3) is said to be approximately controllable on $\mathscr{I}$ if $\overline{\mathscr{R}(T, u_0)} = \mathcal{L}^2(\Omega, \mathcal{H})$, that is, given an arbitrary $\epsilon > 0$, it is possible to steer from the point u_0 to within a distance ϵ from all points in the state space $\mathcal{H}$ at time T.*

Lemma 2.3. *[44] The linear fractional control system (2.1)-(2.2) is approximately controllable on $\mathscr{I}$ if and only if $\mu(\mu\mathcal{I} + \Gamma_0^T) \to 0$ as $\mu \to 0^+$ in the strong operator topology.*

Lemma 2.4. *([31, Lemma 3.2]) For any $\tilde{u}_T \in \mathcal{L}^2(\mathcal{F}_T, \mathcal{H})$, there exists $\tilde{\phi} \in \mathcal{L}^2_{\mathcal{F}}(\Omega; \mathcal{L}^2(0,T;\mathcal{L}^0_2))$ such that $\tilde{u}_T = \mathbb{E}\tilde{u}_T + \int_0^T \tilde{\phi}(s)dw(s)$.*

Now for any $\mu > 0$ and $\tilde{u}_T \in \mathcal{L}^2(\mathcal{F}_T, \mathcal{H})$, we define the control function

$$v^{\mu}(t) = \begin{cases} B^* S^*_\alpha(T-t)(\mu\mathcal{I} + \Gamma^T_0)^{-1}\left[\mathbb{E}\tilde{u}_T + \int_0^T \tilde{\phi}(s)dw(s) - \mathcal{T}_\alpha(T)[\varphi(0) - h(u_T) - \mathscr{G}(0,\varphi)]\right] \\ -B^*\mathcal{S}^*_\alpha(T-t)(\mu\mathcal{I}+\Gamma^T_0)^{-1}\mathscr{G}(T, u_{\varrho(T,u_T)}) \\ -B^*\mathcal{S}^*_\alpha(T-t)\int_0^T(\mu\mathcal{I}+\Gamma^T_s)^{-1}\mathscr{A}\mathcal{S}_\alpha(T-s)\mathscr{G}\left(s, u_{\varrho(s,u_s)}\right)ds \\ -B^*\mathcal{S}^*_\alpha(T-t)\int_0^T(\mu\mathcal{I}+\Gamma^T_s)^{-1}\mathcal{S}_\alpha(T-s)\mathscr{F}\left(s, u_{\varrho(s,u_s)}, \int_0^s e_1(s,\tau,u_{\varrho(\tau,u_\tau)})d\tau\right)ds \\ -B^*S^*_\alpha(T-t)\int_0^T(\mu\mathcal{I}+\Gamma^T_s)^{-1}\mathcal{S}_\alpha(T-s)\Sigma\left(s, u_{\varrho(s,u_s)}, \int_0^s e_2(s,\tau,u_{\varrho(\tau,u_\tau)})d\tau\right)dw(s) \\ -B^*S^*_\alpha(T-t)(\mu\mathcal{I}+\Gamma^T_0)^{-1}\sum_{0<t_k<t}\mathcal{T}_\alpha(T-t_k)\mathcal{I}_k(u(t_k^-)). \end{cases}$$

Lemma 2.5. *[33, Lemma 6] Using $\mathscr{A}$ to denote the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$, then if $\mathscr{F}$ satisfies a uniform Hölder condition with exponent $\beta \in (0,1]$, the solution of the Cauchy system (2.1)-(2.2) are fixed points of the subsequent operator equation:*

$$\Psi x(t) = \mathcal{T}_\alpha(t)x_0 + \int_0^t \mathcal{S}_\alpha(t-s)Bv(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s)dw(s), \tag{2.3}$$

where

$$\mathcal{T}_\alpha(t) = \frac{1}{2\pi i}\int_C e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha, \mathscr{A})d\lambda \quad \text{and} \quad \mathcal{S}_\alpha(t) = \frac{1}{2\pi i}\int_C e^{\lambda t}R(\lambda^\alpha,\mathscr{A})d\lambda.$$

Here C is a suitable path satisfying $\lambda^\alpha \notin \mu + S_\theta$ for some $\lambda \in C$.

Proof. According to the Definitions of 2.1 and 2.2, we modify the Cauchy system (2.1)-(2.2) in the equivalent integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\mathscr{A}x(s)}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{Bv(s)}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\mathscr{F}(s)}{(t-s)^{1-\alpha}}dw(s). \tag{2.4}$$

Let $\lambda > 0$. Making use of the Laplace transform

$$(\mathscr{L}x)(\lambda) = \int_0^\infty e^{-\lambda s}x(s)ds, \quad (\mathscr{L}v(t))(\lambda) = \int_0^\infty e^{-\lambda s}v(s)ds,$$

$$\text{and} \quad (\mathscr{L}\mathscr{F}(t))(\lambda) = \int_0^\infty e^{-\lambda s}\mathscr{F}(s)dw(s)$$

to (2.4) we receive

$$\begin{aligned}(\mathscr{L}x)(\lambda) &= \int_0^\infty e^{-\lambda s}\left[x_0 + \frac{1}{\Gamma(\alpha)}\int_0^s \frac{\mathscr{A}x(\theta)}{(s-\theta)^{1-\alpha}}d\theta + \frac{1}{\Gamma(\alpha)}\int_0^s\frac{Bv(\theta)}{(s-\theta)^{1-\alpha}}d\theta \right.\\ &\quad \left. + \frac{1}{\Gamma(\alpha)}\int_0^s \frac{\mathscr{F}(\theta)}{(s-\theta)^{1-\alpha}}dw(\theta)\right]ds \\ &= \int_0^\infty e^{-\lambda s}x_0 ds + \int_0^\infty e^{-\lambda s}\left[\frac{1}{\Gamma(\alpha)}\int_0^s\frac{\mathscr{A}x(\theta)}{(s-\theta)^{1-\alpha}}d\theta\right]ds \\ &\quad + \int_0^\infty e^{-\lambda s}\left[\frac{1}{\Gamma(\alpha)}\int_0^s \frac{Bv(\theta)}{(s-\theta)^{1-\alpha}}d\theta\right]ds \\ &\quad + \int_0^\infty e^{-\lambda s}\left[\frac{1}{\Gamma(\alpha)}\int_0^s\frac{\mathscr{F}(\theta)}{(s-\theta)^{1-\alpha}}dw(\theta)\right]ds \\ &= \frac{1}{\lambda}\left[e^{-\lambda s}\right]^0_\infty x_0 + \frac{1}{\lambda^\alpha}\mathscr{A}(\mathscr{L}x)(\lambda) + \frac{1}{\lambda^\alpha}B(\mathscr{L}v(t))(\lambda) + \frac{1}{\lambda^\alpha}(\mathscr{L}\mathscr{F}(t))(\lambda)\end{aligned}$$

$$
\begin{aligned}
(\mathscr{L}x)(\lambda)-\frac{1}{\lambda^{\alpha}}\mathscr{A}(\mathscr{L}x)(\lambda)&=\frac{1}{\lambda}x_0++\frac{1}{\lambda^{\alpha}}B(\mathscr{L}v(t))(\lambda)+\frac{1}{\lambda^{\alpha}}(\mathscr{L}\mathscr{F}(t))(\lambda)\\
(\lambda^{\alpha}I-\mathscr{A})(\mathscr{L}x)(\lambda)&=\frac{\lambda^{\alpha}}{\lambda}x_0+B(\mathscr{L}v(t))(\lambda)+(\mathscr{L}\mathscr{F}(t))(\lambda)\\
(\mathscr{L}x)(\lambda)&=\lambda^{\alpha-1}(\lambda^{\alpha}I-\mathscr{A})^{-1}x_0+(\lambda^{\alpha}I-\mathscr{A})^{-1}B(\mathscr{L}v(t))(\lambda)\\
&\quad+(\lambda^{\alpha}I-\mathscr{A})^{-1}(\mathscr{L}\mathscr{F}(t))(\lambda).
\end{aligned}
$$

Using $\lambda^{\alpha}(\lambda^{\alpha}-\mathscr{A})^{-1}=I+\mathscr{A}(\lambda^{\alpha}-\mathscr{A})^{-1}$, the above equation is then inverse Laplace transformed to obtain

$$x(t)=\mathcal{T}_{\alpha}(t)x_0+\int_0^t\mathcal{S}_{\alpha}(t-s)Bv(s)ds+\int_0^t\mathcal{S}_{\alpha}(t-s)\mathscr{F}(s)dw(s).$$

It is noted that $\mathscr{F}$ satisfy a uniform Hölder condition with exponent $\beta\in(0,1)$. Hence, the classical solutions of Cauchy system (2.1)-(2.2) are fixed points of the operator equation (2.3). □

In view of Lemma 2.5, we determine the mild solutions of the system (2.1)-(2.2).

Definition 2.7. *A function $x:\mathscr{I}\to\mathcal{H}$ is considered to be a mild solution of problem (2.1)-(2.2) if $x\in C(\mathscr{I},\mathcal{H})$ fulfills the accompanying integral equation:*

$$x(t)=\mathcal{T}_{\alpha}(t)x_0+\int_0^t\mathcal{S}_{\alpha}(t-s)Bv(s)ds+\int_0^t\mathcal{S}_{\alpha}(t-s)\mathscr{F}(s)dw(s),\quad t\in\mathscr{I}.$$

Remark 2.1. *It is straightforward to confirm that the classical solution of the system (2.1)-(2.2) is a mild solution of the same system. Thus, Definition 2.7 is well defined (see [23, 27]).*

Lemma 2.6. *[33, Lemma 9] Assuming $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup, given by $\{\mathbb{T}(t)\}_{t\geq 0}$ and $0\in\wp(\mathscr{A})$, then we have*

$$\mathcal{S}_{\alpha}(t)=\alpha\int_0^{\infty}r\phi_{\alpha}(r)t^{\alpha-1}\mathbb{T}(t^{\alpha}r)dr\quad\text{and}\quad\mathcal{T}_{\alpha}(t)=\int_0^{\infty}\phi_{\alpha}(r)\mathbb{T}(t^{\alpha}r)dr.\tag{2.5}$$

Here $\phi_{\alpha}(r)$ is the probability density function characterized on $(0,\infty)$ in such a way that its Laplace transform has the form

$$\int_0^{\infty}e^{-rx}\phi_{\alpha}(r)dr=\sum_{j=0}^{\infty}\frac{(-x)^j}{\Gamma(1+\alpha j)},\quad x>0,$$

which fulfills

$$\int_0^{\infty}\phi_{\alpha}(r)dr=1\quad\text{and}\quad\int_0^{\infty}r^{\eta}\phi_{\alpha}(r)dr\leq 1,\quad 0\leq\eta\leq 1.$$

Proof. For all $x\in D(\mathscr{A})\subset\mathcal{H}$, we have

$$(\lambda-\mathscr{A})^{-1}x=\int_0^{\infty}e^{-\lambda s}\mathbb{T}(s)xds.$$

Let

$$\int_0^{\infty}e^{-\lambda r}\psi_{\alpha}(r)dr=e^{-\lambda^{\alpha}},$$

where $\alpha\in(0,1)$, $\psi_{\alpha}(r)=\frac{1}{\pi}\sum\limits_{1\leq n<\infty}(-1)^n r^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha)$, and $r\in(0,\infty)$(see[26]). Thus, we get

$$
\begin{aligned}
(\lambda^{\alpha}-\mathscr{A})^{-1}x&=\int_0^{\infty}e^{-\lambda^{\alpha}s}\mathbb{T}(s)xds\\
&=\int_0^{\infty}\alpha t^{\alpha-1}e^{-(\lambda t)^{\alpha}}\mathbb{T}(t^{\alpha})xdt\\
&=\int_0^{\infty}\alpha t^{\alpha-1}\left[\int_0^{\infty}e^{-\lambda tr}\psi_{\alpha}(r)dr\right]\mathbb{T}(t^{\alpha})xdt\\
&=\int_0^{\infty}\alpha\left[\int_0^{\infty}e^{-\lambda t}\psi_{\alpha}(r)dr\right]\mathbb{T}\left(\frac{t^{\alpha}}{r^{\alpha}}\right)x\frac{t^{\alpha-1}}{r^{\alpha}}dt\\
&=\int_0^{\infty}e^{-\lambda t}\left(\alpha\int_0^{\infty}r\phi_{\alpha}(r)t^{\alpha-1}\mathbb{T}(t^{\alpha}r)xdr\right)dt,
\end{aligned}\tag{2.6}
$$

where $\phi_\alpha(r) = \left(\frac{1}{\alpha}\right) r^{-1-\frac{1}{\alpha}} \psi_\alpha\left(r^{\frac{-1}{\alpha}}\right)$ is the probability density function outlined on $(0, \infty)$ in such a way that

$$\int_0^\infty \phi_\alpha(r)dr = 1 \quad \text{and} \quad \int_0^\infty r^\eta \phi_\alpha(r)dr \leq 1, \quad 0 \leq \eta \leq 1.$$

In perspective of Lemma 2.5 and equation (2.6), we sustain

$$\begin{aligned}
\mathcal{S}_\alpha(t) &= \frac{1}{2\pi i} \int_C e^{\lambda t} R(\lambda^\alpha, \mathscr{A}) d\lambda \\
&= \int_0^\infty e^{\lambda t} (\lambda^\alpha - \mathscr{A})^{-1} dt \\
&= \alpha \int_0^\infty r\phi_\alpha(r) t^{\alpha-1} \mathbb{T}(t^\alpha r) dr.
\end{aligned}$$

Further, we calculate the estimation of $\mathcal{S}_\alpha(t)$:

$$\begin{aligned}
\|\mathcal{S}_\alpha(t)\| &= \left\| \alpha \int_0^\infty r\phi_\alpha(r) t^{\alpha-1} \mathbb{T}(t^\alpha r) dr \right\| \\
&\leq \alpha \left[\int_0^\infty r\phi_\alpha(r) dr \right] t^{\alpha-1} \|\mathbb{T}(t^\alpha r)\| \\
&\leq \alpha \frac{\Gamma(2)}{\Gamma(1+\alpha)} t^{\alpha-1} \mathcal{M} \\
&\leq \frac{\mathcal{M}}{\Gamma(\alpha)} t^{\alpha-1},
\end{aligned}$$

where $\int_0^\infty r^\beta \phi_\alpha(r) dr = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}$.

Then again, for all $x \in D(\mathscr{A}) \subset \mathcal{H}$, we notice that

$$\begin{aligned}
\lambda^{\alpha-1}(\lambda^\alpha - \mathscr{A})^{-1}x &= \int_0^\infty \lambda^{\alpha-1} e^{-\lambda^\alpha s} \mathbb{T}(s) x ds \\
&= \int_0^\infty \alpha(\lambda t)^{\alpha-1} e^{-(\lambda t)^\alpha} \mathbb{T}(t^\alpha) x dt \\
&= \int_0^\infty \frac{-1}{\lambda} \frac{d}{dt} \left[e^{-(\lambda t)^\alpha}\right] \mathbb{T}(t^\alpha) x dt \\
&= \int_0^\infty \frac{-1}{\lambda} \frac{d}{dt} \left[\int_0^\infty e^{-\lambda t r} \psi_\alpha(r) dr \right] \mathbb{T}(t^\alpha) x dt \\
&= \int_0^\infty \left[\int_0^\infty \frac{-1}{\lambda} \left[-\lambda r e^{-\lambda t r}\right] \psi_\alpha(r) dr \right] \mathbb{T}(t^\alpha) x dt \\
&= \int_0^\infty \int_0^\infty r e^{-\lambda t r} \psi_\alpha(r) dr \mathbb{T}(t^\alpha) x dt \\
&= \int_0^\infty e^{-\lambda t} \left[\int_0^\infty \psi_\alpha(r) \mathbb{T}\left(\frac{t^\alpha}{r^\alpha}\right) x dr \right] dt \\
&= \int_0^\infty e^{-\lambda t} \left[\int_0^\infty \phi_\alpha(r) \mathbb{T}(t^\alpha r) x dr \right] dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{T}_\alpha(t) &= \frac{1}{2\pi i} \int_C e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, \mathscr{A}) d\lambda \\
&= \int_0^\infty e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - \mathscr{A})^{-1} dt \\
&= \int_0^\infty \phi_\alpha(r) \mathbb{T}(t^\alpha r) dr.
\end{aligned}$$

Moreover, the estimation of $\mathcal{T}_\alpha(t)$ is

$$\begin{aligned}\|\mathcal{T}_\alpha(t)\| &= \left\| \int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)dr \right\| \\ &\le \left(\int_0^\infty \phi_\alpha(r)dr \right) \|\mathbb{T}(t^\alpha r)\| \\ &\le \mathcal{M},\end{aligned}$$

where $\int_0^\infty \phi_\alpha(r)dr = 1.$ □

Before we characterize the mild solution for the system (1.1)-(1.3), finally, we treat the following system:

$$^C D_t^\alpha \left[x(t) - \mathscr{G}(t, x(t))\right] = \mathscr{A}x(t) + Bv(t) + \mathscr{F}(t, x(t)) + \Sigma(t, x(t))\frac{dw(t)}{dt}, \quad t \neq t_k, \tag{2.7}$$

$$\Delta x(t_k) = \mathcal{I}_k(x(t_k^-)), \quad k = 1, 2, \cdots, n, \tag{2.8}$$

$$x(0) + h(x) = \varphi(0), \tag{2.9}$$

where $^C D_t^\alpha, B, v(t)$ and $\mathscr{A}$ are defined in (1.1)-(1.3) and $\mathscr{F}, \Sigma, \mathscr{G}$ are appropriate functions.

From the Definition of 2.1 and 2.2, the general integral equation of the system (2.7)-(2.9) can be expressed as

$$\begin{aligned}x(t) = {}& \varphi(0) - \mathscr{G}(0, \varphi) - h(x) + \mathscr{G}(t, x(t)) + \sum_{k=1}^{n} \mathcal{I}_k(x(t_k^-)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathscr{A}x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bv(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathscr{F}(s, x(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Sigma(s, x(s))dw(s).\end{aligned} \tag{2.10}$$

Presently, we take after the thought utilized as a part of the paper [46] and apply the Laplace transformation for (2.10), we get

$$\begin{aligned}u(\lambda) = {}& \lambda^{\alpha-1}(\lambda^\alpha I - \mathscr{A})^{-1}[\varphi(0) - \mathscr{G}(0, \varphi) - h(x)] + \lambda^\alpha(\lambda^\alpha I - \mathscr{A})^{-1}w(\lambda) + (\lambda^\alpha I - \mathscr{A})^{-1}v(\lambda) \\ &+ (\lambda^\alpha I - \mathscr{A})^{-1}By(\lambda) + (\lambda^\alpha I - \mathscr{A})^{-1}z(\lambda) + \lambda^{\alpha-1}(\lambda^\alpha I - \mathscr{A})^{-1} \sum_{k=1}^{n} \mathcal{I}_k(x(t_k^-)),\end{aligned}$$

where

$$\begin{aligned}&u(\lambda) = \int_0^\infty e^{-\lambda s}x(s)ds, \quad v(\lambda) = \int_0^\infty e^{-\lambda s}\mathscr{F}(s, x(s))ds, \quad w(\lambda) = \int_0^\infty e^{-\lambda s}\mathscr{G}(s, x(s))ds, \\ &y(\lambda) = \int_0^\infty e^{-\lambda s}v(s)ds, \quad z(\lambda) = \int_0^\infty e^{-\lambda s}\Sigma(s, x(s))dw(s).\end{aligned}$$

At that point by the same calculations in [46] and the properties of the Laplace transform, we obtain the mild solution of the system (2.7)-(2.9) as

$$x(t) = \begin{cases} \mathcal{T}_\alpha(t)[\varphi(0) - \mathscr{G}(0, \varphi) - h(x)] + \mathscr{G}(t, x(t)) + \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, x(s))ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)Bv(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s, x(s))\, ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)\Sigma(s, x(s))\, dw(s) + \sum_{0<t_k<t} \mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(x(t_k^-)). \end{cases} \tag{2.11}$$

where $\mathcal{T}_\alpha$ and $\mathcal{S}_\alpha$ are same as defined in (2.5).

Next, we shall show that this mild solution satisfy the system (2.7)-(2.9). To prove this, first we prove the following crucial lemma.

Lemma 2.7. *[32, Lemma 3.3] Assuming $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup, given by $\{\mathbb{T}(t)\}_{t\geq 0}$ and if $0<\alpha<1$, then*

$$^{C}D_t^{\alpha}[\mathcal{T}_\alpha(t)x_0] = \mathscr{A}[\mathcal{T}_\alpha(t)x_0],$$

and

$$\begin{aligned}
&^{C}D_t^{\alpha}\left(\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}(s,x(s)) + Bv(s) + \mathscr{F}(s,x(s)) + \Sigma(s,x(s))\frac{dw(s)}{ds}\right]ds\right)\\
&= \mathscr{A}\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}(s,x(s)) + Bv(s) + \mathscr{F}(s,x(s)) + \Sigma(s,x(s))\frac{dw(s)}{ds}\right]ds\\
&\quad + \mathscr{A}\mathscr{G}(t,x(t)) + Bv(t) + \mathscr{F}(t,x(t)) + \Sigma(t,x(t))\frac{dw(t)}{dt},
\end{aligned}$$

where $\mathcal{T}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are same as defined in equation (2.5).

Proof. By the well known result from [32, Lemma 3.3], we have

$$^{C}D_t^{\alpha}[\mathcal{T}_\alpha(t)x_0] = \mathscr{A}[\mathcal{T}_\alpha(t)x_0].$$

Furthermore,

$$\begin{aligned}
&\mathscr{L}\left(\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}(s,x(s)) + Bv(s) + \mathscr{F}(s,x(s)) + \Sigma(s,x(s))\frac{dw(s)}{ds}\right]ds\right)\\
&= \mathscr{L}(\mathcal{S}_\alpha(t))\mathscr{L}\left(\mathscr{A}\mathscr{G}(t,x(t)) + Bv(t) + \mathscr{F}(t,x(t)) + \Sigma(t,x(t))\frac{dw(t)}{dt}\right)\\
&= R(\lambda^\alpha,\mathscr{A})\mathscr{L}\left(\mathscr{A}\mathscr{G}(t,x(t)) + Bv(t) + \mathscr{F}(t,x(t)) + \Sigma(t,x(t))\frac{dw(t)}{dt}\right) \qquad (2.12)
\end{aligned}$$

and

$$\begin{aligned}
&\mathscr{L}\left(\ ^{C}D_t^{\alpha}\left(\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}(s,x(s)) + Bv(s) + \mathscr{F}(s,x(s)) + \Sigma(s,x(s))\frac{dw(s)}{ds}\right]ds\right)\right)\\
&= \lambda^\alpha\left[R(\lambda^\alpha,\mathscr{A})\mathscr{L}\left(\mathscr{A}\mathscr{G}(t,x(t)) + Bv(t) + \mathscr{F}(t,x(t)) + \Sigma(t,x(t))\frac{dw(t)}{dt}\right)\right] - \lambda^{\alpha-1}.0\\
&= (\lambda^\alpha I - \mathscr{A} + \mathscr{A})R(\lambda^\alpha,\mathscr{A})\mathscr{L}\left(\mathscr{A}\mathscr{G}(t,x(t)) + Bv(t) + \mathscr{F}(t,x(t)) + \Sigma(t,x(t))\frac{dw(t)}{dt}\right)\\
&= (\lambda^\alpha I - \mathscr{A})R(\lambda^\alpha,\mathscr{A})\mathscr{L}\left(\mathscr{A}\mathscr{G}(t,x(t)) + Bv(t) + \mathscr{F}(t,x(t)) + \Sigma(t,x(t))\frac{dw(t)}{dt}\right)\\
&\quad + \mathscr{A}R(\lambda^\alpha,\mathscr{A})\mathscr{L}\left(\mathscr{A}\mathscr{G}(t,x(t)) + Bv(t) + \mathscr{F}(t,x(t)) + \Sigma(t,x(t))\frac{dw(t)}{dt}\right). \qquad (2.13)
\end{aligned}$$

Thus, it follows from (2.12) and (2.13) that

$$\begin{aligned}
&^{C}D_t^{\alpha}\left(\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}(s,x(s)) + Bv(s) + \mathscr{F}(s,x(s)) + \Sigma(s,x(s))\frac{dw(s)}{ds}\right]ds\right)\\
&= \mathscr{A}\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}(s,x(s)) + Bv(s) + \mathscr{F}(s,x(s)) + \Sigma(s,x(s))\frac{dw(s)}{ds}\right]ds\\
&\quad + \mathscr{A}\mathscr{G}(t,x(t)) + Bv(t) + \mathscr{F}(t,x(t)) + \Sigma(t,x(t))\frac{dw(t)}{dt}.
\end{aligned}$$

□

Now, it is time to show that the mild solution satisfy the model (2.7)-(2.9). From the equation (2.11), we have

$$\begin{aligned}
x(t) - \mathscr{G}(t,x(t)) &= \mathcal{T}_\alpha(t)[\varphi(0) - \mathscr{G}(0,\varphi) - h(x)] + \int_0^t \mathcal{S}_\alpha(t-s)\Big[\mathscr{A}\mathscr{G}(s,x(s)) + Bv(s)\\
&\quad + \mathscr{F}(s,x(s)) + \Sigma(s,x(s))\frac{dw(s)}{ds}\Big]ds + \sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(x(t_k^-)).
\end{aligned}$$

Taking Caputo derivative on both sides and with regard of above Lemma 2.7 , we have

$$
\begin{aligned}
{}^{C}D_t^{\alpha}\Big(x(t)-\mathscr{G}(t,x(t))\Big) &= {}^{C}D_t^{\alpha}\Big(\mathcal{T}_\alpha(t)[\varphi(0)-\mathscr{G}(0,\varphi)-h(x)]\Big)+{}^{C}D_t^{\alpha}\bigg(\int_0^t \mathcal{S}_\alpha(t-s)\Big[\mathscr{A}\mathscr{G}(s,x(s)) \\
&\quad + Bv(s)+\mathscr{F}(s,x(s))+\Sigma(s,x(s))\frac{dw(s)}{ds}\Big]ds\bigg) \\
&\quad +{}^{C}D_t^{\alpha}\Big(\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(x(t_k^-))\Big) \\
&= \mathscr{A}\mathcal{T}_\alpha(t)[\varphi(0)-\mathscr{G}(0,\varphi)-h(x)]+\mathscr{A}\bigg(\int_0^t \mathcal{S}_\alpha(t-s)\Big[\mathscr{A}\mathscr{G}(s,x(s))+Bv(s) \\
&\quad +\mathscr{F}(s,x(s))+\Sigma(s,x(s))\frac{dw(s)}{ds}\Big]ds\bigg)+\mathscr{A}\mathscr{G}(t,x(t))+Bv(t)+\mathscr{F}(t,x(t)) \\
&\quad +\Sigma(t,x(t))\frac{dw(t)}{dt}+\mathscr{A}\Big(\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(x(t_k^-))\Big) \\
&= \mathscr{A}\bigg(\mathcal{T}_\alpha(t)[\varphi(0)-\mathscr{G}(0,\varphi)-h(x)]+\mathscr{G}(t,x(t))+\int_0^t \mathcal{S}_\alpha(t-s)\Big[\mathscr{A}\mathscr{G}(s,x(s)) \\
&\quad +Bv(s)+\mathscr{F}(s,x(s))+\Sigma(s,x(s))\frac{dw(s)}{ds}\Big]ds+\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(x(t_k^-))\bigg) \\
&\quad +Bv(t)+\mathscr{F}(t,x(t))+\Sigma(t,x(t))\frac{dw(t)}{dt} \\
&= \mathscr{A}x(t)+Bv(t)+\mathscr{F}(t,x(t))+\Sigma(t,x(t))\frac{dw(t)}{dt}.
\end{aligned}
$$

That is

$$
{}^{C}D_t^{\alpha}\big(x(t)-\mathscr{G}(t,x(t))\big) = \mathscr{A}x(t)+Bv(t)+\mathscr{F}(t,x(t))+\Sigma(t,x(t))\frac{dw(t)}{dt}.
$$

From the above discussion, we observe that our definition of a mild solution satisfies the given system (2.7)-(2.9).

In accordance with the above discussion, we determine the mild solution of the model (1.1)-(1.3).

Definition 2.8. *[44, Definition 2.1] A stochastic process $u:(-\infty,T]\to\mathcal{H}$ is called a mild solution of the system (1.1)-(1.3) if*

(i) $u(t)$ *is measurable and* $\mathcal{F}_t$*-adapted for each* $t\in\mathscr{I}$;

(ii) $\Delta u(t_k)=u(t_k^+)-u(t_k^-)=\mathcal{I}_k(x(t_k^-)),\quad k=1,2,\dots n;$

(iii) $u(0)+h(u)=\varphi;$

(iv) $u(t)$ *is continuous on* $\mathscr{I}$, *the function* $\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s,u_{\varrho(s,u_s)})$ *is integrable and the following stochastic integral equation is satisfied,*

$$
u(t)=\begin{cases}
\mathcal{T}_\alpha(t)[\varphi(0)-h(u)-\mathscr{G}(0,\varphi)]+\mathscr{G}(t,u_{\varrho(t,u_t)})+\int_0^t\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s,u_{\varrho(s,u_s)})ds \\
+\int_0^t\mathcal{S}_\alpha(t-s)Bv^{\mu}ds+\int_0^t\mathcal{S}_\alpha(t-s)\mathscr{F}\left(s,u_{\varrho(s,u_s)},\int_0^s e_1(s,\tau,u_{\varrho(\tau,u_\tau)})d\tau\right)ds \\
+\int_0^t\mathcal{S}_\alpha(t-s)\Sigma\left(s,u_{\varrho(s,u_s)},\int_0^s e_2(s,\tau,u_{\varrho(\tau,u_\tau)})d\tau\right)dw(s) \\
+\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(u(t_k^-)).
\end{cases}\tag{2.14}
$$

(v) $u_0(\cdot)=\varphi\in\mathcal{B}$ *on* $(-\infty,0]$ *satisfying* $\|\varphi\|_{\mathcal{B}}^2<\infty$.

3 The main results

In this segment, we show and demonstrate the controllability of solutions for the model (1.1)-(1.3) under Schauder [30] fixed point theorem together with operator semigroups and fractional calculus.

Presently, we itemizing the subsequent suppositions:

(H0) $\mathcal{S}_\alpha(t)$, $t > 0$ is compact.

(H1) The function $\mathscr{G} : \mathscr{I} \times \mathcal{B} \to \mathcal{H}$ is continuous and there exist some constants $\beta \in (0,1)$ and $\mathcal{M}_{\mathscr{G}} > 0$ such that $\mathscr{G}$ is $\mathcal{H}_\beta$-valued and it satisfies the following conditions.

$$\mathbb{E}\|\mathscr{A}^\beta \mathscr{G}(t,x)\|^2 \le \mathcal{M}_{\mathscr{G}}(1 + \|x\|^2_{\mathcal{B}}), \quad t \in \mathscr{I}, \quad x \in \mathcal{B}.$$

(H2) The function $\mathscr{F} : \mathscr{I} \times \mathcal{B} \times \mathcal{H} \to \mathcal{H}$ is continuous and there exist two continuous functions $\mathscr{F}_1, \mathscr{F}_2 : \mathscr{I} \to (0,\infty)$ such that

$$\mathbb{E}\|\mathscr{F}(t,x,\phi)\|^2_{\mathcal{H}} \le \mathscr{F}_1(t)\|x\|^2_{\mathcal{B}} + \mathscr{F}_2(t)\mathbb{E}\|\phi\|^2_{\mathcal{H}}, \qquad (t,x,\phi) \in \mathscr{I} \times \mathcal{B} \times \mathcal{H},$$

and $\mathscr{F}_1^* = \sup\limits_{s\in[0,t]} \mathscr{F}_1(s)$, $\quad \mathscr{F}_2^* = \sup\limits_{s\in[0,t]} \mathscr{F}_2(s)$.

(H3) The function $e_i : \mathcal{D} \times \mathcal{B} \to \mathcal{H}$, where $\mathcal{D} = \{(t,s) \in \mathscr{I} \times \mathscr{I}; 0 \le s \le t \le T\}$ satisfies:

(i) For each $(t,s) \in \mathcal{D}$, the function $e_i(t,s,.) : \mathcal{B} \to \mathcal{H}$ is continuous, and for each $\phi \in \mathcal{B}$, the function $e_i(\cdot,\cdot,\phi) : \mathcal{D} \to \mathcal{H}$ is strongly measurable.

(ii) There exist constants $\widetilde{\mathcal{M}}_0, \widetilde{\mathcal{M}}_1 > 0$ such that for all $t,s \in \mathscr{I}$ and $x \in \mathcal{B}$,

$$\mathbb{E}\|e_i(t,s,x)\|^2 \le \widetilde{\mathcal{M}}_j(1 + \|x\|^2_{\mathcal{B}}), \quad \text{for} \quad i = 1,2 \quad \text{and} \quad j = 0,1.$$

(H4) The function Σ: $\mathscr{I} \times \mathcal{B} \times \mathcal{H} \to \mathcal{L}(\mathcal{K},\mathcal{H})$ is continuous and there exist two continuous functions $\Sigma_1, \Sigma_2 : \mathscr{I} \to (0,\infty)$ such that

$$\mathbb{E}\|\Sigma(t,x,\phi)\|^2_{\mathcal{H}} \le \Sigma_1(t)\|x\|^2_{\mathcal{B}} + \Sigma_2(t)\mathbb{E}\|\phi\|^2_{\mathcal{H}}, \qquad (t,x,\phi) \in \mathscr{I} \times \mathcal{B} \times \mathcal{H},$$

and $\Sigma_1^* = \sup\limits_{s\in[0,t]} \Sigma_1(s)$, $\quad \Sigma_2^* = \sup\limits_{s\in[0,t]} \Sigma_2(s)$.

(H5) The function $\mathcal{I}_k : \mathcal{B} \to \mathcal{H}, k = 1,2,\dots,n$ are continuous and there exist non-decreasing continuous functions $\mathcal{M}_{\mathcal{I}_k} : \mathcal{R}^+ \to \mathcal{R}^+$ such that for each $x \in \mathcal{B}$,

$$\mathbb{E}\|\mathcal{I}_k(x)\|^2 \le \mathcal{M}_{\mathcal{I}_k}(\mathbb{E}\|x\|^2_{\mathcal{B}}), \qquad \liminf_{r\to\infty} \frac{\mathcal{M}_{\mathcal{I}_k}(r)}{r} = \gamma_k < \infty.$$

(H6) The function $h : \mathcal{B} \to \mathcal{H}$ is continuous and there exists a constant $\mathcal{M}_h > 0$ such that for each $x \in \mathcal{B}$, we sustain

$$\mathbb{E}\|h(x)\|^2 \le \mathcal{M}_h\|x\|^2_{\mathcal{B}}.$$

Presently, we are in a position to derive the controllability results for the model (1.1)-(1.3).

Theorem 3.1. *Assume that the assumptions (H0)-(H6) hold. Then the system (1.1)-(1.3) has a mild solution on $\mathscr{I}$ provided that*

$$32\left(1 + \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^4 \frac{T^{4\alpha-2}}{\alpha^2}\right)\Bigg[\mathcal{M}^2\left(\mathcal{M}_h + H^2 n\sum_{k=1}^n \gamma_k\right) + \mathcal{M}_{\mathscr{G}}\left(\mathcal{N}_0{}^2 + \left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)T^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right)^2\right) + \left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\left[\mathscr{F}_1^* + tr(Q)\Sigma_1^* + \left(\mathscr{F}_2^*\widetilde{\mathcal{M}}_0 + \Sigma_2^* tr(Q)\widetilde{\mathcal{M}}_1\right)T\right]\Bigg]\mathscr{E}_1^{*2} < 1 \tag{3.1}$$

where $\mathcal{N}_0 = \|\mathscr{A}^{-\beta}\|$.

Proof. We will transform the model (1.1)-(1.3) into a fixed-point problem. Recognize the operator $\Upsilon : \mathscr{B}_T \to \mathscr{B}_T$ specified by

$$(\Upsilon u)(t) = \begin{cases} \varphi(t), \quad t \in (-\infty, 0], \\ \mathcal{T}_\alpha(t)[\varphi(0) - h(u) - \mathscr{G}(0,\varphi)] + \mathscr{G}(t, u_{\varrho(t,u_t)}) + \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, u_{\varrho(s,u_s)})ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)Bv^\mu(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}\left(s, u_{\varrho(s,u_s)}, \int_0^s e_1(s,\tau,u_{\varrho(\tau,u_\tau)})d\tau\right)ds \\ + \int_0^t \mathcal{S}_\alpha(t-s)\Sigma\left(s, u_{\varrho(s,u_s)}, \int_0^s e_2(s,\tau,u_{\varrho(\tau,u_\tau)})d\tau\right)dw(s) \\ + \sum_{0<t_k<t} \mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(u(t_k^-)), \quad t \in \mathscr{I}. \end{cases}$$

In perspective of Lemma 2.1 and for any $u \in \mathcal{H}$ and $\beta \in (0,1)$, we have

$$\begin{aligned} &\|\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, u_{\varrho(s,u_s)})\|^2_{\mathcal{H}} \\ &= \|\mathscr{A}^{1-\beta}\mathcal{S}_\alpha(t-s)\mathscr{A}^\beta\mathscr{G}(s, u_{\varrho(s,u_s)})\|^2_{\mathcal{H}} \\ &\le \left\| \left[\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathscr{A}^{1-\beta}\mathbb{T}((t-s)^\alpha r)dr\right]\mathscr{A}^\beta\mathscr{G}(s, u_{\varrho(s,u_s)})\right\|^2_{\mathcal{H}} \\ &\le \left(\alpha\mathcal{C}_{1-\beta}(t-s)^{\alpha\beta-1}\right)^2\left[\int_0^\infty r^\beta\phi_\alpha(r)dr\right]^2\|\mathscr{A}^\beta\mathscr{G}(s, u_{\varrho(s,u_s)})\|^2_{\mathcal{H}}. \end{aligned} \tag{3.2}$$

On the other hand, from $\int_0^\infty r^{-q}\psi_\alpha(r)dr = \frac{\Gamma(1+\frac{q}{\alpha})}{\Gamma(1+q)}$, for all $q \in [0,1]$ (see [46, Lemma 3.2]), we have

$$\int_0^\infty r^\beta\phi_\alpha(r)dr = \int_0^\infty \frac{1}{r^{\beta\alpha}}\psi_\alpha(r)dr = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}. \tag{3.3}$$

Then, by (3.2) and (3.3) , it is easy to see that

$$\|\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, u_{\varrho(s,u_s)})\|^2_{\mathcal{H}} \le \left(\frac{\alpha\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)(t-s)^{1-\alpha\beta}}\right)^2\|\mathscr{A}^\beta\mathscr{G}(s, u_{\varrho(s,u_s)})\|^2_{\mathcal{H}}. \tag{3.4}$$

It is obvious that the function $s \to \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, u_{\varrho(s,u_s)})$ is integrable on [0, t) for every $t > 0$.

It is evident that the fixed points of the operator Υ are mild solutions of the model (1.1)-(1.3). We express the function $x(\cdot) : (-\infty, T] \to \mathcal{H}$ by

$$x(t) = \begin{cases} \varphi(t), \quad t \le 0; \\ \mathcal{T}_\alpha(t)\varphi(0), \quad t \in \mathscr{I}, \end{cases}$$

then $x_0 = \varphi$. For every function $z \in C(\mathscr{I}, \mathcal{R}^+)$ with $z(0) = 0$, we allocate as $\bar{z}$ is characterized by

$$\bar{z}(t) = \begin{cases} 0, \quad t \le 0; \\ z(t), \quad t \in \mathscr{I}. \end{cases}$$

If $u(\cdot)$ fulfills (2.14), we are able to split it as $u(t) = z(t) + x(t)$, $t \in \mathscr{I}$, which suggests $u_t = z_t + x_t$, for each $t \in \mathscr{I}$ and also the function $z(\cdot)$ fulfills

$$z(t) = \begin{cases} \mathcal{T}_\alpha(t)[-h(z_t+x_t) - \mathscr{G}(0,\varphi)] + \mathscr{G}(t, z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}) \\ + \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\right)ds + \int_0^t \mathcal{S}_\alpha(t-s)Bv^\mu(s)ds \\ + \int_0^t \mathcal{S}_\alpha(t-s) \\ \quad (\times)\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)ds \\ + \int_0^t \mathcal{S}_\alpha(t-s) \\ \quad (\times)\Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)dw(s) \\ + \sum_{0<t_k<t} \mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(z(t_k^-) + x(t_k^-)), \quad t \in \mathscr{I}. \end{cases}$$

Let $\mathscr{B}_T^0 = \{z \in \mathscr{B}_T \colon z_0 = 0 \in \mathcal{B}\}$. Let $\|\cdot\|_{\mathscr{B}_T^0}$ be the seminorm in $\mathscr{B}_T^0$ described by

$$\|z\|_{\mathscr{B}_T^0} = \sup_{s\in\mathscr{I}} \left(\mathbb{E}\|z(s)\|^2\right)^{\frac{1}{2}} + \|z_0\|_{\mathcal{B}} = \sup_{s\in\mathscr{I}} \left(\mathbb{E}\|z(s)\|^2\right)^{\frac{1}{2}}, \quad z \in \mathscr{B}_T^0,$$

as a result $(\mathscr{B}_T^0, \|\cdot\|_{\mathscr{B}_T^0})$ is a Banach space. Set $B_r = \{z \in \mathscr{B}_T^0 : \|z\|^2 \leq r\}$ for some $r \geq 0$; then for each r, $B_r \subset \mathscr{B}_T^0$ is clearly a bounded closed convex set. For $z \in B_r$, from Lemma 2.2 and along with the above discussion, we get

$$\begin{aligned}
&\mathbb{E}\|z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}\|_{\mathcal{B}}^2 \\
&\leq 2\left(\mathbb{E}\|z_{\varrho(t,z_t+x_t)}\|_{\mathcal{B}}^2 + \mathbb{E}\|x_{\varrho(t,z_t+x_t)}\|_{\mathcal{B}}^2\right) \\
&\leq 4\Bigg(\mathscr{E}_1^{*2} \sup_{\substack{0\leq s\leq \max(0,t)\\ t\in\mathcal{R}(\varrho^-)\cup\mathscr{I}}} \mathbb{E}\|z(s)\|^2 + (\mathscr{E}_2^* + J^\varphi)^2\mathbb{E}\|z_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2} \sup_{\substack{0\leq s\leq \max(0,t)\\ t\in\mathcal{R}(\varrho^-)\cup\mathscr{I}}} \mathbb{E}\|x(s)\|^2 \\
&\quad + (\mathscr{E}_2^* + J^\varphi)^2\mathbb{E}\|x_0\|_{\mathcal{B}}^2\Bigg) \\
&\leq 4\left(\mathscr{E}_1^{*2} r + (\mathscr{E}_2^* + J^\varphi)^2\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\mathbb{E}\|\mathcal{T}_\alpha(t)\varphi(0)\|^2\right) \\
&\leq 4\left(\mathscr{E}_1^{*2} r + (\mathscr{E}_2^* + J^\varphi)^2\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\left\|\int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)dr\right\|^2 \mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2\right) \\
&\leq 4\mathscr{E}_1^{*2}\left(r + \mathcal{M}^2\mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2\right) + 4(\mathscr{E}_2^* + J^\varphi)^2\mathbb{E}\|\varphi\|_{\mathcal{B}}^2 \\
&\leq 4\mathscr{E}_1^{*2} r + c_n = r^*,
\end{aligned} \tag{3.5}$$

where $c_n = 4\left[\mathscr{E}_1^{*2}\mathcal{M}^2\mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2 + (\mathscr{E}_2^* + J^\varphi)^2\mathbb{E}\|\varphi\|_{\mathcal{B}}^2\right]$ and

$$\begin{aligned}
\mathbb{E}\|z_t + x_t\|_{\mathcal{B}}^2 &\leq 2(\mathbb{E}\|z_t\|_{\mathcal{B}}^2 + \mathbb{E}\|x_t\|_{\mathcal{B}}^2) \\
&\leq 4\left(\mathscr{E}_2^{*2}\mathbb{E}\|z_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\sup_{s\in\mathscr{I}}\mathbb{E}\|z(s)\|^2 + \mathscr{E}_2^{*2}\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\sup_{s\in\mathscr{I}}\mathbb{E}\|x(s)\|^2\right) \\
&\leq 4\left(\mathscr{E}_1^{*2} r + \mathscr{E}_2^{*2}\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\mathbb{E}\|\mathcal{T}_\alpha(t)\varphi(0)\|^2\right) \\
&\leq 4\left(\mathscr{E}_1^{*2} r + \mathscr{E}_2^{*2}\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\left\|\int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)dr\right\|^2\mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2\right) \\
&\leq 4\left(\mathscr{E}_1^{*2}(r + \mathcal{M}^2\mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2) + \mathscr{E}_2^{*2}\mathbb{E}\|\varphi\|_{\mathcal{B}}^2\right) \\
&\leq 4\mathscr{E}_1^{*2} r + \widetilde{c}_n = \widetilde{r},
\end{aligned} \tag{3.6}$$

where $\widetilde{c}_n = 4\left[\mathscr{E}_1^{*2}\mathcal{M}^2\mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2 + \mathscr{E}_2^{*2}\mathbb{E}\|\varphi\|_{\mathcal{B}}^2\right]$. We delimit the operator $\overline{\Upsilon} : \mathscr{B}_T^0 \to \mathscr{B}_T^0$ by

$$\begin{aligned}
(\overline{\Upsilon}z)(t) &= \mathcal{T}_\alpha(t)[-h(z_t + x_t) - \mathscr{G}(0,\varphi)] + \mathscr{G}(t, z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}) \\
&\quad + \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\right)ds + \int_0^t \mathcal{S}_\alpha(t-s)Bv^\mu(s)ds \\
&\quad + \int_0^t \mathcal{S}_\alpha(t-s) \\
&\qquad (\times)\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)ds \\
&\quad + \int_0^t \mathcal{S}_\alpha(t-s) \\
&\qquad (\times)\Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)dw(s) \\
&\quad + \sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(z(t_k^-) + x(t_k^-)), \quad t \in \mathscr{I}.
\end{aligned}$$

It is vindicated that the operator Υ has a fixed point if and only if $\overline{\Upsilon}$ has a fixed point. Thus, let us demonstrate that $\overline{\Upsilon}$ has a fixed point.

The facts of the theorem is lengthy and technical. Therefore it is practical to split it into several steps.

Step 1: $\overline{\Upsilon}(B_r) \subset B_r$ for some $r > 0$.

We assert that there exists a positive integer r in ways that $\overline{\Upsilon}(B_r) \subset B_r$. If it is not true, then for each positive number r, we can find a function $z^r(\cdot) \in B_r$, but $\overline{\Upsilon}(z^r) \notin B_r$, i.e., $\mathbb{E}\|\overline{\Upsilon}(z^r)(t)\|^2 > r$ for some $t \in \mathscr{I}$, we sustain

$$
\begin{aligned}
&\mathbb{E}\|v^\mu(s)\|^2 \\
&\leq 10\mathbb{E}\left\| B^*\mathcal{S}_\alpha^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\left[\mathbb{E}\tilde{u}_T + \int_0^T \tilde{\phi}(s)dw(s) - \mathcal{T}_\alpha(T)[\varphi(0)-h(z_T+x_T)]\right] \right\|^2 \\
&\quad + 10\mathbb{E}\left\| B^*\mathcal{S}_\alpha^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\mathcal{T}_\alpha(T)\mathscr{G}(0,\varphi)\right\|^2 \\
&\quad + 10\mathbb{E}\left\| B^*\mathcal{S}_\alpha^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\mathscr{G}(T, z_{\varrho(T,z_T+x_T)} + x_{\varrho(T,z_T+x_T)})\right\|^2 \\
&\quad + 10\mathbb{E}\left\| B^*\mathcal{S}_\alpha^*(T-t)\int_0^T(\mu\mathcal{I}+\Gamma_s^T)^{-1}\mathscr{A}\mathcal{S}_\alpha(T-s)\mathscr{G}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\right)ds\right\|^2 \\
&\quad + 10\mathbb{E}\left\| B^*\mathcal{S}_\alpha^*(T-t)\int_0^T(\mu\mathcal{I}+\Gamma_s^T)^{-1}\mathcal{S}_\alpha(T-s)\right. \\
&\qquad \left.(\times)\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)ds\right\|^2 \\
&\quad + 10\mathbb{E}\left\| B^*\mathcal{S}_\alpha^*(T-t)\int_0^T(\mu\mathcal{I}+\Gamma_s^T)^{-1}\mathcal{S}_\alpha(T-s)\right. \\
&\qquad \left.(\times)\Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)dw(s)\right\|^2 \\
&\quad + 10\mathbb{E}\left\| B^*\mathcal{S}_\alpha^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\sum_{0<t_k<t}\mathcal{T}_\alpha(T-t_k)\mathcal{I}_k(z(t_k^-)+x(t_k^-))\right\|^2 \\
&= \sum_{i=1}^{7} J_i.
\end{aligned} \tag{3.7}
$$

By using (3.4), (3.5), (3.6),(H1)-(H6) and Holder's inequality, we receive

$$
\begin{aligned}
J_1 &= 10\mathbb{E}\left\| B^*\mathcal{S}_\alpha^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\left[\mathbb{E}\tilde{u}_T + \int_0^T \tilde{\phi}(s)dw(s) - \mathcal{T}_\alpha(T)[\varphi(0)-h(z_T+x_T)]\right] \right\|^2 \\
&\leq 10\mathcal{M}_B^2\left\|\alpha\int_0^\infty r\phi_\alpha(r)(T-t)^{\alpha-1}\mathbb{T}((T-t)^\alpha r)dr\right\|^2\frac{1}{\mu^2}\left[\mathbb{E}\|\tilde{u}_T\|^2 + \int_0^T\mathbb{E}\|\tilde{\phi}(s)\|^2ds\right. \\
&\quad \left. + \left\|\int_0^\infty \phi_\alpha(r)\mathbb{T}(T^\alpha r)dr\right\|^2\left[\mathbb{E}\|\varphi(0)\|^2 + \mathbb{E}\|h(z_T+x_T)\|^2\right]\right] \\
&\leq 10\mathcal{M}_B^2\left(\frac{\alpha\mathcal{M}T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2\frac{1}{\mu^2}\left[\mathbb{E}\|\tilde{u}_T\|^2 + \int_0^T\mathbb{E}\|\tilde{\phi}(s)\|^2ds + \mathcal{M}^2\left[\mathbb{E}\|\varphi(0)\|^2 + \mathbb{E}\|h(z_T+x_T)\|^2\right]\right] \\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_BT^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2(\times)\left[\mathbb{E}\|\tilde{u}_T\|^2 + \int_0^T\mathbb{E}\|\tilde{\phi}(s)\|^2ds + \mathcal{M}^2\left[\mathbb{E}\|\varphi(0)\|^2 + \mathcal{M}_h\tilde{r}\right]\right], \\
&\qquad \text{where}\|B^*\| = \mathcal{M}_B.
\end{aligned}
$$

$$
\begin{aligned}
J_2 &= 10\mathbb{E}\left\| B^*\mathcal{S}_\alpha^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\mathcal{T}_\alpha(T)\mathscr{G}(0,\varphi)\right\|^2 \\
&\leq 10\mathcal{M}_B^2\left(\frac{\alpha\mathcal{M}T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2\frac{1}{\mu^2}\mathbb{E}\|\mathscr{G}(0,\varphi)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq 10\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \frac{1}{\mu^2}\|\mathscr{A}^{-\beta}\|^2\mathbb{E}\|\mathscr{A}^{\beta}\mathscr{G}(0,\varphi)\|^2\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\mathcal{M}^2\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|_{\mathcal{B}}^2).\\
J_3 &= 10\mathbb{E}\left\|B^*\mathcal{S}_\alpha^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\mathscr{G}(T,z_{\varrho(T,z_T+x_T)}+x_{\varrho(T,z_T+x_T)})\right\|^2\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \mathbb{E}\left\|\mathscr{G}(T,z_{\varrho(T,z_T+x_T)}+x_{\varrho(T,z_T+x_T)})\right\|^2\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \|A^{-\beta}\|^2\mathbb{E}\left\|A^{\beta}\mathscr{G}(T,z_{\varrho(T,z_T+x_T)}+x_{\varrho(T,z_T+x_T)})\right\|^2\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+r^*).\\
J_4 &= 10\mathbb{E}\left\|B^*\mathcal{S}_\alpha^*(T-t)\int_0^T(\mu\mathcal{I}+\Gamma_s^T)^{-1}\mathscr{A}\mathcal{S}_\alpha(T-s)\mathscr{G}\left(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\right)ds\right\|^2\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \left\|\int_0^T\mathscr{A}^{1-\beta}\mathcal{S}_\alpha(T-s)ds\right\|^2\mathbb{E}\|\mathscr{A}^{\beta}\mathscr{G}\left(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\right)\|^2\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \left(\frac{\alpha\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\right)^2\int_0^T(T-s)^{\alpha\beta-1}ds\int_0^T(T-s)^{\alpha\beta-1}\mathcal{M}_{\mathscr{G}}(1+r^*)ds\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2\mathcal{M}_{\mathscr{G}}(1+r^*).\\
J_5 &= 10\mathbb{E}\Bigg\|B^*\mathcal{S}_\alpha^*(T-t)\int_0^T(\mu\mathcal{I}+\Gamma_s^T)^{-1}\mathcal{S}_\alpha(T-s)\\
&\qquad(\times)\mathscr{F}\left(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)ds\Bigg\|^2\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \frac{T^\alpha}{\alpha}\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\int_0^T(T-s)^{\alpha-1}\Bigg[\mathscr{F}_1(s)\|z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\|_{\mathcal{B}}^2\\
&\quad+\mathscr{F}_2(s)\mathbb{E}\left\|\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right\|_{\mathcal{H}}^2\Bigg]ds\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\left[\mathscr{F}_1^*r^*+\mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+r^*)T\right].\\
J_6 &= 10\mathbb{E}\Bigg\|B^*\mathcal{S}_\alpha^*(T-t)\int_0^T(\mu\mathcal{I}+\Gamma_s^T)^{-1}\mathcal{S}_\alpha(T-s)\\
&\qquad(\times)\Sigma\left(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)dw(s)\Bigg\|^2\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \frac{T^\alpha}{\alpha}\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\int_0^T(T-s)^{\alpha-1}tr(Q)\Bigg[\Sigma_1(s)\|z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\|_{\mathcal{B}}^2\\
&\quad+\Sigma_2(s)\mathbb{E}\left\|\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right\|_{\mathcal{H}}^2\Bigg]ds\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 tr(Q)\left[\Sigma_1^*r^*+\Sigma_2^*\widetilde{\mathcal{M}}_1(1+r^*)T\right].\\
J_7 &= 10\mathbb{E}\left\|B^*\mathcal{S}_\alpha^*(T-t)(\mu\mathcal{I}+\Gamma_0^T)^{-1}\sum_{0<t_k<t}\mathcal{T}_\alpha(T-t_k)\mathcal{I}_k(z(t_k^-)+x(t_k^-))\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \|\mathcal{T}_\alpha(T-t_k)\|^2 n\sum_{k=1}^{n}\mathbb{E}\|\mathcal{I}_k(z(t_k^-)+x(t_k^-))\|^2\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 \mathcal{M}^2 n\sum_{k=1}^{n}\mathcal{M}_{\mathcal{I}_k}H^2\|z_t+x_t\|^2\\
&\leq \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\mathcal{M}^2 H^2 n\sum_{k=1}^{n}\mathcal{M}_{\mathcal{I}_k}\tilde{r}.
\end{aligned}
$$

By combining the estimations $(J_1)-(J_7)$ together with (3.7), we sustain

$$
\begin{aligned}
\mathbb{E}\|v^\mu(s)\|^2 \leq{}& \frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left[\mathbb{E}\|\tilde{u}_T\|^2+\int_0^T\mathbb{E}\|\tilde{\phi}(s)\|^2ds+\mathcal{M}^2\left[\mathbb{E}\|\varphi(0)\|^2+\mathcal{M}_h\tilde{r}\right]\right]\\
&+\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\mathcal{M}^2\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|_{\mathcal{B}}^2)\\
&+\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+r^*)\\
&+\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2\mathcal{M}_{\mathscr{G}}(1+r^*)\\
&+\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\left[\mathscr{F}_1^* r^*+\mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+r^*)T\right]\\
&+\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 tr(Q)\left[\Sigma_1^* r^*+\Sigma_2^*\widetilde{\mathcal{M}}_1(1+r^*)T\right]\\
&+\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2 (\times)\mathcal{M}^2 H^2 n\sum_{k=1}^{n}\mathcal{M}_{\mathcal{I}_k}\tilde{r}.
\end{aligned}
$$

$$
\begin{aligned}
r \leq{}& \mathbb{E}\|\overline{\Upsilon}(z^r)(t)\|^2\\
\leq{}& 8\mathbb{E}\|\mathcal{T}_\alpha(t)[-h(z_t^r+x_t)-\mathscr{G}(0,\varphi)]\|^2+8\mathbb{E}\|\mathscr{G}(t,z^r_{\varrho(t,z_t^r+x_t)}+x_{\varrho(t,z_t^r+x_t)})\|^2\\
&+8\mathbb{E}\left\|\int_0^t\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s,z^r_{\varrho(s,z_s^r+x_s)}+x_{\varrho(s,z_s^r+x_s)}\right)ds\right\|^2\\
&+8\mathbb{E}\left\|\int_0^t\mathcal{S}_\alpha(t-s)Bv^\mu(s)ds\right\|^2+8\mathbb{E}\left\|\int_0^t\mathcal{S}_\alpha(t-s)\right.\\
&\left.(\times)\mathscr{F}\left(s,z^r_{\varrho(s,z_s^r+x_s)}+x_{\varrho(s,z_s^r+x_s)},\int_0^s e_1(s,\tau,z^r_{\varrho(\tau,z_\tau^r+x_\tau)}+x_{\varrho(\tau,z_\tau^r+x_\tau)})d\tau\right)ds\right\|^2\\
&+8\mathbb{E}\left\|\int_0^t\mathcal{S}_\alpha(t-s)\right.\\
&\left.(\times)\Sigma\left(s,z^r_{\varrho(s,z_s^r+x_s)}+x_{\varrho(s,z_s^r+x_s)},\int_0^s e_2(s,\tau,z^r_{\varrho(\tau,z_\tau^r+x_\tau)}+x_{\varrho(\tau,z_\tau^r+x_\tau)})d\tau\right)dw(s)\right\|^2\\
&+8\mathbb{E}\|\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(z^r(t_k^-)+x(t_k^-))\|^2\\
={}&\sum_{i=8}^{14}J_i. \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
J_8 ={}& 8\mathbb{E}\|\mathcal{T}_\alpha(t)[-h(z_t^r+x_t)-\mathscr{G}(0,\varphi)]\|^2\\
\leq{}& 8\|\mathcal{T}_\alpha(t)\|^2\left[\mathbb{E}\|h(z_t^r+x_t)\|_{\mathcal{B}}^2+\mathbb{E}\|\mathscr{G}(0,\varphi)\|^2\right]\\
\leq{}& 8\left\|\int_0^\infty\phi_\alpha(r)\mathbb{T}(t^\alpha r)dr\right\|^2\left[\mathbb{E}\|h(z_t^r+x_t)\|_{\mathcal{B}}^2+\mathbb{E}\|\mathscr{G}(0,\varphi)\|^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq 8\mathcal{M}^2\left[\mathcal{M}_h\|z_t^r+x_t\|_{\mathcal{B}}^2+\|\mathscr{A}^{-\beta}\|^2\mathbb{E}\|\mathscr{A}^{\beta}\mathscr{G}(0,\varphi)\|^2\right]\\
&\leq 8\mathcal{M}^2\left[\mathcal{M}_h\left(4\mathscr{E}_1^{*2}r+\widetilde{c}_n\right)+\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|_{\mathcal{B}}^2)\right]\\
&\leq 32\mathcal{M}^2\mathcal{M}_h\mathscr{E}_1^{*2}r+C_1,
\end{aligned}
$$

where $\mathcal{N}_0=\|A^{-\beta}\|$ and $C_1=8\mathcal{M}^2\mathcal{M}_h\widetilde{c}_n+8\mathcal{M}^2\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|_{\mathcal{B}}^2)$.

$$
\begin{aligned}
J_9&=8\mathbb{E}\|\mathscr{G}(t,z^r_{\varrho(t,z_t^r+x_t)}+x_{\varrho(t,z_t^r+x_t)})\|^2\\
&\leq 8\|\mathscr{A}^{-\beta}\|^2\mathbb{E}\|\mathscr{A}^{\beta}\mathscr{G}(t,z^r_{\varrho(t,z_t^r+x_t)}+x_{\varrho(t,z_t^r+x_t)})\|^2\\
&\leq 8\|\mathscr{A}^{-\beta}\|^2\mathcal{M}_{\mathscr{G}}\left(1+\|z^r_{\varrho(t,z_t^r+x_t)}+x_{\varrho(t,z_t^r+x_t)}\|_{\mathcal{B}}^2\right)\\
&\leq 32\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}r+C_2,
\end{aligned}
$$

where $C_2=8\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+c_n)$.

$$
\begin{aligned}
J_{10}&=8\mathbb{E}\left\|\int_0^t\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\left(s,z^r_{\varrho(s,z_s^r+x_s)}+x_{\varrho(s,z_s^r+x_s)}\right)ds\right\|^2\\
&\leq 8\left\|\int_0^t\left\{\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathscr{A}^{1-\beta}\mathbb{T}((t-s)^\alpha r)dr\right\}ds\right\|^2\\
&\quad(\times)\mathbb{E}\left\|\mathscr{A}^{\beta}\mathscr{G}\left(s,z^r_{\varrho(s,z_s^r+x_s)}+x_{\varrho(s,z_s^r+x_s)}\right)\right\|^2\\
&\leq 8\mathcal{M}_{\mathscr{G}}\left(\frac{\alpha\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\right)^2\int_0^t(t-s)^{\alpha\beta-1}ds\int_0^t(t-s)^{\alpha\beta-1}\left(1+\|z^r_{\varrho(s,z_s^r+x_s)}+x_{\varrho(s,z_s^r+x_s)}\|_{\mathcal{B}}^2\right)ds\\
&\leq 8\mathcal{M}_{\mathscr{G}}\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2\left(1+4\mathscr{E}_1^{*2}r+c_n\right)\\
&\leq 32\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}r\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2+C_3,
\end{aligned}
$$

where $C_3=8\mathcal{M}_{\mathscr{G}}\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2(1+c_n)$.

$$
\begin{aligned}
J_{11}&=8\mathbb{E}\left\|\int_0^t\mathcal{S}_\alpha(t-s)Bv^\mu(s)ds\right\|^2\\
&\leq 8\left\|\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathbb{T}((t-s)^\alpha r)dr\right\|^2\mathbb{E}\left\|\int_0^t Bv^\mu(s)ds\right\|^2\\
&\leq 8\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\mathcal{M}_B^2\frac{T^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}\mathbb{E}\|v^\mu(s)\|^2ds\\
&\leq 8\left(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^2\frac{T^{2\alpha}}{\alpha^2}(\times)\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_BT^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2\mathcal{M}_v,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{M}_v=&\mathbb{E}\|\tilde{u}_T\|^2+\int_0^T\mathbb{E}\|\tilde{\phi}(s)\|^2ds+\mathcal{M}^2\left[\mathbb{E}\|\varphi(0)\|^2+\mathcal{M}_h\tilde{r}\right]+\mathcal{M}^2\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|_{\mathcal{B}}^2)+\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+r^*)\\
&+\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2\mathcal{M}_{\mathscr{G}}(1+r^*)+\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\left[\mathscr{F}_1^*r^*+\mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+r^*)T\right]\\
&+\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2tr(Q)\left[\Sigma_1^*r^*+\Sigma_2^*\widetilde{\mathcal{M}}_1(1+r^*)T\right]+\mathcal{M}^2H^2n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}\tilde{r}.
\end{aligned}
$$

$$\begin{aligned}
J_{12} &= 8\mathbb{E}\Big\| \int_0^t \mathcal{S}_\alpha(t-s) \\
&\quad (\times)\mathscr{F}\left(s, z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}, \int_0^s e_1(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)})d\tau\right) ds\Big\|^2 \\
&\leq 8\left\|\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathbb{T}((t-s)^\alpha r)dr\right\|^2 \\
&\quad (\times)\mathbb{E}\Big\|\int_0^t \mathscr{F}\Big(s, z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}, \int_0^s e_1(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)})d\tau\Big)ds\Big\|^2 \\
&\leq 8\Big(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\Big)\frac{T^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}\Big[\mathscr{F}_1(s)\|z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}\|^2_{\mathcal{B}} \\
&\quad + \mathscr{F}_2(s)\int_0^s \mathbb{E}\|e_1(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)})\|^2_{\mathcal{H}}d\tau\Big]ds \\
&\leq 32\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 \mathscr{E}_1^{*2} r(\mathscr{F}_1^* + \mathscr{F}_2^*\widetilde{\mathcal{M}}_0 T) + C_4,
\end{aligned}$$

where $C_4 = 8\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\left(\mathscr{F}_1^* c_n + \mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+c_n)T\right)$.

$$\begin{aligned}
J_{13} &= 8\mathbb{E}\Big\| \int_0^t \mathcal{S}_\alpha(t-s) \\
&\quad (\times)\Sigma\left(s, z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}, \int_0^s e_2(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)})d\tau\right) dw(s)\Big\|^2 \\
&\leq 8\left\|\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathbb{T}((t-s)^\alpha r)dr\right\|^2 tr(Q) \\
&\quad (\times)\mathbb{E}\Big\|\int_0^t \Sigma\Big(s, z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}, \int_0^s e_2(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)})d\tau\Big)ds\Big\|^2 \\
&\leq 8\Big(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\Big)\frac{T^\alpha}{\alpha}\frac{T^\alpha}{\alpha}tr(Q)\Big[\Sigma_1^*(4\mathscr{E}_1^{*2} + c_n) + \Sigma_2^*\widetilde{\mathcal{M}}_1(1+4\mathscr{E}_1^{*2} + c_n)T\Big] \\
&\leq 32\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 tr(Q)\mathscr{E}_1^{*2} r(\Sigma_1^* + \Sigma_2^*\widetilde{\mathcal{M}}_1 T) + C_5,
\end{aligned}$$

where $C_5 = 8\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 tr(Q)\Big(\Sigma_1^* c_n + \Sigma_2^*\widetilde{\mathcal{M}}_1(1+c_n)T\Big)$.

$$\begin{aligned}
J_{14} &= 8n\sum_{k=1}^n \|\mathcal{T}_\alpha(t-t_k)\|^2\mathbb{E}\|\mathcal{I}_k(z^r(t_k^-) + x(t_k^-))\|^2 \\
&\leq 8n\sum_{k=1}^n \Big\|\int_0^\infty \mathbb{T}((t-t_k)^\alpha r)\phi_\alpha(r)dr\Big\|^2\mathbb{E}\|\mathcal{I}_k(z^r(t_k^-) + x(t_k^-))\|^2 \\
&\leq 8\mathcal{M}^2 n\sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k}\mathbb{E}\|(z^r(t_k^-) + x(t_k^-))\|^2 \\
&\leq 8\mathcal{M}^2 n\sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k}\left(\sup_{t\in\mathscr{I}}\mathbb{E}\|z^r(t) + x(t)\|^2\right) \\
&\leq 8\mathcal{M}^2 H^2 n\sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k}\mathbb{E}\|z^r_t + x_t\|^2_{\mathcal{B}} \\
&\leq 8\mathcal{M}^2 H^2 n\sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k}\left[4\mathscr{E}_1^{*2} r + \widetilde{c}_n\right] \\
&\leq 32\mathcal{M}^2 H^2\mathscr{E}_1^{*2} n\sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k} r + C_6,
\end{aligned}$$

where $C_6 = 8\mathcal{M}^2 H^2\widetilde{c}_n n\sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k}$.

By combining the estimations $(J_8)-(J_{14})$ together with (3.8), we sustain

$$
\begin{aligned}
r &\le \mathbb{E}\|\overline{\Upsilon}(z^r)(t)\|^2\\
&\le 32\mathcal{M}^2\mathcal{M}_h\mathscr{E}_1^{*2}r + C_1 + 32\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}r + C_2 + 32\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}r\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2 + C_3\\
&\quad + 8\left(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^2\frac{T^{2\alpha}}{\alpha^2}(\times)\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B T^{\alpha-1}}{\Gamma(1+\alpha)}\right)^2\mathcal{M}_v + 32\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\mathscr{E}_1^{*2}r(\mathscr{F}_1^* + \mathscr{F}_2^*\widetilde{\mathcal{M}}_0 T)\\
&\quad + C_4 + 32\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 tr(Q)\mathscr{E}_1^{*2}r(\Sigma_1^* + \Sigma_2^*\widetilde{\mathcal{M}}_1 T) + C_5 + 32\mathcal{M}^2H^2\mathscr{E}_1^{*2}n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}r + C_6,
\end{aligned}
$$

where $C_1 - C_6$ are independent of r. Dividing both sides by r and taking the limit as $r\to\infty$, we sustain

$$
\begin{aligned}
&32\left(1+\frac{10}{\mu^2}\left(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^4\frac{T^{4\alpha-2}}{\alpha^2}\right)\left[\mathcal{M}^2\left(\mathcal{M}_h + H^2 n\sum_{k=1}^n\gamma_k\right) + \mathcal{M}_{\mathscr{G}}\left(\mathcal{N}_0{}^2 + \left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)T^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right)^2\right)\right.\\
&\qquad\left. + \left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\left[\mathscr{F}_1^* + tr(Q)\Sigma_1^* + \left(\mathscr{F}_2^*\widetilde{\mathcal{M}}_0 + \Sigma_2^* tr(Q)\widetilde{\mathcal{M}}_1\right)T\right]\right]\mathscr{E}_1^{*2} \ge 1
\end{aligned}
$$

which is a contradiction to (3.1). For this reason for some positive number r in a way that $\overline{\Upsilon}(B_r)\subset B_r$.
Step 2: Now we prove that for each $\mu>0$, the operator $\overline{\Upsilon}$ maps B_r into a relatively compact subset of B_r. First we prove that the set $\mathcal{V}(t) = \{(\overline{\Upsilon}z)(t) : z\in B_r\}$ is relatively compact in $\mathcal{H}$ for every $t\in\mathscr{I}$. The case $t=0$ is obvious. For $0<\epsilon<t\le T$, define $(\overline{\Upsilon}^\epsilon z)(t) = \mathcal{S}_\alpha(\epsilon)Q(t-\epsilon)$, where

$$
\begin{aligned}
Q(t-\epsilon) &= \mathcal{T}_\alpha(t)[-h(z_t+x_t) - \mathscr{G}(0,\varphi)] + \mathscr{G}(t, z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)})\\
&\quad + \int_0^{t-\epsilon}\mathscr{A}\mathcal{S}_\alpha(t-\epsilon-s)\mathscr{G}(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)})ds + \int_0^{t-\epsilon}\mathcal{S}_\alpha(t-\epsilon-s)Bv^\mu(s)ds\\
&\quad + \int_0^{t-\epsilon}\mathcal{S}_\alpha(t-\epsilon-s)\mathscr{F}\Big(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\Big)ds\\
&\quad + \int_0^{t-\epsilon}\mathcal{S}_\alpha(t-\epsilon-s)\Sigma\Big(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, e_2(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\Big)dw(s)\\
&\quad + \sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(z(t_k^-) + x(t_k^-)).
\end{aligned}
$$

Since $\mathcal{S}_\alpha(t)$ is compact and $Q(t-\epsilon)$ is bounded on B_r, the set $\mathcal{V}_\epsilon(t) = \{(\overline{\Upsilon}^\epsilon z)(t) : z(\cdot)\in B_r\}$ is relatively compact in $\mathcal{H}$. Also for every $z\in B_r$, we have

$$
\begin{aligned}
&\mathbb{E}\|(\overline{\Upsilon}z)(t) - (\overline{\Upsilon}^\epsilon z)(t)\|_{\mathcal{H}}^2\\
&\le 4\mathbb{E}\left\|\int_{t-\epsilon}^t\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)})ds\right\|^2\\
&\quad + 4\mathbb{E}\left\|\int_{t-\epsilon}^t\mathcal{S}_\alpha(t-s)Bv^\mu(s)ds\right\|^2 + 4\mathbb{E}\left\|\int_{t-\epsilon}^t\mathcal{S}_\alpha(t-s)\right.\\
&\qquad\left.(\times)\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)ds\right\|^2\\
&\quad + 4\mathbb{E}\left\|\int_{t-\epsilon}^t\mathcal{S}_\alpha(t-s)\right.\\
&\qquad\left.(\times)\Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)dw(s)\right\|^2\\
&\le 4\left(\frac{\alpha\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\right)^2\frac{\epsilon^{\alpha\beta}}{\alpha\beta}\int_{t-\epsilon}^t(t-s)^{\alpha\beta-1}\mathcal{M}_{\mathscr{G}}(1+r^*)ds + 4\left(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^2\frac{\epsilon^\alpha}{\alpha}\\
&\qquad(\times)\int_{t-\epsilon}^t(t-s)^{\alpha-1}\mathcal{M}_v ds + 4\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\frac{\epsilon^\alpha}{\alpha}\int_{t-\epsilon}^t(t-s)^{\alpha-1}\left[\mathscr{F}_1 r^* + \mathscr{F}_2^*(1+\widetilde{\mathcal{M}}_0)T\right]ds
\end{aligned}
$$

$$+4\bigg(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\bigg)^2\frac{\epsilon^\alpha}{\alpha}tr(Q)\int_{t-\epsilon}^{t}(t-s)^{\alpha-1}\big[\Sigma_1 r^*+\Sigma_2^*(1+\widetilde{\mathcal{M}}_1)T\big]ds$$
$$\to 0, \quad \text{as} \quad \epsilon\to 0^+.$$

This implies that there are relatively compact sets arbitrarily close to the set $\mathcal{V}(t), t>0$. As a result, $\mathcal{V}(t)=\{(\overline{\Upsilon}z)(t): z\in B_r\}$ is also relatively compact in $\mathcal{H}$.
Step 3: Next we shall show that $\mathcal{V}(t)=\{(\overline{\Upsilon}z)(t): z\in B_r\}$ is equicontinuous in $[0,T]$. For $0\le t_1\le t_2\le T$ such that $\|\mathbb{T}(t_1^\alpha)-\mathbb{T}(t_2^\alpha)\|<\epsilon$, we get

$$\begin{aligned}
&\mathbb{E}\|(\overline{\Upsilon}z)(t_2)-(\overline{\Upsilon}z)(t_1)\|^2\\
&\le 16\mathbb{E}\,\big\|\big[\mathbb{T}(t_2^\alpha r)-\mathbb{T}(t_1^\alpha r)\big]\big[-h(z_t+x_t)-\mathscr{G}(0,\varphi)\big]\big\|^2\\
&\quad+16\mathbb{E}\,\big\|\mathscr{G}\big(t_2,z_{\varrho(t_2,z_{t_2}+x_{t_2})}+x_{\varrho(t_2,z_{t_2}+x_{t_2})}\big)-\mathscr{G}\big(t_1,z_{\varrho(t_1,z_{t_1}+x_{t_1})}+x_{\varrho(t_1,z_{t_1}+x_{t_1})}\big)\big\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)^2\int_0^{t_1}(t_1-s)^{\alpha-1}\mathscr{A}\big[\mathbb{T}((t_2-s)^\alpha r)-\mathbb{T}((t_1-s)^\alpha r)\big]\\
&\qquad(\times)\mathscr{G}\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\bigg)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)^2\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\mathscr{A}\mathbb{T}((t_2-s)^\alpha r)\\
&\qquad(\times)\mathscr{G}\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\bigg)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)^2\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathscr{A}\mathbb{T}((t_2-s)^\alpha r)\\
&\qquad(\times)\mathscr{G}\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\bigg)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)\int_0^{t_1}(t_1-s)^{\alpha-1}\big[\mathbb{T}((t_2-s)^\alpha r)-\mathbb{T}((t_1-s)^\alpha r)\big]Bv^\mu(s)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\mathbb{T}((t_2-s)^\alpha r)Bv^\mu(s)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{T}((t_2-s)^\alpha r)Bv^\mu(s)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)\int_0^{t_1}(t_1-s)^{\alpha-1}\big[\mathbb{T}((t_2-s)^\alpha r)-\mathbb{T}((t_1-s)^\alpha r)\big]\\
&\qquad(\times)\mathscr{F}\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\bigg)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathbb{T}((t_2-s)^\alpha r)\\
&\qquad(\times)\mathscr{F}\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\bigg)ds\bigg\|^2\\
&\quad+16\mathbb{E}\bigg\|\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{T}((t_2-s)^\alpha r)\\
&\qquad(\times)\mathscr{F}\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_1(s,\tau,z_{\varrho(s,z_s+x_s)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\bigg)ds\bigg\|^2
\end{aligned}$$

$$
\begin{aligned}
&+16\mathbb{E}\bigg\|\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)\int_0^{t_1}(t_1-s)^{\alpha-1}\big[\mathbb{T}((t_2-s)^\alpha r)-\mathbb{T}((t_1-s)^\alpha r)\big]\\
&\quad(\times)\Sigma\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\bigg)dw(s)\bigg\|^2\\
&+16\mathbb{E}\bigg\|\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathbb{T}((t_2-s)^\alpha r)\\
&\quad(\times)\Sigma\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\bigg)dw(s)\bigg\|^2\\
&+16\mathbb{E}\bigg\|\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{T}((t_2-s)^\alpha r)\\
&\quad(\times)\Sigma\bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_2(s,\tau,z_{\varrho(s,z_s+x_s)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\bigg)dw(s)\bigg\|^2\\
&+16\mathbb{E}\bigg\|\sum_{0<t_k<t_1}\big[\mathbb{T}((t_2-t_k)^\alpha r)-\mathbb{T}((t_1-t_k)^\alpha r)\big]\mathbb{E}\big[\mathcal{I}_k(z(t_k^-)+x(t_k^-))\big]\bigg\|^2\\
&+16\mathbb{E}\bigg\|\sum_{t_1<t_k<t_2}\mathbb{T}((t_2-t_k)^\alpha r)\mathbb{E}\big[\mathcal{I}_k(z(t_k^-)+x(t_k^-))\big]\bigg\|^2\\
\le\;& 32\epsilon^2\big[\mathcal{M}_h\widetilde{r}+\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|_{\mathcal{B}}^2)\big]+16\mathcal{N}_0^2\mathbb{E}\bigg\|\mathscr{A}^\beta\mathscr{G}\big(t_2,z_{\varrho(t_2,z_{t_2}+x_{t_2})}+x_{\varrho(t_2,z_{t_2}+x_{t_2})}\big)\\
&\quad-\mathscr{A}^\beta\mathscr{G}\big(t_1,z_{\varrho(t_1,z_{t_1}+x_{t_1})}+x_{\varrho(t_1,z_{t_1}+x_{t_1})}\big)\bigg\|^2\\
&+16\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)^2\frac{t_1^\alpha}{\alpha}\mathcal{M}_{\mathscr{G}}(1+r^*)\int_0^{t_1}(t_1-s)^{\alpha-1}\big\|\mathscr{A}^{1-\beta}\big[\mathbb{T}((t_2-s)^\alpha r)-\mathbb{T}((t_1-s)^\alpha r)\big]\big\|^2ds\\
&+16\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)^2\|\mathscr{A}^{1-\beta}\mathbb{T}((t_2-s)^\alpha r)\|^2\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds\\
&\quad(\times)\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mathcal{M}_{\mathscr{G}}(1+r^*)ds\\
&+16\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)^2\frac{(t_2-t_1)^\alpha}{\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|\mathscr{A}^{1-\beta}\mathbb{T}((t_2-s)^\alpha r)\|^2\mathcal{M}_{\mathscr{G}}(1+r^*)ds\\
&+16\left(\frac{\alpha\epsilon\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^2\frac{t_1^\alpha}{\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}\mathbb{E}\|v^\mu(s)\|^2ds\\
&+16\left(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^2\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]ds\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\mathbb{E}\|v^\mu(s)\|^2ds\\
&+16\left(\frac{\alpha\mathcal{M}\mathcal{M}_B}{\Gamma(1+\alpha)}\right)^2\frac{(t_2-t_1)^\alpha}{\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{E}\|v^\mu(s)\|^2ds\\
&+16\left(\frac{\alpha\epsilon}{\Gamma(1+\alpha)}\right)^2\frac{t_1^\alpha}{\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}\big[\mathscr{F}_1^*r^*+\mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+r^*)T\big]ds\\
&+16\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\big[\mathscr{F}_1^*r^*+\mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+r^*)T\big]\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]ds\\
&\quad(\times)\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]ds\\
&+16\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\frac{(t_2-t_1)^\alpha}{\alpha}\big[\mathscr{F}_1^*r^*+\mathscr{F}_2^*\widetilde{\mathcal{M}}_0(1+r^*)T\big]\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\\
&+16\left(\frac{\alpha\epsilon}{\Gamma(1+\alpha)}\right)^2\frac{t_1^\alpha}{\alpha}tr(Q)\big[\Sigma_1^*r^*+\Sigma_2^*\widetilde{\mathcal{M}}_1(1+r^*)T\big]\int_0^{t_1}(t_1-s)^{\alpha-1}ds\\
&+16\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2tr(Q)\big[\Sigma_1^*r^*+\Sigma_2^*\widetilde{\mathcal{M}}_1(1+r^*)T\big]\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]ds\\
&\quad(\times)\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]ds
\end{aligned}
$$

$$+16\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\frac{(t_2-t_1)^\alpha}{\alpha}tr(Q)\left[\Sigma_1^* r^* + \Sigma_2^*\widetilde{\mathcal{M}}_1(1+r^*)T\right]\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds$$
$$+16\epsilon^2H^2n\sum_{k=1}^{n}\mathcal{M}_{\mathcal{I}_k}\widetilde{r}+16\mathcal{M}^2H^2n\sum_{k=1}^{n}\mathcal{M}_{\mathcal{I}_k}\widetilde{r}.$$

Therefore, for ϵ sufficiently small, the right-hand side of the above inequality tends to zero as $t_1 \to t_2$. Since the compactness of $\mathcal{T}_\alpha(t)$ for $t > 0$ implies the continuity in the uniform operator topology. This proves that $\mathcal{V}$ is right equicontinuous at $t \in (0, T)$. Similarly, we can prove that the right equicontinuity at zero and the left equicontinuity at $t \in (0, T]$. Thus $(\overline{\Upsilon} z)$ is equicontinuous on $[0, T]$. By using a procedure similar to that used in [1], we can easily prove that the map $(\overline{\Upsilon} z)$ is continuous on z which completes the proof that $\Upsilon(\cdot)$ is completely continuos. Hence from the schauder fixed point theorem Υ has a fixed point and consequently the systems (1.1) -(1.3) has a mild solution on $[0, T]$. □

Theorem 3.2. *Assume that the conditions of above theorem hold and, in addition, the function $\mathscr{G}, \mathscr{F}, \Sigma, e_i, \{i = 1,2\}$ and h are uniformly bounded on their respective domains. If $\mathbb{T}(t)$ is compact, then the impulsive fractional neutral stochastic integro-differential equations (1.1)-(1.3) is approximately controllable on $\mathscr{I}$*

Proof. Let $u^\mu(\cdot)$ be fixed point of $\overline{\Upsilon}$. By using the stochastic Fubini theorem, any fixed point of $\overline{\Upsilon}$ is a mild solution of (1.1)-(1.3), if the control $v^\mu(t)$ satisfies

$$u^\mu(T) = \tilde{u}_T - \mu\Phi(v^\mu(\cdot)), \tag{3.9}$$

where

$$\Phi v^\mu(t) = \begin{cases} (\mu\mathcal{I}+\Gamma_0^T)^{-1}\left[\mathbb{E}\tilde{u}_T + \int_0^T \tilde{\phi}(s)dw(s) - \mathcal{T}_\alpha(T)[\varphi(0) - h(z_T + x_T) - \mathscr{G}(0,\varphi)]\right] \\ -(\mu\mathcal{I}+\Gamma_0^T)^{-1}\mathscr{G}(T, z_{\varrho(T,z_T+x_T)} + x_{\varrho(T,z_T+x_T)}) \\ -\int_0^T (\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathscr{A}\mathcal{S}_\alpha(T-s)\mathscr{G}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\right)ds \\ -\int_0^T (\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathcal{S}_\alpha(T-s) \\ \quad (\times)\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)ds \\ -\int_0^T (\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathcal{S}_\alpha(T-s) \\ \quad (\times)\Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)dw(s) \\ -(\mu\mathcal{I}+\Gamma_0^T)^{-1}\sum_{0<t_k<t}\mathcal{T}_\alpha(T-t_k)\mathcal{I}_k(z(t_k^-)+x(t_k^-)). \end{cases}$$

Further, by assumption, $\mathscr{G}, \mathscr{F}, \Sigma, e_i, \{i = 1,2\}$ and h are uniformly bounded on $\mathscr{I}$. Then there are subsequences still denoted by

$$\left\{\mathscr{A}^\beta\mathscr{G}(s, u^\mu_{\varrho(s,u_s)}), \mathscr{F}\left(s, u^\mu_{\varrho(s,u_s)}, \int_0^s e_1(s,\tau,u^\mu_{\varrho(\tau,u_\tau)})d\tau\right), \Sigma\left(s, u^\mu_{\varrho(s,u_s)}, \int_0^s e_2(s,\tau,u^\mu_{\varrho(\tau,u_\tau)})d\tau\right)\right\},$$

which converge weakly to $\{\mathscr{G}(s), \mathscr{F}(s), \Sigma(s)\}$, respectively. Thus from the (3.9), we have

$$\begin{aligned} &\mathbb{E}\|u^\mu(T) - \tilde{u}_T\|^2 \\ &\le 9\mathbb{E}\left\|\mu(\mu\mathcal{I}+\Gamma_0^T)^{-1}\left[\mathbb{E}\tilde{u}_T + \int_0^T \tilde{\phi}(s)dw(s) - \mathcal{T}_\alpha(T)[\varphi(0) - h(z_T + x_T) - \mathscr{G}(0,\varphi)]\right]\right\|^2 \\ &\quad + 9\mathbb{E}\left\|\mu(\mu\mathcal{I}+\Gamma_0^T)^{-1}\mathscr{G}(T, z_{\varrho(T,z_T+x_T)} + x_{\varrho(T,z_T+x_T)})\right\|^2 \end{aligned}$$

$$
\begin{aligned}
&+9\mathbb{E}\left\| \int_0^T \mu(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathscr{A}\mathcal{S}_\alpha(T-s)\left[\mathscr{G}\left(s, z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\right)-\mathscr{G}(s)\right]ds\right\|^2 \\
&+9\mathbb{E}\left\| \int_0^T \mu(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathscr{A}\mathcal{S}_\alpha(T-s)\mathscr{G}(s)ds\right\|^2 \\
&+9\mathbb{E}\left\| \int_0^T \mu(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathcal{S}_\alpha(T-s)\right. \\
&\quad(\times)\left.\left[\mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)-\mathscr{F}(s)\right]ds\right\|^2 \\
&+9\mathbb{E}\left\| \int_0^T \mu(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathcal{S}_\alpha(T-s)\mathscr{F}(s)ds\right\|^2 \\
&+9\mathbb{E}\left\| \int_0^T \mu(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathcal{S}_\alpha(T-s)\right. \\
&\quad(\times)\left.\left[\Sigma\left(s, z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})d\tau\right)-\Sigma(s)\right]dw(s)\right\|^2 \\
&+9\mathbb{E}\left\| \int_0^T \mu(\mu\mathcal{I}+\Gamma_s^T)^{-1}(T-s)^{\alpha-1}\mathcal{S}_\alpha(T-s)\Sigma(s)dw(s)\right\|^2 \\
&+9\mathbb{E}\left\| \mu(\mu\mathcal{I}+\Gamma_0^T)^{-1}\sum_{0<t_k<t}\mathcal{T}_\alpha(T-t_k)\mathcal{I}_k(z(t_k^-)+x(t_k^-))\right\|^2.
\end{aligned}
$$

On the other hand, by lemma 2.3 for all $0 \le s \le T$, the operator $\mu(\mu\mathcal{I}+\Gamma_s^T)^{-1} \to 0$ strongly as $\mu \to 0^+$, and moreover $\|\mu(\mu\mathcal{I}+\Gamma_0^T)^{-1}\| \le 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of $\mathcal{S}_\alpha(t)$, we obtain $\mathbb{E}\|u^\mu(T)-\tilde{u}_T\|^2 \to 0$ as $\mu \to 0^+$. This gives the approximate controllability of (1.1)-(1.3). The proof is now completed. □

4 Application

In this section an illustration is provided for the existence results to the following IFNSIDS with SDD of the structure

$$
\begin{aligned}
&D_t^\alpha\left[u(t,x)-\int_{-\infty}^t \mu_1(s-t)u(s-\varrho_1(t)\varrho_2(\|u(t)\|),x)ds\right] \\
&=\frac{\partial^2}{\partial x^2}u(t,x)+\mu(t,x)+\int_{-\infty}^t \mu_2(t,x,s-t)P_1\Big(u(s-\varrho_1(t)\varrho_2(\|u(t)\|),x)\Big)ds \\
&\quad+\int_0^t\int_{-\infty}^s k_1(s-\tau)P_2\Big(u(\tau-\varrho_1(\tau)\varrho_2(\|u(\tau)\|),x)\Big)d\tau ds \\
&\quad+\left[\int_{-\infty}^t \mu_3(t,x,s-t)Q_1\Big(u(s-\varrho_1(t)\varrho_2(\|u(t)\|),x)\Big)ds\right. \\
&\quad\left.+\int_0^t\int_{-\infty}^s k_2(s-\tau)Q_2\Big(u(\tau-\varrho_1(\tau)\varrho_2(\|u(\tau)\|),x)\Big)d\tau ds\right]\frac{d\beta(t)}{dt},\ x\in[0,\pi],\ 0\le t\le T, \qquad (4.1)
\end{aligned}
$$

$$u(t,0)=0=u(t,\pi), \quad t\ge 0, \qquad (4.2)$$

$$u(0,x)+\int_0^\pi k_3(x,z)u(t,z)dz=\varphi(t,x), \quad t\in(-\infty,0], \quad 0\le x\le\pi, \qquad (4.3)$$

$$\Delta u(t_k,x)=\int_{-\infty}^{t_k}\eta_k(s-t_k)u(s,x)ds, \quad k=1,2,\ldots,n, \qquad (4.4)$$

where $\beta(t)$ is a standard cylindrical Wiener process in $\mathcal{H}$ defined on a stochastic space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathcal{P})$; D_t^α is Caputo's fractional derivative of order $0<\alpha<1$; φ is continuous; and $0<t_1<t_2<\cdots<t_n<T$

are prefixed numbers. We consider $\mathcal{H} = \mathcal{K} = L^2[0,\pi]$ having the norm $\|\cdot\|_{\mathcal{L}^2}$ and define the operator $\mathscr{A} : D(\mathscr{A}) \subset \mathcal{H} \to \mathcal{H}$ by $\mathscr{A}w = w''$ with the domain

$$D(\mathscr{A}) = \{w \in \mathcal{H} : w, w' \quad \text{are absolutely continuous}, \quad w'' \in \mathcal{H}, \, w(0) = w(\pi) = 0\}.$$

Then

$$\mathscr{A}w = \sum_{n=1}^{\infty} n^2 \langle w, w_n \rangle w_n, \quad w \in D(\mathscr{A}),$$

in which $w_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns)$, $n = 1, 2, \ldots, .$ is the orthogonal set of eigenvectors of $\mathscr{A}$. It is long familiar that $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t \geq 0}$ in $\mathcal{H}$ and is provided by

$$\mathbb{T}(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} \langle w, w_n \rangle w_n, \quad \text{for all} \quad w \in \mathcal{H}, \quad \text{and every} \quad t > 0.$$

If we fix $\beta = \frac{1}{2}$, then the operator $(\mathscr{A})^{\frac{1}{2}}$ is given by

$$(\mathscr{A})^{\frac{1}{2}} w = \sum_{n=1}^{\infty} n \langle w, w_n \rangle w_n, \;\; w \in (D(\mathscr{A})^{\frac{1}{2}}),$$

in which $(D(\mathscr{A})^{\frac{1}{2}}) = \left\{ \omega(\cdot) \in \mathcal{H} : \sum_{n=1}^{\infty} n \langle \omega, w_n \rangle w_n \in \mathcal{H} \right\}$ and $\|(\mathscr{A})^{-\frac{1}{2}}\| = 1$. Let $\gamma < 0$, define the phase space

$$\mathcal{B} = \left\{ \varphi \in C((-\infty, 0], \mathcal{H}) : \lim_{\theta \to -\infty} e^{\gamma\theta} \varphi(\theta) \quad \text{exists in} \quad \mathcal{H} \right\},$$

and let $\|\varphi\|_{\mathcal{B}} = \sup_{\theta \in (-\infty, 0]} \left\{ e^{\gamma\theta} \|\varphi(\theta)\|_{\mathcal{L}^2} \right\}$, then $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space satisfies $(P_1) - (P_3)$ with $H = 1, \mathscr{E}_1(t) = \max\{1, e^{-\gamma t}\}, \mathscr{E}_2(t) = e^{-\gamma t}$. Therefore, for $(t, \varphi) \in [0, T] \times \mathcal{B}$, where $\varphi(\theta)(x) = \varphi(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times [0, \pi]$. Set

$$u(t)(x) = u(t, x), \quad \varrho(t, \varphi) = \varrho_1(t) \varrho_2(\|\varphi(0)\|),$$

we have

$$\mathscr{G}(t, \varphi)(x) = \int_{-\infty}^{0} \mu_1(\theta) \varphi(\theta)(x) d\theta,$$

$$\mathscr{F}(t, \varphi, \mathscr{H}\varphi)(x) = \int_{-\infty}^{0} \mu_2(t, x, \theta) P_1(\varphi(\theta)(x)) d\theta + \mathscr{H}\varphi(x),$$

$$\Sigma(t, \varphi, \overline{\mathscr{H}}\varphi)(x) = \int_{-\infty}^{0} \mu_3(t, x, \theta) Q_1(\varphi(\theta)(x)) d\theta + \overline{\mathscr{H}}\varphi(x)$$

and

$$\mathcal{I}_k(\varphi)(x) = \int_{-\infty}^{0} \eta_k(\theta) \varphi(\theta)(x) d\theta, \quad k = 1, 2, \ldots, n,$$

where

$$\mathscr{H}\varphi(x) = \int_0^t \int_{-\infty}^{0} k_1(s - \theta) P_2(\varphi(\theta)(x)) d\theta ds, \quad \overline{\mathscr{H}}\varphi(x) = \int_0^t \int_{-\infty}^{0} k_2(s - \theta) Q_2(\varphi(\theta)(x)) d\theta ds.$$

Further, define the bounded linear operator $B : U \to \mathcal{H}$ by $Bv(t)(x) = \mu(t, x)$, $0 \leq x \leq \pi, u \in U$, where $\mu : [0, 1] \times [0, \pi] \to [0, \pi]$ is continuous. Now, under the above conditions, we can represent the system (4.1) - (4.4) in the abstract form (1.1) - (1.3) . Hence , according to Theorem 3.2, system (4.1) - (4.4) is approximately controllable on $[0, T]$.

5 Conclusion

In this manuscript, we have studied the approximate controllability results for impulsive stochastic fractional neutral integro-differential systems with non-local and state-dependent delay conditions in Hilbert space. More precisely, by utilizing the stochastic analysis theory, fractional powers of operators and Schauder fixed point theorem, we investigate the IFNSIDS with NLCs and SDD in Hilbert space. To validate the obtained theoretical results, one example is analyzed. The FDEs are very efficient to describe the real-life phenomena; thus, it is essential to extend the present study to establish the other qualitative and quantitative properties such as stability and controllability.

There are two direct issues which require further study. First, we will investigate the approximate controllability of fractional neutral stochastic integro-differential systems with state-dependent delay both in the case of a Poisson jumps and a normal topological space. Secondly, we will be devoted to studying the approximate controllability of a new class of impulsive fractional stochastic differential equations with state-dependent delay and non-instantaneous impulses as discussed in [15].

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