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General Solution and Two Methods of Generalized Ulam - Hyers Stability of $n-$ Dimensional AQCQ Functional Equation

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Abstract

In this paper, we achieve the general solution and generalized Ulam - Hyers stability of a $n-$ dimensional additive-quadratic-cubic-quartic (AQCQ) functional equation

$$f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) = 4f\left(\sum_{i=1}^{n} v_i\right) + 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) - 6f\left(\sum_{i=1}^{n-1} v_i\right) \\ + f(2v_n) + f(-2v_n) - 4f(v_n) - 4f(-v_n)$$

where n is a positive integer with $n \geq 3$ in Banach Space (**BS**) via direct and fixed point methods. The stability results are discussed in two different ways by assuming n is an odd positive integer and n is an even positive integer.

Keywords: AQCQ functional equation, generalized Ulam - Hyers stability, Banach space, fixed point.

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1 Introduction

The education of stability problems for functional equations is tied to a question of Ulam [61] regarding the stability of group homomorphisms and certainly answered for a additive functional equation on Banach spaces by Hyers [30] and Aoki [3]. It was further generalized and marvelous outcome obtained by number of authors [24, 44, 53, 58].

The general solution and the generalized Hyers-Ulam-Rassias stability of the generalized mixed type of functional equation

$$f(x+ay) + f(x-ay) = a^2\left[f(x+y) + f(x-y)\right] + 2\left(1-a^2\right)f(x) \\ + \frac{\left(a^4-a^2\right)}{12}\left[f(2y) + f(-2y) - 4f(y) - 4f(-y)\right]. \tag{1.1}$$

for fixed integers a with $a \neq 0, \pm 1$ having solution **additive, quadratic, cubic and quartic** was discussed by K. Ravi et. al., [59] . Its generalized Ulam-Hyers stability in multi-Banach spaces and non-Archimedean normed spaces via fixed point approach was respectively investigated by T.Z. Xu et. al [62, 63].

Very recently, Choonkil Park and Jung Rye Lee [42] proved the Hyers - Ulam stability of the following additive - quadratic - cubic - quartic functional equation

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y) \tag{1.2}$$

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in paranormed spaces.

During the last seven decades, the stability problems of various functional equations in several spaces have been broadly investigated by number of mathematicians [4] - [18], [20] - [23], [25] - [29], [32] - [34], [39, 40, 43], [48] - [52], [56, 57].

Now, we will recall the fundamental results in fixed point theory [36].

Theorem 1.1. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ for some (Lipschitz constant) $L < 1$. Then,
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is

(A2) $lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in X$;
(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L} d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ x \in X$;

(A4) $d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X$.

Theorem 1.2. *Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 such that
(FP1) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;
(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T
(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;
(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

In this paper, we established the generalized Ulam - Hyers stability of a $n-$ dimensional additive-quadratic-cubic-quartic (AQCQ) functional equation

$$\begin{aligned} f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) = 4f\left(\sum_{i=1}^{n} v_i\right) + 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) - 6f\left(\sum_{i=1}^{n-1} v_i\right) \\ + f\,(2v_n) + f\,(-2v_n) - 4f\,(v_n) - 4f\,(-v_n) \end{aligned} \tag{1.3}$$

where n is a positive integer with $n \geq 3$ in Banach Space (**BS**) via direct and fixed point methods. The stability results are discussed in two different ways by assuming n is an odd positive integer and n is an even positive integer.

In section 2, the general solution of (1.3) is present.

In Sections 3 and 4, the generalized Ulam-Hyers stability of the functional equation (1.3) where n is an odd positive integer and n is an even positive integer in Banach space using direct method are discussed, respectively.

In Sections 5 and 6, we investigate the generalized Ulam-Hyers stability of the functional equation (1.3) where n is an odd positive integer and n is an even positive integer in Banach space using fixed point methods, respectively.

In Section 7, we conclude with the non stable cases for the functional equation (1.3).

2 General Solution

In this section, we provide the general solution of the function equation (1.3). To prove this, let us take $\mathcal{I}$ and $\mathcal{J}$ be real vector spaces.

Lemma 2.1. *If a function $f : \mathcal{I} \to \mathcal{J}$ fulfills (1.3) for all $v_1, \cdots, v_n \in \mathcal{I}$ if and only if $f : \mathcal{I} \to \mathcal{J}$ satisfies (1.2) for all $x, y \in \mathcal{I}$.*

Proof. Let $f : \mathcal{I} \to \mathcal{J}$ be a function fulfills (1.3). Replacing $(v_1, v_2, v_3 \cdots, v_{n-1}, v_n)$ by $(x, 0, 0, \cdots, 0, y)$ in (1.3), we get (1.2) as desired. Conversely, let $f : \mathcal{I} \to \mathcal{J}$ be a function satisfying (1.2). Changing (x, y) by $(v_1 + v_2 + v_3 \cdots + v_{n-1}, v_n)$ in (1.2), we arrive (1.3) as desired. □

Lemma 2.2. *If $f : \mathcal{I} \to \mathcal{J}$ be an odd mapping fulfills (1.3) and let $a : \mathcal{I} \to \mathcal{J}$ be a mapping given by*

$$a(v) = f(2v) - 8f(v) \tag{2.1}$$

for all $v \in \mathcal{I}$ then

$$a(2v) = 2a(v) \tag{2.2}$$

for all $v \in \mathcal{I}$ such that a is additive.

Proof. Using oddness of f in (1.3), we arrive

$$f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) = 4f\left(\sum_{i=1}^{n} v_i\right) + 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) - 6f\left(\sum_{i=1}^{n-1} v_i\right) \tag{2.3}$$

for all $v_1, \cdots, v_n \in \mathcal{I}$. Letting $(v_1, \cdots, v_n)$ by $(0, \cdots, 0)$ in (2.3), we find that

$$f(0) = 0. \tag{2.4}$$

Also, replacing $(v_2, v_3, \cdots, v_{n-1})$ by $(0, 0, \cdots, 0)$ in (2.3), we get

$$f(v_1 + 2v_n) + f(v_1 - 2v_n) = 4f(v_1 + v_n) + 4f(v_1 - v_n) - 6f(v_1) \tag{2.5}$$

for all $v_1, v_n \in \mathcal{I}$. Changing (v_1, v_n) by (v, v) in (2.5), we obtain

$$f(3v) = 4f(2v) - 5f(v) \tag{2.6}$$

for all $v \in \mathcal{I}$. Again changing (v_1, v_n) by $(2v, v)$ in (2.5) and using (2.4), (2.6), we arrive

$$f(4v) = 4f(3v) - 6f(2v) + 4f(v) \tag{2.7}$$

for all $v \in \mathcal{I}$. Using (2.6) in (2.7), we get

$$f(4v) = 10f(2v) - 16f(v) \tag{2.8}$$

for all $v \in \mathcal{I}$. From (2.1), we have

$$a(2v) - 2a(v) = f(4v) - 10f(2v) + 16f(v) \tag{2.9}$$

for all $v \in \mathcal{I}$. Using (2.8) in (2.9), we desired our result. □

Lemma 2.3. *If $f : \mathcal{I} \to \mathcal{J}$ be an odd mapping fulfills (1.3) and let $c : \mathcal{I} \to \mathcal{J}$ be a mapping given by*

$$c(v) = f(2v) - 2f(v) \tag{2.10}$$

for all $v \in \mathcal{I}$ then

$$c(2v) = 8c(v) \tag{2.11}$$

for all $v \in \mathcal{I}$ such that c is cubic.

Proof. It follows from (2.11) that

$$c(2v) - 8c(v) = f(4v) - 10f(2v) + 16f(v) \tag{2.12}$$

for all $u \in \mathcal{I}$. Using (2.8) in (2.12), we desired our result. □

Lemma 2.4. *If $f : \mathcal{I} \to \mathcal{J}$ be an even mapping fulfills (1.3) and let $q_2 : \mathcal{I} \to \mathcal{J}$ be a mapping given by*

$$q_2(v) = f(2v) - 16f(v) \tag{2.13}$$

for all $v \in \mathcal{I}$ then

$$q_2(2v) = 4q_2(v) \tag{2.14}$$

for all $v \in \mathcal{I}$ such that q_2 is quadratic.

Proof. Using evenness of f in (1.3), we get

$$f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) = 4f\left(\sum_{i=1}^{n} v_i\right) + 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) - 6f\left(\sum_{i=1}^{n-1} v_i\right) + 2f(2v_n) - 8f(v_n) \tag{2.15}$$

for all $v_1, \cdots, v_n \in \mathcal{I}$. Letting $(v_1, \cdots, v_n)$ by $(0, \cdots, 0)$ in (2.15), we obtain

$$f(0) = 0. \tag{2.16}$$

Replacing $(v_2, v_3, \cdots, v_{n-1})$ by $(0, 0, \cdots, 0)$ in (2.15), we arrive

$$f(v_1 + 2v_n) + f(v_1 - 2v_n) = 4f(v_1 + v_n) + 4f(v_1 - v_n) - 6f(v_1) + 2f(2v_n) - 8f(v_n) \tag{2.17}$$

for all $v_1, v_n \in \mathcal{I}$. Setting (v_1, v_n) by (v, v) in (2.17), we have

$$f(3v) = 6f(2v) - 15f(v) \tag{2.18}$$

for all $v \in \mathcal{I}$. Again setting (v_1, v_n) by $(2v, v)$ in (2.17) and using (2.16), (2.18), we arrive

$$f(4v) = 4f(3v) - 4f(2v) - 4f(v) \tag{2.19}$$

for all $v \in \mathcal{I}$. Using (2.18) in (2.19), we get

$$f(4v) = 20f(2v) - 64f(v) \tag{2.20}$$

for all $v \in \mathcal{I}$. From (2.13), we establish

$$q_2(2v) - 4q_2(v) = f(4v) - 20f(2v) + 64f(v) \tag{2.21}$$

for all $v \in \mathcal{I}$. Using (2.20) in (2.21), we desired our result. □

Lemma 2.5. *If $f : \mathcal{I} \to \mathcal{J}$ be an even mapping fulfills (1.3) and let $q_4 : \mathcal{I} \to \mathcal{J}$ be a mapping given by*

$$q_4(v) = f(2v) - 4f(v) \tag{2.22}$$

for all $v \in \mathcal{I}$ then

$$q_4(2v) = 16q_4(v) \tag{2.23}$$

for all $v \in \mathcal{I}$ such that q_4 is quartic.

Proof. It follows from (2.23) that

$$q_4(2v) - 4q_4(v) = f(4v) - 20f(2v) + 64f(v) \tag{2.24}$$

for all $v \in \mathcal{I}$. Using (2.20) in (2.24), we desired our result. □

Remark 2.1. *If $f : \mathcal{I} \to \mathcal{J}$ be a mapping fulfills (1.3) then there exists $f_o, f_e : \mathcal{I} \to \mathcal{J}$ and let $a, q_2, c, q_4 : \mathcal{I} \to \mathcal{J}$ be a mapping defined in (2.1), (2.10), (2.13) and (2.22), we have*

$$f_e(v) = \frac{1}{12}(q_4(v) - q_2(v)) \tag{2.25}$$

and

$$f_o(v) = \frac{1}{6}(c(v) - a(v)) \tag{2.26}$$

for all $v \in \mathcal{I}$. Also if we define

$$f(v) = f_e(v) + f_o(v) \tag{2.27}$$

we arrive

$$f(v) = \frac{1}{12}(q_4(v) - q_2(v)) + \frac{1}{6}(c(v) - a(v)) \tag{2.28}$$

for all $v \in \mathcal{I}$.

Throughout this paper, let we consider $\mathcal{Y}$ be a normed space and $\mathcal{Z}$ be a Banach space. Define a mapping $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ by

$$\begin{aligned} Df_{aqcq}(v_1, \cdots, v_n) = f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) \\ - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) + 6f\left(\sum_{i=1}^{n-1} v_i\right) \\ - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \end{aligned}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$.

3 Stability Results - Direct Method: n Odd Positive Integer

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.3) where n is an odd positive integer in Banach space using direct method.

3.1 f IS AN ODD FUNCTION

Theorem 3.3. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$\lim_{p\to\infty} \frac{\omega\left(2^{pq} v_1, \cdots, 2^{pq} v_n\right)}{2^{pq}} = 0 \tag{3.1}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd function satisfying the inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{3.2}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{A}(v) - a(v)\| = \|\mathcal{A}(v) - \{f(2v) - 8f(v)\}\| \leq \frac{1}{2} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{rq}} \tag{3.3}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)$ and $\mathcal{A}(v)$ are defined by

$$\begin{aligned} &\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v) \\ &= 4\omega_1(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v) + \omega_1(2 \cdot 2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v) \\ &= 4\omega\left(2^{rq}v, \underbrace{2^{rq}v, 2^{rq}v, \cdots, 2^{rq}v}_{\frac{n-3}{2}\ times}, \underbrace{-2^{rq}v, -2^{rq}v, \cdots, -2^{rq}v}_{\frac{n-3}{2}\ times}, 0, 2^{rq}v\right) \\ &\quad + \omega\left(2 \cdot 2^{rq}v, \underbrace{2^{rq}v, 2^{rq}v, \cdots, 2^{rq}v}_{\frac{n-3}{2}\ times}, \underbrace{-2^{rq}v, -2^{rq}v, \cdots, -2^{rq}v}_{\frac{n-3}{2}\ times}, 0, 2^{rq}v\right) \end{aligned} \tag{3.4}$$

and

$$\mathcal{A}(v) = \lim_{p\to\infty} \frac{a(2^{pq}v)}{2^{pq}} = \lim_{p\to\infty} \frac{f(2 \cdot 2^{pq}v) - 8f(2^{pq}v)}{2^{pq}} \tag{3.5}$$

for all $v \in \mathcal{Y}$, respectively.

Proof. **Case (i):** Assume $q = 1$.

Given, f is an odd function. Using oddness of f in (3.2), we arrive

$$\begin{aligned} \Bigg\| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) \\ -4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) + 6f\left(\sum_{i=1}^{n-1} v_i\right)\Bigg\| \leq \omega\left(v_1, \cdots, v_n\right) \end{aligned} \tag{3.6}$$

for all $v_1, v_2, v_3, v_4, \cdots, v_{n-1}, v_n \in \mathcal{Y}$. Replacing

$$(v_1, v_2, v_3, v_4, \cdots, v_{n-1}, v_n) = \left(v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2} \ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2} \ times}, 0, v \right)$$

in (3.6), we get

$$\begin{aligned} \left\| f(3v) - 4f(2v) + 5f(v) \right\| &\leq \omega \left(v, \ \underbrace{v, v, \cdots, v}_{\frac{n-3}{2} \ times} \ , \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2} \ times} \ , 0, v \right) \\ &= \omega_1(v, v, \cdots, -v, 0, v) \end{aligned} \tag{3.7}$$

for all $v \in \mathcal{Y}$. Again replacing

$$(v_1, v_2, v_3, v_4, \cdots, v_{n-1}, v_n) = \left(2v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2} \ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2} \ times}, 0, v \right)$$

in (3.6), we obtain

$$\begin{aligned} \left\| f(4v) - 4f(3v) + 6f(2v) - 4f(v) \right\| &\leq \omega \left(2v, \ \underbrace{v, v, \cdots, v}_{\frac{n-3}{2} \ times} \ , \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2} \ times} \ , 0, v \right) \\ &= \omega_1(2v, v, \cdots, -v, 0, v) \end{aligned} \tag{3.8}$$

for all $v \in \mathcal{Y}$. It follows from (3.7) and (3.8),

$$\begin{aligned} &\left\| f(4v) - 10f(2v) + 16f(v) \right\| \\ &\quad = \left\| f(4v) - 4f(3v) + 4f(3v) + 6f(2v) - 16f(2v) + 20f(v) - 4f(v) \right\| \\ &\quad \leq 4 \left\| f(3v) - 4f(2v) + 5f(v) \right\| + \left\| f(4v) - 4f(3v) + 6f(2v) - 4f(v) \right\| \\ &\quad \leq 4\omega_1(v, v, \cdots, -v, 0, v) + \omega_1(2v, v, \cdots, -v, 0, v) \end{aligned} \tag{3.9}$$

for all $v \in \mathcal{Y}$. Define

$$\Omega(v, v, \cdots, -v, 0, v) = 4\omega_1(v, v, \cdots, -v, 0, v) + \omega_1(2v, v, \cdots, -v, 0, v) \tag{3.10}$$

for all $v \in \mathcal{Y}$. Using (3.10) in (3.9), we have

$$\left\| f(4v) - 10f(2v) + 16f(v) \right\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{3.11}$$

for all $v \in \mathcal{Y}$. It follows from (3.11), we reach

$$\left\| \Big\{ f(4v) - 8f(2v) \Big\} - 2 \Big\{ f(2v) - 8f(v) \Big\} \right\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{3.12}$$

for all $v \in \mathcal{Y}$. Using (2.1) in (3.12), we land

$$\left\| a(2v) - 2a(v) \right\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{3.13}$$

for all $v \in \mathcal{Y}$. It follows from (3.13) that

$$\left\| \frac{a(2v)}{2} - a(v) \right\| \leq \frac{\Omega(v, v, \cdots, -v, 0, v)}{2} \tag{3.14}$$

for all $v \in \mathcal{Y}$. Now, replacing v by $2v$ and dividing by 2 in (3.14), we have

$$\left\| \frac{a(2^2 v)}{2^2} - \frac{a(2v)}{2} \right\| \leq \frac{\Omega(2v, 2v, \cdots, -2v, 0, 2v)}{2^2} \tag{3.15}$$

for all $v \in \mathcal{Y}$. From (3.14) and (3.15), we obtain

$$\begin{aligned} \left\| \frac{a(2^2 v)}{2^2} - a(v) \right\| &\leq \left\| \frac{a(2^2 v)}{2^2} - \frac{a(2v)}{2} \right\| + \left\| \frac{a(2v)}{2} - a(v) \right\| \\ &\leq \frac{1}{2} \left[\Omega(v, v, \cdots, -v, 0, v) + \frac{\Omega(2v, 2v, \cdots, -2v, 0, 2v)}{2} \right] \end{aligned} \tag{3.16}$$

for all $v \in \mathcal{Y}$. Generalizing, for a positive integer p, we reach

$$\left\| \frac{a(2^p v)}{2^p} - a(v) \right\| \leq \frac{1}{2} \sum_{r=0}^{p-1} \frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^r} \tag{3.17}$$

for all $v \in \mathcal{Y}$. Thus, the sequence $\left\{ \frac{a(2^p v)}{2^p} \right\}$ is a Cauchy in $\mathcal{Z}$ and so it converges.

Indeed, to prove the convergence of the sequence $\left\{ \frac{a(2^p v)}{2^p} \right\}$, replacing v by $2^s v$ and dividing by 2^s in (3.17), for any $p, s > 0$, we get

$$\begin{aligned} \left\| \frac{a(2^{p+s} v)}{2^{(p+s)}} - \frac{a(2^s v)}{2^s} \right\| &= \frac{1}{2^s} \left\| \frac{a(2^p \cdot 2^s v)}{2^p} - a(2^s v) \right\| \\ &\leq \frac{1}{2^s} \frac{1}{2} \sum_{r=0}^{p-1} \frac{\Omega(2^r \cdot 2^s v, 2^r \cdot 2^s v, \cdots, -2^r \cdot 2^s v, 0, 2^r \cdot 2^s v)}{2^r} \\ &\leq \frac{1}{2} \sum_{r=0}^{\infty} \frac{\Omega(2^r \cdot 2^s v, 2^r \cdot 2^s v, \cdots, -2^r \cdot 2^s v, 0, 2^r \cdot 2^s v)}{2^r \cdot 2^s} \\ &\to 0 \quad as \;\; s \to \infty \end{aligned}$$

for all $v \in \mathcal{Y}$. Since $\mathcal{Z}$ is complete, we see that a mapping $\mathcal{A}(v) : \mathcal{Y} \to \mathcal{Z}$ defined by

$$\mathcal{A}(v) = \lim_{p \to \infty} \frac{a(2^p v)}{2^p}$$

for all $v \in \mathcal{Y}$. Letting $p \to \infty$ in (3.17), we see that (3.3) holds for all $v \in \mathcal{Y}$. In order to show that $\mathcal{A}$ satisfies (1.3), replacing $(v_1, \cdots, v_n)$ by $(2^p v_1, \cdots, 2^p v_n)$ and dividing by 2^p in (3.2), we have

$$\|\mathcal{A}(v_1, \cdots, v_n)\| = \lim_{p \to \infty} \frac{1}{2} \left\| Df_{aqcq}(2^p v_1, \cdots, 2^p v_n) \right\| \leq \lim_{p \to \infty} \frac{1}{2} \omega(2^p v_1, \cdots, 2^p v_n)$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ and so the mapping $\mathcal{A}$ is additive. Hence, $\mathcal{A}$ satisfies (1.3), for all $v_1, \cdots, v_n \in \mathcal{Y}$.

To prove that $\mathcal{A}$ is unique, we assume now that there is $\mathcal{A}'$ as another additive mapping satisfying (1.3) and the inequality (3.3). Then it follows easily that

$$\mathcal{A}(2^s v) = 2^s \mathcal{A}(v), \qquad \mathcal{A}'(2^s v) = 2^s \mathcal{A}'(v)$$

for all $v \in \mathcal{Y}$ and all $s \in \mathbb{N}$. Thus

$$\begin{aligned} \|\mathcal{A}(v) - \mathcal{A}'(v)\| &= \frac{1}{2^s} \|\mathcal{A}(2^s v) - \mathcal{A}'(2^s v)\| \\ &= \frac{1}{2^s} \left\{ \|\mathcal{A}(2^s v) - a(2^s v) + a(2^s v) - \mathcal{A}'(2^s v)\| \right\} \\ &\leq \frac{1}{2^s} \left\{ \|\mathcal{A}(2^s v) - a(2^s v)\| + \|a(2^s v) - \mathcal{A}'(2^s v)\| \right\} \\ &\leq \sum_{r=0}^{\infty} \frac{\Omega(2^r \cdot 2^s v, 2^r \cdot 2^s v, \cdots, -2^r \cdot 2^s v, 0, 2^r \cdot 2^s v)}{2^{(r+s)}} \end{aligned}$$

for all $v \in \mathcal{Y}$. Letting $s \to \infty$, in the above inequality, we achieve uniqueness of $\mathcal{A}$. Hence the theorem holds for $q = 1$.

Case (ii): Assume $q = -1$.

Now replacing v by $\frac{v}{2}$ in (3.13), we get

$$\left\| a(v) - 2a\left(\frac{v}{2}\right) \right\| \leq \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right) \tag{3.18}$$

for all $v \in \mathcal{Y}$. Now, replacing v by $\frac{v}{2}$ and multiply by 2 in (3.18), we have

$$\left\| 2a\left(\frac{v}{2}\right) - 2^2 a\left(\frac{v}{2^2}\right) \right\| \leq 2\Omega\left(\frac{v}{2^2}, \frac{v}{2^2}, \cdots, -\frac{v}{2^2}, 0, \frac{v}{2^2}\right) \tag{3.19}$$

for all $v \in \mathcal{Y}$. From (3.18) and (3.19), we obtain

$$\begin{aligned} \left\| a(v) - 2^2 a\left(\frac{v}{2^2}\right) \right\| &\leq \left\| a(v) - 2a\left(\frac{v}{2}\right) \right\| + \left\| 2a\left(\frac{v}{2}\right) - 2^2 a\left(\frac{v}{2^2}\right) \right\| \\ &\leq \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right) + 2\Omega\left(\frac{v}{2^2}, \frac{v}{2^2}, \cdots, -\frac{v}{2^2}, 0, \frac{v}{2^2}\right) \end{aligned} \tag{3.20}$$

for all $v \in \mathcal{Y}$. Generalizing, for a positive integer p, we reach

$$\left\| a(v) - 2^p a\left(\frac{v}{2^p}\right) \right\| \leq \sum_{r=1}^{p-1} 2^r \Omega\left(\frac{v}{2^r}, \frac{v}{2^r}, \cdots, -\frac{v}{2^r}, 0, \frac{v}{2^r}\right) \tag{3.21}$$

for all $v \in \mathcal{Y}$. The rest of the proof is similar to that of case $q = 1$. Hence for $q = -1$ also the theorem holds. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.3 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.1. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d such that*

$$\left\| Df_{aqcq}(v_1, \cdots, v_n) \right\| \leq \begin{cases} b, & \\ b \sum\limits_{i=1}^{n} ||v_i||^d, & d \neq 1; \\ b \left\{ \prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd} \right\}, & nd \neq 1; \end{cases} \tag{3.22}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\| a(v) - \mathcal{A}(v) \| \leq \begin{cases} 5b, \\ \dfrac{b||v||^d (5n - 6 + 2^{rd})}{|2 - 2^d|}, \\ \dfrac{b||v||^{nd} (5n - 6 + 2^{rnd})}{|2 - 2^{nd}|} \end{cases} \tag{3.23}$$

for all $v \in \mathcal{Y}$.

Theorem 3.4. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$\lim_{p \to \infty} \frac{\omega\left(2^{pq} v_1, \cdots, 2^{pq} v_n\right)}{2^{3pq}} = 0 \tag{3.24}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd function satisfying the inequality

$$\left\| Df_{aqcq}(v_1, \cdots, v_n) \right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{3.25}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\| \mathcal{C}(v) - c(v) \| = \| \mathcal{C}(v) - \{ f(2v) - 2f(v) \} \| \leq \frac{1}{8} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{rq}} \tag{3.26}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)$ is defined in (3.4) and $\mathcal{C}(v)$ is defined by

$$\mathcal{C}(v) = \lim_{p \to \infty} \frac{c(2^{pq}v)}{2^{3pq}} = \lim_{p \to \infty} \frac{f(2 \cdot 2^{pq}v) - 2f(2^{pq}v)}{2^{3pq}} \tag{3.27}$$

for all $v \in \mathcal{Y}$, respectively.

Proof. It follows from (3.11), we reach

$$\left\|\left\{f(4v)-2f(2v)\right\}-8\left\{f(2v)-2f(v)\right\}\right\| \leq \Omega(v,v,\cdots,-v,0,v) \tag{3.28}$$

for all $v \in \mathcal{Y}$. Using (2.10) in (3.28), we land

$$\left\|c(2v)-8c(v)\right\| \leq \Omega(v,v,\cdots,-v,0,v) \tag{3.29}$$

for all $v \in \mathcal{Y}$. The rest of the proof is similar to that of Theorem 3.3 . This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.4 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.2. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\| \leq \begin{cases} b, & \\ b\sum\limits_{i=1}^{n}||v_i||^d, & d \neq 3; \\ b\left\{\prod\limits_{i=1}^{n}||v_i||^d+\sum\limits_{i=1}^{n}||v_i||^{nd}\right\}, & nd \neq 3; \end{cases} \tag{3.30}$$

for all $v_1,\cdots,v_n \in \mathcal{Y}$, then there existss a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|c(v)-\mathcal{C}(v)\| \leq \begin{cases} \dfrac{5b}{|7|}, \\ \dfrac{b||v||^d(5n-6+2^{rd})}{|8-2^d|}, \\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{|8-2^{nd}|} \end{cases} \tag{3.31}$$

for all $v \in \mathcal{Y}$.

Theorem 3.5. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0,\infty)$ be functions satisfying (3.1) and (3.24) for all $v_1,\cdots,v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd function satisfying the inequality*

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\| \leq \omega\left(v_1,\cdots,v_n\right) \tag{3.32}$$

for all $v_1,\cdots,v_n \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ and a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v)-\mathcal{A}(v)-\mathcal{C}(v)\| \leq \frac{1}{6}\left\{\frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)}{2^{rq}} + \frac{1}{8}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)}{2^{rq}}\right\} \tag{3.33}$$

where $\Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)$, $\mathcal{A}(v)$ and $\mathcal{C}(v)$ is defined in (3.4), (3.5) and (3.27) for all $v \in \mathcal{Y}$, respectively.

Proof. **Case (i):** For $q = 1$. Given f is an odd function.

If $f : \mathcal{Y} \to \mathcal{Z}$ satisfies (3.32) then by Theorem 3.3, there exists a unique additive function $\mathcal{A}' : \mathcal{Y} \to \mathcal{Z}$ such that

$$\left\|\mathcal{A}'(v)-(f(2v)-8f(v))\right\| \leq \frac{1}{2}\sum_{r=0}^{\infty}\frac{\Omega(2^rv,2^rv,\cdots,-2^rv,0,2^rv)}{2^r} \tag{3.34}$$

for all $v \in \mathcal{Y}$.

Also, if $f : \mathcal{Y} \to \mathcal{Z}$ satisfies (3.32) then by Theorem 3.4, there exists a unique cubic function $\mathcal{C}' : \mathcal{Y} \to \mathcal{Z}$ such that

$$\left\|\mathcal{C}'(v)-(f(2v)-2f(v))\right\| \leq \frac{1}{8}\sum_{r=0}^{\infty}\frac{\Omega(2^rv,2^rv,\cdots,-2^rv,0,2^rv)}{2^r} \tag{3.35}$$

for all $v \in \mathcal{Y}$. Combining (3.34) and (3.35), we achieve

$$
\begin{aligned}
&\left\| \frac{1}{6}\mathcal{A}'(v) - \frac{1}{6}\mathcal{C}'(v) - f(v) \right\| \\
&\quad = \left\| \frac{1}{6}\mathcal{A}'(v) - \frac{1}{6}f(2v) - \frac{8}{6}f(v) - \frac{1}{6}\mathcal{C}'(v) + \frac{1}{6}f(2v) - \frac{2}{6}f(v) \right\| \\
&\quad \leq \left\| \frac{1}{6}\mathcal{A}'(v) - \frac{1}{6}(f(2v) - 8f(v)) \right\| + \left\| \frac{1}{6}\mathcal{C}'(v) - \frac{1}{6}(f(2v) + 2f(v)) \right\| \\
&\quad \leq \frac{1}{6}\left\{ \left\|\mathcal{A}'(v) - (f(2v) - 8f(v))\right\| + \left\|\mathcal{C}'(v) - (f(2v) - 2f(v))\right\| \right\} \\
&\quad \leq \frac{1}{6}\left\{ \frac{1}{2}\sum_{r=0}^{\infty} \frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^r} + \frac{1}{8}\sum_{r=0}^{\infty} \frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^r} \right\}
\end{aligned}
$$

for all $v \in \mathcal{Y}$. Defining

$$
\mathcal{A}(v) = \frac{1}{6}\mathcal{A}'(v); \qquad \mathcal{C}(v) = \frac{-1}{6}\mathcal{C}(v)
$$

we arrive (3.33) as desired. Similarly, we can prove for $j = -1$. Hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 3.5 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.3. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d such that*

$$
\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, & \\ b\sum\limits_{i=1}^{n} ||v_i||^d, & d \neq 1,3; \\ b\left\{ \prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd} \right\}, & nd \neq 1,3; \end{cases} \tag{3.36}
$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ and a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that

$$
\|f(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \leq \begin{cases} \frac{5b}{6}\left(1 + \frac{1}{|7|}\right), \\ \frac{b||v||^d(5n - 6 + 2^{rd})}{6}\left(\frac{1}{|2 - 2^d|} + \frac{1}{|8 - 2^d|}\right), \\ \frac{b||v||^{nd}(5n - 6 + 2^{rnd})}{6}\left(\frac{1}{|2 - 2^{nd}|} + \frac{1}{|8 - 2^{nd}|}\right) \end{cases} \tag{3.37}
$$

for all $v \in \mathcal{Y}$.

3.2 f IS AN EVEN FUNCTION

Theorem 3.6. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$
\lim_{p \to \infty} \frac{\omega\left(2^{pq}v_1, \cdots, 2^{pq}v_n\right)}{2^{2pq}} = 0 \tag{3.38}
$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even function satisfying the inequality

$$
\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{3.39}
$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$
\|\mathcal{Q}_2(v) - q_2(v)\| = \|\mathcal{Q}_2(v) - \{f(2v) - 16f(v)\}\| \leq \frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{rq}} \tag{3.40}
$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)$ is defined in (3.4) and $\mathcal{Q}_2(v)$ are defined by

$$
\mathcal{Q}_2(v) = \lim_{p \to \infty} \frac{q_2(2^{pq}v)}{2^{2pq}} = \lim_{p \to \infty} \frac{f(2 \cdot 2^{pq}v) - 16f(2^{pq}v)}{2^{2pq}} \tag{3.41}
$$

for all $v \in \mathcal{Y}$, respectively.

Proof. Given, f is an even function. Using evenness of f in (3.39), we arrive

$$\Bigg\| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) + 6f\left(\sum_{i=1}^{n-1} v_i\right) - 2f(2v_n) + 8f(v_n)\Bigg\| \leq \omega(v_1, \cdots, v_n) \tag{3.42}$$

for all $v_1, v_2, v_3, v_4, \cdots, v_{n-1}, v_n \in \mathcal{Y}$. Replacing

$$(v_1, v_2, v_3, v_4, \cdots, v_{n-1}, v_n) = \Bigg(v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2}\ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2}\ times}, 0, v \Bigg)$$

in (3.42), we get

$$\begin{aligned}\Big\| f(3v) - 6f(2v) + 15f(v) \Big\| &\leq \omega \Bigg(v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2}\ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2}\ times}, 0, v \Bigg) \\ &= \omega_1(v, v, \cdots, -v, 0, v)\end{aligned} \tag{3.43}$$

for all $v \in \mathcal{Y}$. Again replacing

$$(v_1, v_2, v_3, v_4, \cdots, v_{n-1}, v_n) = \Bigg(2v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2}\ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2}\ times}, 0, v \Bigg)$$

in (3.42), we obtain

$$\begin{aligned}\Big\| f(4v) - 4f(3v) + 4f(2v) + 4f(v) \Big\| &\leq \omega \Bigg(2v, \underbrace{v, v, \cdots, v}_{\frac{n-3}{2}\ times}, \underbrace{-v, -v, \cdots, -v}_{\frac{n-3}{2}\ times}, 0, v \Bigg) \\ &= \omega_1(2v, v, \cdots, -v, 0, v)\end{aligned} \tag{3.44}$$

for all $v \in \mathcal{Y}$. It follows from (3.43) and (3.44),

$$\begin{aligned}&\Big\| f(4v) - 20f(2v) + 64f(v) \Big\| \\ &= \Big\| f(4v) - 4f(3v) + 4f(3v) + 4f(2v) - 24f(2v) + 60f(v) + 4f(v) \Big\| \\ &\leq 4\Big\| f(3v) - 6f(2v) + 15f(v) \Big\| + \Big\| f(4v) - 4f(3v) + 4f(2v) + 4f(v) \Big\| \\ &\leq 4\omega_1(v, v, \cdots, -v, 0, v) + \omega_1(2v, v, \cdots, -v, 0, v)\end{aligned} \tag{3.45}$$

for all $v \in \mathcal{Y}$. Define

$$\Omega(v, v, \cdots, -v, 0, v) = 4\omega_1(v, v, \cdots, -v, 0, v) + \omega_1(2v, v, \cdots, -v, 0, v) \tag{3.46}$$

for all $v \in \mathcal{Y}$. Using (3.46) in (3.45), we have

$$\Big\| f(4v) - 20f(2v) + 64f(v) \Big\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{3.47}$$

for all $v \in \mathcal{Y}$. It follows from (3.47), we reach

$$\Big\| \Big\{ f(4v) - 16f(2v) \Big\} - 4\Big\{ f(2v) - 16f(v) \Big\} \Big\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{3.48}$$

for all $v \in \mathcal{Y}$. Using (2.13) in (3.48), we land

$$\Big\| q_2(2v) - 4q_2(v) \Big\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{3.49}$$

for all $v \in \mathcal{Y}$. The rest of the proof is similar to that of Theorem 3.3 . This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.6 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.4. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \le \begin{cases} b, & \\ b \sum\limits_{i=1}^{n} ||v_i||^d, & d \neq 2; \\ b \left\{ \prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd} \right\}, & nd \neq 2; \end{cases} \tag{3.50}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|q_2(v) - \mathcal{Q}_2(v)\| \le \begin{cases} \dfrac{5b}{|3|}, \\ \dfrac{b||v||^d(5n-6+2^{rd})}{|4-2^d|}, \\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{|4-2^{nd}|} \end{cases} \tag{3.51}$$

for all $v \in \mathcal{Y}$.

Theorem 3.7. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$\lim_{p\to\infty} \frac{\omega\left(2^{pq}v_1, \cdots, 2^{pq}v_n\right)}{2^{4pq}} = 0 \tag{3.52}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even function satisfying the inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \le \omega\left(v_1, \cdots, v_n\right) \tag{3.53}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{Q}_4(v) - q_4(v)\| = \|\mathcal{Q}_4(v) - \{f(2v) - 4f(v)\}\| \le \frac{1}{16} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{rq}} \tag{3.54}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)$ is defined in (3.4) and $\mathcal{Q}_4(v)$ is defined by

$$\mathcal{Q}_4(v) = \lim_{p\to\infty} \frac{q_4(2^{pq}v)}{2^{4pq}} = \lim_{p\to\infty} \frac{f(2 \cdot 2^{pq}v) - 4f(2^{pq}v)}{2^{4pq}} \tag{3.55}$$

for all $v \in \mathcal{Y}$, respectively.

Proof. It follows from (3.47), we reach

$$\left\|\left\{f(4v) - 4f(2v)\right\} - 16\left\{f(2v) - 4f(v)\right\}\right\| \le \Omega(v, v, \cdots, -v, 0, v) \tag{3.56}$$

for all $v \in \mathcal{Y}$. Using (2.22) in (3.56), we land

$$\left\|q_4(2v) - 16q_4(v)\right\| \le \Omega(v, v, \cdots, -v, 0, v) \tag{3.57}$$

for all $v \in \mathcal{Y}$. The rest of the proof is similar to that of Theorem 3.3 . This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.7 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.5. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \le \begin{cases} b, & \\ b \sum\limits_{i=1}^{n} ||v_i||^d, & d \neq 4; \\ b \left\{ \prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd} \right\}, & nd \neq 4; \end{cases} \tag{3.58}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|q_4(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \dfrac{5b}{|15|}, \\ \dfrac{b||v||^d(5n-6+2^{rd})}{|16-2^d|}, \\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{|16-2^{nd}|} \end{cases} \tag{3.59}$$

for all $v \in \mathcal{Y}$.

Theorem 3.8. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions satisfying (3.38) and (3.52) for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even function satisfying the inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{3.60}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\begin{aligned} \|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \leq \frac{1}{12}\Bigg\{ &\frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{2rq}} \\ &+ \frac{1}{16}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{4rq}} \Bigg\} \end{aligned} \tag{3.61}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)$, $\mathcal{Q}_2(v)$ and $\mathcal{Q}_4(v)$ is defined in (3.4), (3.41) and (3.55) for all $v \in \mathcal{Y}$, respectively.

Proof. **Case (i):** For $q = 1$. Given f is an even function.

If $f : \mathcal{Y} \to \mathcal{Z}$ satisfies (3.60) then by Theorem 3.6, there exists a unique quadratic function $\mathcal{Q}_2' : \mathcal{Y} \to \mathcal{Z}$ such that

$$\left\|\mathcal{Q}_2'(v) - (f(2v) - 4f(v))\right\| \leq \frac{1}{4}\sum_{r=0}^{\infty} \frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^{2r}} \tag{3.62}$$

for all $v \in \mathcal{Y}$.

Also, if $f : \mathcal{Y} \to \mathcal{Z}$ satisfies (3.60) then by Theorem 3.7, there exists a unique cubic function $\mathcal{Q}_4' : \mathcal{Y} \to \mathcal{Z}$ such that

$$\left\|\mathcal{Q}_4'(v) - (f(2v) - 16f(v))\right\| \leq \frac{1}{16}\sum_{r=0}^{\infty} \frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^{4r}} \tag{3.63}$$

for all $v \in \mathcal{Y}$. Combining (3.62) and (3.63), we achieve

$$\begin{aligned} &\left\|\frac{1}{12}\mathcal{Q}_2'(v) - \frac{1}{12}\mathcal{Q}_4'(v) - f(v)\right\| \\ &\quad = \left\|\frac{1}{12}\mathcal{Q}_2'(v) - \frac{1}{12}f(2v) - \frac{16}{12}f(v) - \frac{1}{12}\mathcal{Q}_4'(v) + \frac{1}{12}f(2v) - \frac{4}{12}f(v)\right\| \\ &\quad \leq \left\|\frac{1}{12}\mathcal{Q}_2'(v) - \frac{1}{12}(f(2v) - 16f(v))\right\| + \left\|\frac{1}{12}\mathcal{Q}_4'(v) - \frac{1}{12}(f(2v) + 4f(v))\right\| \\ &\quad \leq \frac{1}{12}\left\{\left\|\mathcal{Q}_2'(v) - (f(2v) - 16f(v))\right\| + \left\|\mathcal{Q}_4'(v) - (f(2v) - 4f(v))\right\|\right\} \\ &\quad \leq \frac{1}{12}\left\{\frac{1}{2}\sum_{r=0}^{\infty} \frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^{2r}} + \frac{1}{8}\sum_{r=0}^{\infty} \frac{\Omega(2^r v, 2^r v, \cdots, -2^r v, 0, 2^r v)}{2^{4r}}\right\} \end{aligned}$$

for all $v \in \mathcal{Y}$. Defining

$$\mathcal{Q}_2(v) = \frac{1}{12}\mathcal{Q}_2'(v); \qquad \mathcal{Q}_4(v) = \frac{-1}{12}\mathcal{Q}_4'(v)$$

we arrive (3.61) as desired. Similarly, we can prove for $j = -1$. Hence the proof is complete. $\square$

The following corollary is an immediate consequence of Theorem 3.8 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.6. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exist real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\| \leq \begin{cases} b, & \\ b\sum\limits_{i=1}^{n}||v_i||^d, & d \neq 2,4; \\ b\left\{\prod\limits_{i=1}^{n}||v_i||^d + \sum\limits_{i=1}^{n}||v_i||^{nd}\right\}, & nd \neq 2,4; \end{cases} \tag{3.64}$$

for all $v_1,\cdots,v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \frac{5b}{12}\left(\frac{1}{|3|}+\frac{1}{|15|}\right), \\ \frac{b||v||^d(5n-6+2^{rd})}{12}\left(\frac{1}{|4-2^d|}+\frac{1}{|16-2^d|}\right), \\ \frac{b||v||^{nd}(5n-6+2^{rnd})}{12}\left(\frac{1}{|4-2^{nd}|}+\frac{1}{|16-2^{nd}|}\right) \end{cases} \tag{3.65}$$

for all $v \in \mathcal{Y}$.

3.3 f IS AN ODD - EVEN FUNCTION

Theorem 3.9. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0,\infty)$ be functions satisfying (3.32) and (3.60) for all $v_1,\cdots,v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a function satisfying the inequality*

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\| \leq \omega\left(v_1,\cdots,v_n\right) \tag{3.66}$$

for all $v_1,\cdots,v_n \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$, a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$, a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\begin{aligned} &\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\| \\ &\quad \leq \frac{1}{2}\left\{\frac{1}{6}\left[\frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega_1(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)}{2^{rq}}+\frac{1}{8}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega_3(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)}{2^{3rq}}\right]\right. \\ &\qquad \left.+\frac{1}{12}\left[\frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega_2(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)}{2^{2rq}}+\frac{1}{16}\sum_{r=\frac{1-q}{2}}^{\infty}\frac{\Omega_4(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)}{2^{4rq}}\right]\right\} \end{aligned} \tag{3.67}$$

where

$$\Omega_t(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v) = \Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v) + \Omega(-2^{rq}v,-2^{rq}v,\cdots 2^{rq}v,0,-2^{rq}v)$$

for $t = 1,2,3,4$ and $\Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,0,2^{rq}v)$, $\mathcal{A}(v)$, $\mathcal{Q}_2(v)$, $\mathcal{C}(v)$ and $\mathcal{Q}_4(v)$ is defined in (3.4), (3.5), (3.41), (3.27), and (3.55) for all $v \in \mathcal{Y}$, respectively.

Proof. Let we define

$$f_{ODD}(v) = \frac{f(v) - f(-v)}{2}$$

for all $v \in \mathcal{Y}$. Then $f_{ODD}(0) = 0$ and $f_{ODD}(-v) = -f_{ODD}(v)$ for all $v \in \mathcal{Y}$. Hence

$$\|Df_{ODD}(v_1,\cdots,v_n)\| \leq \frac{\omega(v_1,\cdots,v_n)}{2} + \frac{\omega(-v_1,\cdots,-v_n)}{2} \tag{3.68}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. By Theorem 3.5, we have

$$\begin{aligned}
&\|f_{ODD}(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \\
&\leq \frac{1}{2}\left\{\frac{1}{6}\left(\frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{rq}} + \frac{\Omega(-2^{rq}v, -2^{rq}v, \cdots 2^{rq}v, 0, -2^{rq}v)}{2^{rq}}\right]\right.\right. \\
&\qquad \left.\left. + \frac{1}{8}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{3rq}} + \frac{\Omega(-2^{rq}v, -2^{rq}v, \cdots 2^{rq}v, 0, -2^{rq}v)}{2^{3rq}}\right]\right)\right\} \\
&\leq \frac{1}{2}\left\{\frac{1}{6}\left(\frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega_1(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{rq}}\right] + \frac{1}{8}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega_3(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{3rq}}\right]\right)\right\}
\end{aligned} \tag{3.69}$$

for all $v \in \mathcal{Y}$.

Also, let

$$f_{EVEN}(v) = \frac{f(v) + f(-v)}{2}$$

for all $v \in \mathcal{Y}$. Then $f_{EVEN}(0) = 0$ and $f_{EVEN}(-v) = f_{EVEN}(v)$ for all $v \in \mathcal{Y}$. Hence

$$\|Df_{EVEN}(v_1, \cdots, v_n)\| \leq \frac{\omega(v_1, \cdots, v_n)}{2} + \frac{\omega(-v_1, \cdots, -v_n)}{2} \tag{3.70}$$

for all $v \in \mathcal{Y}$. By Theorem 3.8, we have

$$\begin{aligned}
&\|f_{EVEN}(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \\
&\leq \frac{1}{2}\left\{\frac{1}{12}\left(\frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{2rq}} + \frac{\Omega(-2^{rq}v, -2^{rq}v, \cdots 2^{rq}v, 0, -2^{rq}v)}{2^{2rq}}\right]\right.\right. \\
&\qquad \left.\left. + \frac{1}{16}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{4rq}} + \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{4rq}}\right]\right)\right\} \\
&\leq \frac{1}{2}\left\{\frac{1}{12}\left(\frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega_2(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{2rq}}\right] + \frac{1}{16}\sum_{r=\frac{1-q}{2}}^{\infty}\left[\frac{\Omega_4(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 0, 2^{rq}v)}{2^{4rq}}\right]\right)\right\}
\end{aligned} \tag{3.71}$$

for all $v \in \mathcal{Y}$. Define

$$f(v) = f_{EVEN}(v) + f_{ODD}(v) \tag{3.72}$$

for all $v \in \mathcal{Y}$. From (3.69),(3.71) and (3.72), we arrive our result. □

The following corollary is an immediate consequence of Theorem 3.9 concerning the Hyers - Ulam, Hyers - Ulam - Rassias and Ulam - JMRassias stabilities of (1.3).

Corollary 3.7. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a mapping. If there exists real numbers b and d such that*

$$\|Df_{aqcq}(v_1, \cdots, v_n)\| \leq \begin{cases} b, & \\ b\sum_{i=1}^{n} ||v_i||^d, & d \neq 1,2,3,4; \\ b\left\{\prod_{i=1}^{n} ||v_i||^d + \sum_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 1,2,3,4; \end{cases} \tag{3.73}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there existss a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$, a unique quadratic function

$\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$, a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \dfrac{5b}{2}\left\{\dfrac{1}{12}\left[\dfrac{1}{|3|}+\dfrac{1}{|15|}\right]+\dfrac{1}{6}\left[1+\dfrac{1}{|7|}\right]\right\}, \\ \dfrac{b||v||^d(5n-6+2^{rd})}{2}\left\{\dfrac{1}{12}\left[\dfrac{1}{|4-2^d|}+\dfrac{1}{|16-2^d|}\right]+\dfrac{1}{6}\left[\dfrac{1}{|2-2^d|}+\dfrac{1}{|8-2^d|}\right]\right\}, \\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{2}\left\{\dfrac{1}{12}\left[\dfrac{1}{|4-2^{nd}|}+\dfrac{1}{|16-2^{nd}|}\right]+\dfrac{1}{6}\left[\dfrac{1}{|2-2^{nd}|}+\dfrac{1}{|8-2^{nd}|}\right]\right\}, \end{cases} \tag{3.74}$$

for all $v \in \mathcal{Y}$.

4 Stability Results - Banach Space : n is an Even Positive Integer

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.3) where n is an even positive integer in Banach space using direct method.

The proof of the following theorems and corollaries are similar to that of proofs of Section 3. Hence the details of the proofs are omitted.

4.1 f IS AN ODD FUNCTION

Theorem 4.10. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$\lim_{p\to\infty} \frac{\omega\left(2^{pq}v_1, \cdots, 2^{pq}v_n\right)}{2^{pq}} = 0 \tag{4.1}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd function satisfying the inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{4.2}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{A}(v) - a(v)\| = \|\mathcal{A}(v) - \{f(2v) - 8f(v)\}\| \leq \frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{rq}} \tag{4.3}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$ and $\mathcal{A}(v)$ are defined by

$$\begin{aligned} &\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v) \\ &= 4\omega_1(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v) + \omega_1(2\cdot 2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v) \\ &= 4\omega\left(2^{rq}v, \underbrace{2^{rq}v, 2^{rq}v, \cdots, 2^{rq}v}_{\frac{n-2}{2}\ times}, \underbrace{-2^{rq}v, -2^{rq}v, \cdots, -2^{rq}v}_{\frac{n-2}{2}\ times}, 2^{rq}v\right) \\ &\quad + \omega\left(2\cdot 2^{rq}v, \underbrace{2^{rq}v, 2^{rq}v, \cdots, 2^{rq}v}_{\frac{n-2}{2}\ times}, \underbrace{-2^{rq}v, -2^{rq}v, \cdots, -2^{rq}v}_{\frac{n-2}{2}\ times}, 2^{rq}v\right) \end{aligned} \tag{4.4}$$

and

$$\mathcal{A}(v) = \lim_{p\to\infty} \frac{a(2^{pq}v)}{2^{pq}} = \lim_{p\to\infty} \frac{f(2\cdot 2^{pq}v) - 8f(2^{pq}v)}{2^{pq}} \tag{4.5}$$

for all $v \in \mathcal{Y}$, respectively.

Corollary 4.8. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, & \\ b\sum_{i=1}^{n} ||v_i||^d, & d \neq 1; \\ b\prod_{i=1}^{n} ||v_i||^d, & nd \neq 1; \\ b\left\{\prod_{i=1}^{n} ||v_i||^d + \sum_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 1; \end{cases} \tag{4.6}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there existss a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|a(v) - \mathcal{A}(v)\| \leq \begin{cases} 5b, \\ \dfrac{b||v||^d(5n-1+2^{rd})}{|2-2^d|}, \\ \dfrac{b||v||^d(4+2^{rnd})}{|2-2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n-3+2^{rd}+2^{rnd})}{|2-2^{nd}|} \end{cases} \tag{4.7}$$

for all $v \in \mathcal{Y}$.

Theorem 4.11. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$\lim_{p\to\infty} \frac{\omega\left(2^{pq}v_1, \cdots, 2^{pq}v_n\right)}{2^{3pq}} = 0 \tag{4.8}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd function satisfying the inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{4.9}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{C}(v) - c(v)\| = \|\mathcal{C}(v) - \{f(2v) - 2f(v)\}\| \leq \frac{1}{8} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{rq}} \tag{4.10}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$ is defined in (4.4) and $\mathcal{C}(v)$ is defined by

$$\mathcal{C}(v) = \lim_{p\to\infty} \frac{c(2^{pq}v)}{2^{3pq}} = \lim_{p\to\infty} \frac{f(2\cdot 2^{pq}v) - 2f(2^{pq}v)}{2^{3pq}} \tag{4.11}$$

for all $v \in \mathcal{Y}$, respectively.

Corollary 4.9. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, \\ b\sum\limits_{i=1}^{n} ||v_i||^d, & d \neq 3; \\ b\prod\limits_{i=1}^{n} ||v_i||^d, & nd \neq 3; \\ b\left\{\prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 3; \end{cases} \tag{4.12}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|c(v) - \mathcal{C}(v)\| \leq \begin{cases} \dfrac{5b}{|7|}, \\ \dfrac{b||v||^d(5n-1+2^{rd})}{|8-2^d|}, \\ \dfrac{b||v||^d(4+2^{rnd})}{|8-2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n-3+2^{rd}+2^{rnd})}{|8-2^{nd}|} \end{cases} \tag{4.13}$$

for all $v \in \mathcal{Y}$.

Theorem 4.12. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions satisfying (4.1) and (4.8) for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd function satisfying the inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{4.14}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ and a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \leq \frac{1}{6}\left\{\frac{1}{2}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{rq}} + \frac{1}{8}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{3rq}}\right\} \tag{4.15}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$, $\mathcal{A}(v)$ and $\mathcal{C}(v)$ is defined in (4.4), (4.5) and (4.11) for all $v \in \mathcal{Y}$, respectively.

Corollary 4.10. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d such that*

$$\|Df_{aqcq}(v_1, \cdots, v_n)\| \leq \begin{cases} b, & \\ b\sum_{i=1}^{n} ||v_i||^d, & d \neq 1, 3; \\ b\prod_{i=1}^{n} ||v_i||^d, & nd \neq 1, 3; \\ b\left\{\prod_{i=1}^{n} ||v_i||^d + \sum_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 1, 3; \end{cases} \tag{4.16}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ and a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|f(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \leq \begin{cases} \frac{5b}{6}\left(1 + \frac{1}{|7|}\right), \\ \frac{b||v||^d(5n - 1 + 2^{rd})}{6}\left(\frac{1}{|2 - 2^d|} + \frac{1}{|8 - 2^d|}\right), \\ \frac{b||v||^d(4 + 2^{rnd})}{6}\left(\frac{1}{|2 - 2^d|} + \frac{1}{|8 - 2^d|}\right), \\ \frac{b||v||^{nd}(5n + 3 + 2^{rd} + 2^{rnd})}{6}\left(\frac{1}{|2 - 2^{nd}|} + \frac{1}{|8 - 2^{nd}|}\right) \end{cases} \tag{4.17}$$

for all $v \in \mathcal{Y}$.

4.2 f IS AN EVEN FUNCTION

Theorem 4.13. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be function such that*

$$\lim_{p\to\infty} \frac{\omega\left(2^{pq}v_1, \cdots, 2^{pq}v_n\right)}{2^{2pq}} = 0 \tag{4.18}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even function satisfying the inequality

$$\|Df_{aqcq}(v_1, \cdots, v_n)\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{4.19}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{Q}_2(v) - q_2(v)\| = \|\mathcal{Q}_2(v) - \{f(2v) - 16f(v)\}\| \leq \frac{1}{4}\sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{2rq}} \tag{4.20}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$ is defined in (4.4) and $\mathcal{Q}_2(v)$ are defined by

$$\mathcal{Q}_2(v) = \lim_{p\to\infty} \frac{q_2(2^{pq}v)}{2^{2pq}} = \lim_{p\to\infty} \frac{f(2 \cdot 2^{pq}v) - 16f(2^{pq}v)}{2^{2pq}} \tag{4.21}$$

for all $v \in \mathcal{Y}$, respectively.

Corollary 4.11. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, & \\ b\sum\limits_{i=1}^{n} ||v_i||^d, & d \neq 2; \\ b\prod\limits_{i=1}^{n} ||v_i||^d, & nd \neq 2; \\ b\left\{\prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 2; \end{cases} \tag{4.22}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|a(v) - \mathcal{Q}_{\in}(v)\| \leq \begin{cases} \dfrac{5b}{|3|}, \\ \dfrac{b||v||^d(5n-1+2^{rd})}{|4-2^d|}, \\ \dfrac{b||v||^{nd}(4+2^{rnd})}{|4-2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{|4-2^{nd}|} \end{cases} \tag{4.23}$$

for all $v \in \mathcal{Y}$.

Theorem 4.14. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions such that*

$$\lim_{p\to\infty} \frac{\omega\left(2^{pq}v_1, \cdots, 2^{pq}v_n\right)}{2^{4pq}} = 0 \tag{4.24}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even function satisfying the inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{4.25}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{Q}_4(v) - q_4(v)\| = \|\mathcal{Q}_4(v) - \{f(2v) - 4f(v)\}\| \leq \frac{1}{16} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{4rq}} \tag{4.26}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$ is defined in (4.4) and $\mathcal{Q}_4(v)$ is defined by

$$\mathcal{Q}_4(v) = \lim_{p\to\infty} \frac{q_4(2^{pq}v)}{2^{4pq}} = \lim_{p\to\infty} \frac{f(2\cdot 2^{pq}v) - 4f(2^{pq}v)}{2^{4pq}} \tag{4.27}$$

for all $v \in \mathcal{Y}$, respectively.

Corollary 4.12. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \begin{cases} b, & \\ b\sum\limits_{i=1}^{n} ||v_i||^d, & d \neq 4; \\ b\prod\limits_{i=1}^{n} ||v_i||^d, & nd \neq 4; \\ b\left\{\prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd}\right\}, & nd \neq 4; \end{cases} \tag{4.28}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|q_4(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \dfrac{5b}{|15|}, \\ \dfrac{b||v||^d(5n-1+2^{rd})}{|16-2^d|}, \\ \dfrac{b||v||^{nd}(4+2^{rnd})}{|16-2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{|16-2^{nd}|} \end{cases} \tag{4.29}$$

for all $v \in \mathcal{Y}$.

Theorem 4.15. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions satisfying (4.18) and (4.24) for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even function satisfying the inequality*

$$\left\| Df_{aqcq}(v_1, \cdots, v_n) \right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{4.30}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\begin{aligned} \|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \leq \frac{1}{12} \Bigg\{ & \frac{1}{4} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{2rq}} \\ & + \frac{1}{16} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{4rq}} \Bigg\} \end{aligned} \tag{4.31}$$

where $\Omega(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)$, $\mathcal{Q}_2(v)$ and $\mathcal{Q}_4(v)$ is defined in (4.4), (4.21) and (4.27) for all $v \in \mathcal{Y}$, respectively.

Corollary 4.13. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d such that*

$$\left\| Df_{aqcq}(v_1, \cdots, v_n) \right\| \leq \begin{cases} b, & \\ b \sum\limits_{i=1}^{n} ||v_i||^d, & d \neq 2, 4; \\ b \prod\limits_{i=1}^{n} ||v_i||^d, & nd \neq 2, 4; \\ b \left\{ \prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd} \right\}, & nd \neq 2, 4; \end{cases} \tag{4.32}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \dfrac{5b}{12}\left(\dfrac{1}{|3|} + \dfrac{1}{|15|}\right), \\ \dfrac{b||v||^d(5n - 1 + 2^{rd})}{12}\left(\dfrac{1}{|4 - 2^d|} + \dfrac{1}{|16 - 2^d|}\right), \\ \dfrac{b||v||^{nd}(4 + 2^{rnd})}{12}\left(\dfrac{1}{|4 - 2^d|} + \dfrac{1}{|16 - 2^d|}\right), \\ \dfrac{b||v||^{nd}(5n + 3 + 2^{rd} + 2^{rnd})}{12}\left(\dfrac{1}{|4 - 2^{nd}|} + \dfrac{1}{|16 - 2^{nd}|}\right) \end{cases} \tag{4.33}$$

for all $v \in \mathcal{Y}$.

4.3 f IS AN ODD - EVEN FUNCTION

Theorem 4.16. *Let $q = \pm 1$ and $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ be functions satisfying (4.14) and (4.30) for all $v_1, \cdots, v_n \in \mathcal{Y}$. Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a function satisfying the inequality*

$$\left\| Df_{aqcq}(v_1, \cdots, v_n) \right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{4.34}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$, a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$, a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\begin{aligned} & \|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\| \\ & \quad \leq \frac{1}{2} \Bigg\{ \frac{1}{6} \left[\frac{1}{2} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega_{11}(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{rq}} + \frac{1}{8} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega_{33}(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{3rq}} \right] \\ & \qquad + \frac{1}{12} \left[\frac{1}{4} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega_{22}(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{2rq}} + \frac{1}{16} \sum_{r=\frac{1-q}{2}}^{\infty} \frac{\Omega_{44}(2^{rq}v, 2^{rq}v, \cdots, -2^{rq}v, 2^{rq}v)}{2^{4rq}} \right] \Bigg\} \end{aligned} \tag{4.35}$$

where

$$\Omega_{tt}(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,2^{rq}v)=\Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,2^{rq}v)+\Omega(-2^{rq}v,-2^{rq}v,\cdots 2^{rq}v,-2^{rq}v)$$

for $tt=1,2,3,4$ *and* $\Omega(2^{rq}v,2^{rq}v,\cdots,-2^{rq}v,2^{rq}v)$, $\mathcal{A}(v)$, $\mathcal{Q}_2(v)$, $\mathcal{C}(v)$ *and* $\mathcal{Q}_4(v)$ *is defined in (4.4), (4.5), (4.21), (4.11) and (4.27) for all* $v\in\mathcal{Y}$, *respectively.*

Corollary 4.14. *Let* $Df_{aqcq}:\mathcal{Y}\to\mathcal{Z}$ *be a mapping. If there exists real numbers b and d such that*

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\|\le\begin{cases} b, & \\ b\sum\limits_{i=1}^{n}||v_i||^d, & d\neq 1,2,3,4; \\ b\prod\limits_{i=1}^{n}||v_i||^d, & nd\neq 1; \\ b\left\{\prod\limits_{i=1}^{n}||v_i||^d+\sum\limits_{i=1}^{n}||v_i||^{nd}\right\}, & nd\neq 1,2,3,4; \end{cases}\tag{4.36}$$

for all $v_1,\cdots,v_n\in\mathcal{Y}$, *then there exists a unique additive function* $\mathcal{A}:\mathcal{Y}\to\mathcal{Z}$, *a unique quadratic function* $\mathcal{Q}_2:\mathcal{Y}\to\mathcal{Z}$, *a unique cubic function* $\mathcal{C}:\mathcal{Y}\to\mathcal{Z}$ *and a unique quartic function* $\mathcal{Q}_4:\mathcal{Y}\to\mathcal{Z}$ *such that*

$$\begin{aligned}&\|f(v)-\mathcal{A}(v)-\mathcal{Q}_2(v)-\mathcal{C}(v)-\mathcal{Q}_4(v)\|\\ &\le\begin{cases}\dfrac{5b}{2}\left\{\dfrac{1}{12}\left[\dfrac{1}{|3|}+\dfrac{1}{|15|}\right]+\dfrac{1}{6}\left[1+\dfrac{1}{|7|}\right]\right\},\\ \dfrac{b||v||^d(5n-1+2^{rd})}{2}\left\{\dfrac{1}{12}\left[\dfrac{1}{|4-2^d|}+\dfrac{1}{|16-2^d|}\right]+\dfrac{1}{6}\left[\dfrac{1}{|2-2^d|}+\dfrac{1}{|8-2^d|}\right]\right\},\\ \dfrac{b||v||^{nd}(4+2^{rnd})}{2}\left\{\dfrac{1}{12}\left[\dfrac{1}{|4-2^d|}+\dfrac{1}{|16-2^d|}\right]+\dfrac{1}{6}\left[\dfrac{1}{|2-2^d|}+\dfrac{1}{|8-2^d|}\right]\right\},\\ \dfrac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{2}\left\{\dfrac{1}{12}\left[\dfrac{1}{|4-2^{nd}|}+\dfrac{1}{|16-2^{nd}|}\right]+\dfrac{1}{6}\left[\dfrac{1}{|2-2^{nd}|}+\dfrac{1}{|8-2^{nd}|}\right]\right\},\end{cases}\end{aligned}\tag{4.37}$$

for all $v\in\mathcal{Y}$.

5 Stability Results - Fixed Point Method: n Odd Positive Integer

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.3) where n is an odd positive integer in Banach space using fixed point method.

5.1 f IS AN ODD FUNCTION

Theorem 5.17. *Let* $Df_{aqcq}:\mathcal{Y}\to\mathcal{Z}$ *be a odd mapping for which there exist a function* $\omega,\Omega:\mathcal{Y}^n\to[0,\infty)$ *with the condition*

$$\lim_{p\to\infty}\frac{\omega\left(\kappa_j^p v_1,\cdots,\kappa_j^p v_n\right)}{\kappa_j^p}=0\tag{5.1}$$

for all $v_1,\cdots,v_n\in\mathcal{Y}$ *where*

$$\kappa_j=\begin{cases}2 & if \quad j=0;\\ \frac{1}{2} & if \quad j=1,\end{cases}\tag{5.2}$$

such that the functional inequality

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\|\le\omega\left(v_1,\cdots,v_n\right)\tag{5.3}$$

for all $v_1,\cdots,v_n\in\mathcal{Y}$. *If there exists* $L=L(i)$ *such that the function*

$$\Psi(v,v,\cdots,-v,0,v)=\Omega\left(\frac{v}{2},\frac{v}{2},\cdots,-\frac{v}{2},0,\frac{v}{2}\right),$$

has the property

$$\Psi(v,v,\cdots,-v,0,v)=\frac{L}{\kappa_j}\,\Psi\left(\kappa_j v,\kappa_j v,\cdots,-\kappa_j v,0,\kappa_j v\right).\tag{5.4}$$

for all $v \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{A}(v) - a(v)\| = \|\mathcal{A}(v) - \{f(2v) - 8f(v)\}\| \leq \frac{L^{1-j}}{1-L}\Psi(v, v, \cdots, -v, 0, v) \tag{5.5}$$

for all $v \in \mathcal{Y}$.

Proof. Consider the set

$$\Gamma = \{p/p : \mathcal{Y} \to \mathcal{Z},\ p(0) = 0\}$$

and introduce the generalized metric on Γ,

$$d(p, q) = \inf\{K \in (0, \infty) :\| p(v) - q(v) \|\leq K\psi(v), v \in \mathcal{Y}\}.$$

It is easy to see that (Γ, d) is complete.

Define $\Upsilon : \Gamma \to \Gamma$ by

$$\Upsilon p(v) = \frac{1}{\kappa_j} p(\kappa_j v),$$

for all $v \in \mathcal{Y}$. Now $p, q \in \Gamma$, by [36], we have $d(\Upsilon p, \Upsilon q) \leq Ld(p, q)$, i.e., T is a strictly contractive mapping on Γ with Lipschitz constant L.

From (3.13), we arrive

$$\left\|a(2v) - 2a(v)\right\| \leq \Omega(v, v, \cdots, -v, 0, v) \tag{5.6}$$

for all $v \in \mathcal{Y}$. It follows from (5.6) that

$$\left\|\frac{a(2v)}{2} - a(v)\right\| \leq \frac{\Omega(v, v, \cdots, -v, 0, v)}{2} \tag{5.7}$$

for all $v \in \mathcal{Y}$. Using (5.4) for the case $j = 0$ it reduces to

$$\left\|\frac{a(2v)}{2} - a(v)\right\| \leq\ L\Psi(v, v, \cdots, -v, 0, v)$$

for all $v \in \mathcal{Y}$,

$$\text{i.e.,}\quad d(\Upsilon a, a) \leq L \Rightarrow d(\Upsilon a, a) \leq L = L^1 < \infty. \tag{5.8}$$

Again replacing $v = \dfrac{v}{2}$ in (5.6), we get

$$\left\|a(v) - 2a\left(\frac{v}{2}\right)\right\| \leq \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right) \tag{5.9}$$

for all $v \in \mathcal{Y}$. Using (5.4) for the case $j = 1$ it reduces to

$$\left\|a(v) - 2a\left(\frac{v}{2}\right)\right\| \leq \Psi(v, v, \cdots, -v, 0, v)$$

for all $v \in \mathcal{Y}$,

$$\text{i.e.,}\quad d(a, \Upsilon a) \leq 1 \Rightarrow d(a, \Upsilon a) \leq 1 = L^0 < \infty. \tag{5.10}$$

From (5.8) and (5.10), we arrive

$$d(a, \Upsilon a) \leq L^{1-j}.$$

Therefore (FP1) holds.

By (FP2), it follows that there exists a fixed point A of Υ in Γ such that

$$A(v) = \lim_{p\to\infty} \frac{a(\kappa_j^p v)}{\kappa_j^p}, \qquad \forall\ v \in \mathcal{Y}. \tag{5.11}$$

To order to prove A satisfies the functional equation (1.3), the proof is similar to the of Theorem 3.3. By (FP3), A is the unique fixed point of Υ in the set

$$\Delta = \{A \in \Gamma : d(a, A) < \infty\},$$

such that

$$\|a(v) - A(v)\| \leq K\psi(v)$$

for all $v \in \mathcal{Y}$ and $K > 0$. Finally by (FP4), we obtain

$$d(a, A) \leq \frac{1}{1-L} d(a, \Upsilon a)$$

this implies

$$d(a, A) \leq \frac{L^{1-j}}{1-L}$$

which yields

$$\|a(v) - A(v)\| \leq \frac{L^{1-j}}{1-L} \Psi(v, v, \cdots, -v, 0, v)$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 5.17 concerning the stability of (1.3).

Corollary 5.15. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d satisfying (3.22) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|a(v) - \mathcal{A}(v)\| \leq \begin{cases} 5b, \\ \dfrac{b||v||^d (5n - 6 + 2^{rd})}{|2 - 2^d|}, \\ \dfrac{b||v||^{nd} (5n - 6 + 2^{rnd})}{|2 - 2^{nd}|} \end{cases} \tag{5.12}$$

for all $v \in \mathcal{Y}$.

Proof. Taking

$$\omega\left(v_1, \cdots, v_n\right) = \begin{cases} b, \\ b \sum\limits_{i=1}^{n} ||v_i||^d, \\ b \left\{ \prod\limits_{i=1}^{n} ||v_i||^d + \sum\limits_{i=1}^{n} ||v_i||^{nd} \right\}, \end{cases}$$

for all $v \in \mathcal{Y}$. Now,

$$\frac{1}{\kappa_j^p} \omega\left(\kappa_j^p v_1, \kappa_j^p \cdots, \kappa_j^p v_n\right) = \begin{cases} \frac{b}{\kappa_j^p}, \\ \frac{b}{\kappa_j^p} \sum\limits_{i=1}^{n} ||\kappa_j^p v_i||^d, \\ \frac{b}{\kappa_j^p} \left\{ \prod\limits_{i=1}^{n} ||\kappa_j^p v_i||^d + \sum\limits_{i=1}^{n} ||\kappa_j^p v_i||^{nd} \right\}, \end{cases} = \begin{cases} \to 0 \text{ as } p \to \infty, \\ \to 0 \text{ as } p \to \infty, \\ \to 0 \text{ as } p \to \infty. \end{cases}$$

Thus, (5.1) is holds. But we have

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots - \frac{v}{2}, 0, \frac{v}{2}\right),$$

has the property

$$\Psi(v, v, \cdots, -v, 0, v) = \frac{L}{\kappa_j} \Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, 0, \kappa_j v\right)$$

for all $v \in \mathcal{Y}$. Hence

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots - \frac{v}{2}, 0, \frac{v}{2}\right) = \begin{cases} 5b, \\ \dfrac{b(5n - 6 + 2^{rd})}{2^d} ||v||^d, \\ \dfrac{b(5n - 6 + 2^{rnd})}{2^d} ||v||^d. \end{cases}$$

Now,

$$\frac{1}{\kappa_j}\Psi\left(\kappa_j v,\kappa_j v,\cdots,-\kappa_j v,0,\kappa_j v\right)=\begin{cases}\dfrac{5b}{\kappa_j}, \\ \dfrac{b(5n-6+2^{rd})}{\kappa_j}||\kappa_j v||^d, \\ \dfrac{b(5n-6+2^{rnd})}{\kappa_j}||\kappa_j v||^{nd},\end{cases}=\begin{cases}\kappa_j^{-1}\Psi(v,v,\cdots,-v,0,v), \\ \kappa_j^{d-1}\Psi(v,v,\cdots,-v,0,v) \\ \kappa_j^{nd-1}\Psi(v,v,\cdots,-v,0,v).\end{cases}$$

Hence the inequality (5.4) holds either, $L=2^{-1}$ if $i=0$ and $L=\frac{1}{2^{-1}}$ if $i=1$. Now from (5.5), we prove the following cases for condition (i).

Case:1 $L=2^{-1}$ if $i=0$

$$\|a(v)-A(v)\|\leq\frac{\left(2^{-1}\right)^{1-0}}{1-2^{-1}}\kappa_j^{d-1}\Psi(v,v,\cdots,-v,0,v)=5b.$$

Case:2 $L=\frac{1}{2^{-1}}$ if $i=1$

$$\|a(v)-A(v)\|\leq\frac{\left(\frac{1}{2^{-1}}\right)^{1-1}}{1-\frac{1}{2^{-1}}}\kappa_j^{d-1}\Psi(v,v,\cdots,-v,0,v)=-5b.$$

Also the inequality (5.4) holds either, $L=2^{d-1}$ for $d<1$ if $i=0$ and $L=\frac{1}{2^{d-1}}$ for $d>1$ if $i=1$. Now from (5.5), we prove the following cases for condition (ii).

Case:3 $L=2^{d-1}$ for $d<1$ if $i=0$

$$\|a(v)-A(v)\|\leq\frac{\left(2^{(d-1)}\right)^{1-0}}{1-2^{(d-1)}}\Psi(v,v,\cdots,-v,0,v)=\frac{b(5n-6+2^{rd})||v||^d}{2-2^d}.$$

Case:4 $L=\frac{1}{2^{d-1}}$ for $d>1$ if $i=1$

$$\|a(v)-A(v)\|\leq\frac{\left(\frac{1}{2^{(d-1)}}\right)^{1-1}}{1-\frac{1}{2^{(d-1)}}}\Psi(v,v,\cdots,-v,0,v)=\frac{b(5n-6+2^{rd})||v||^d}{2^d-2}.$$

The proof of condition (iii) is similar to that of condition (ii). Hence the proof is complete. □

The proofs of the subsequent theorems and corollaries are similar to that of Theorem 5.17 and 5.15. Hence the details of the proofs are omitted.

Theorem 5.18. *Let $Df_{aqcq}:\mathcal{Y}\to\mathcal{Z}$ be a odd mapping for which there exists a function $\omega,\Omega:\mathcal{Y}^n\to[0,\infty)$ with the condition*

$$\lim_{p\to\infty}\frac{\omega\left(\kappa_j^p v_1,\cdots,\kappa_j^p v_n\right)}{\kappa_j^{3p}}=0 \tag{5.13}$$

for all $v_1,\cdots,v_n\in\mathcal{Y}$ where κ_j is defined (5.2) such that the functional inequality

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\|\leq\omega\left(v_1,\cdots,v_n\right) \tag{5.14}$$

for all $v_1,\cdots,v_n\in\mathcal{Y}$. If there exists $L=L(i)$ such that the function

$$\Psi(v,v,\cdots,-v,0,v)=\Omega\left(\frac{v}{2},\frac{v}{2},\cdots,-\frac{v}{2},0,\frac{v}{2}\right),$$

has the property

$$\Psi(v,v,\cdots,-v,0,v)=\frac{L}{\kappa_j^3}\Psi\left(\kappa_j v,\kappa_j v,\cdots,-\kappa_j v,0,\kappa_j v\right). \tag{5.15}$$

for all $v\in\mathcal{Y}$. Then there exists a unique cubic function $\mathcal{C}:\mathcal{Y}\to\mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{C}(v)-c(v)\|=\|\mathcal{C}(v)-\{f(2v)-2f(v)\}\|\leq\frac{L^{1-j}}{1-L}\Psi(v,v,\cdots,-v,0,v) \tag{5.16}$$

for all $x\in\mathcal{Y}$.

Corollary 5.16. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d satisfying (3.30) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|c(v) - \mathcal{C}(v)\| \leq \begin{cases} 5b, \\ \dfrac{b||v||^d(5n-6+2^{rd})}{|8-2^d|}, \\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{|8-2^{nd}|} \end{cases} \tag{5.17}$$

for all $v \in \mathcal{Y}$.

Theorem 5.19. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a odd mapping for which there exist a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the conditions (5.1) and (5.13) for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined (5.2) such that the functional inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{5.18}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right),$$

has the properties (5.4) and (5.15) and

$$\Psi(v, v, \cdots, -v, 0, v) = \frac{L}{\kappa_j}\Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, 0, \kappa_j v\right). \tag{5.19}$$

for all $v \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ and a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \leq \frac{L^{1-j}}{6(1-L)}\Psi(v, v, \cdots, -v, 0, v) \tag{5.20}$$

for all $x \in \mathcal{Y}$.

Corollary 5.17. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d satisfying (3.36) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|f(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \leq \begin{cases} \dfrac{5b}{6}\left(1 + \dfrac{1}{|7|}\right), \\ \dfrac{b||v||^d(5n-6+2^{rd})}{6}\left(\dfrac{1}{|2-2^d|} + \dfrac{1}{|8-2^d|}\right), \\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{6}\left(\dfrac{1}{|2-2^{nd}|} + \dfrac{1}{|8-2^{nd}|}\right) \end{cases} \tag{5.21}$$

for all $v \in \mathcal{Y}$.

5.2 f IS AN EVEN FUNCTION

Theorem 5.20. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a even mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the condition*

$$\lim_{p\to\infty} \frac{\omega\left(\kappa_j^p v_1, \cdots, \kappa_j^p v_n\right)}{\kappa_j^{2p}} = 0 \tag{5.22}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined (5.2) such that the functional inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{5.23}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right),$$

has the property

$$\Psi(v,v,\cdots,-v,0,v) = \frac{L}{\kappa_j^2}\,\Psi\left(\kappa_j v,\kappa_j v,\cdots,-\kappa_j v,0,\kappa_j v\right). \tag{5.24}$$

for all $v \in \mathcal{Y}$. *Then there exists a unique quadratic function* $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ *which satisfies (1.3) and*

$$\|\mathcal{Q}_2(v) - q_2(v)\| = \|\mathcal{Q}_2(v) - \{f(2v) + 16f(v)\}\| \le \frac{L^{1-j}}{1-L}\Psi(v,v,\cdots,-v,0,v) \tag{5.25}$$

for all $x \in \mathcal{Y}$.

Corollary 5.18. *Let* $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ *be an even mapping. If there exists real numbers b and d satisfying (3.50) for all* $v_1,\cdots,v_n \in \mathcal{Y}$, *then there exists a unique quadratic function* $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ *such that*

$$\|q_2(v) - \mathcal{Q}_2(v)\| \le \begin{cases} \dfrac{5b}{7}, \\ \dfrac{b||v||^d(5n-6+2^{rd})}{|4-2^d|}, \\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{|4-2^{nd}|} \end{cases} \tag{5.26}$$

for all $v \in \mathcal{Y}$.

Theorem 5.21. *Let* $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ *be a even mapping for which there exists a function* $\omega, \Omega : \mathcal{Y}^n \to [0,\infty)$ *with the condition*

$$\lim_{p\to\infty} \frac{\omega\left(\kappa_j^p v_1,\cdots,\kappa_j^p v_n\right)}{\kappa_j^{4p}} = 0 \tag{5.27}$$

for all $v_1,\cdots,v_n \in \mathcal{Y}$ *where* κ_j *is defined (5.2)*
such that the functional inequality

$$\left\|Df_{aqcq}(v_1,\cdots,v_n)\right\| \le \omega\left(v_1,\cdots,v_n\right) \tag{5.28}$$

for all $v_1,\cdots,v_n \in \mathcal{Y}$. *If there exists* $L = L(i)$ *such that the function*

$$\Psi(v,v,\cdots,-v,0,v) = \Omega\left(\frac{v}{2},\frac{v}{2},\cdots,-\frac{v}{2},0,\frac{v}{2}\right),$$

has the property

$$\Psi(v,v,\cdots,-v,0,v) = \frac{L}{\kappa_j^4}\,\Psi\left(\kappa_j v,\kappa_j v,\cdots,-\kappa_j v,0,\kappa_j v\right). \tag{5.29}$$

for all $v \in \mathcal{Y}$. *Then there exists a unique quartic function* $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ *which satisfies (1.3) and*

$$\|\mathcal{Q}_4(v) - q_4(v)\| = \|\mathcal{Q}_4(v) - \{f(2v) - 4f(v)\}\| \le \frac{L^{1-j}}{1-L}\Psi(v,v,\cdots,-v,0,v) \tag{5.30}$$

for all $x \in \mathcal{Y}$.

Corollary 5.19. *Let* $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ *be an even mapping. If there exists real numbers b and d satisfying (3.58) for all* $v_1,\cdots,v_n \in \mathcal{Y}$, *then there exists a unique quartic function* $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ *such that*

$$\|q_4(v) - \mathcal{Q}_4(v)\| \le \begin{cases} \dfrac{5b}{15}, \\ \dfrac{b||v||^d(5n-6+2^{rd})}{|16-2^d|}, \\ \dfrac{b||v||^{nd}(5n-6+2^{rnd})}{|16-2^{nd}|} \end{cases} \tag{5.31}$$

for all $v \in \mathcal{Y}$.

Theorem 5.22. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a even mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the conditions (5.22) and (5.27) for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined (5.2) such that the functional inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{5.32}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right),$$

has the properties (5.24) and (5.29) and

$$\Psi(v, v, \cdots, -v, 0, v) = \frac{L}{\kappa_j}\, \Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, 0, \kappa_j v\right). \tag{5.33}$$

for all $v \in \mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \leq \frac{L^{1-j}}{12(1-L)} \Psi(v, v, \cdots, -v, 0, v) \tag{5.34}$$

for all $v \in \mathcal{Y}$.

Corollary 5.20. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d satisfying (3.64) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \dfrac{5b}{12}\left(\dfrac{1}{|3|} + \dfrac{1}{|15|}\right), \\ \dfrac{b||v||^d(5n - 1 + 2^{rd})}{12}\left(\dfrac{1}{|4 - 2^d|} + \dfrac{1}{|16 - 2^d|}\right), \\ \dfrac{b||v||^{nd}(5n - 1 + 2^{rnd})}{12}\left(\dfrac{1}{|4 - 2^{nd}|} + \dfrac{1}{|16 - 2^{nd}|}\right) \end{cases} \tag{5.35}$$

for all $v \in \mathcal{Y}$.

5.3 f IS AN ODD - EVEN FUNCTION

Theorem 5.23. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the conditions (5.1), (5.22), (5.13) and (5.27) for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined (5.2) such that the functional inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{5.36}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, 0, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, 0, \frac{v}{2}\right),$$

has the properties (5.4), (5.24), (5.15) and (5.29) and

$$\Psi(v, v, \cdots, -v, 0, v) = \frac{L}{\kappa_j}\, \Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, 0, \kappa_j v\right). \tag{5.37}$$

for all $v \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\| \leq \frac{L^{1-j}}{(1-L)}\left(\frac{1}{6} + \frac{1}{12}\right) \Psi(v, v, \cdots, -v, 0, v) \tag{5.38}$$

for all $v \in \mathcal{Y}$.

Corollary 5.21. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a mapping. If there exists real numbers b and d satisfying (3.73) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\| \leq \begin{cases} \frac{5b}{2}\left\{\frac{1}{12}\left[\frac{1}{|3|}+\frac{1}{|15|}\right]+\frac{1}{6}\left[1+\frac{1}{|7|}\right]\right\}, \\ \frac{b||v||^d(5n-1+2^{rd})}{2}\left\{\frac{1}{12}\left[\frac{1}{|4-2^d|}+\frac{1}{|16-2^d|}\right]+\frac{1}{6}\left[\frac{1}{|2-2^d|}+\frac{1}{|8-2^d|}\right]\right\}, \\ \frac{b||v||^{nd}(5n-1+2^{rnd})}{2}\left\{\frac{1}{12}\left[\frac{1}{|4-2^{nd}|}+\frac{1}{|16-2^{nd}|}\right]+\frac{1}{6}\left[\frac{1}{|2-2^{nd}|}+\frac{1}{|8-2^{nd}|}\right]\right\}, \end{cases} \tag{5.39}$$

for all $v \in \mathcal{Y}$.

6 Stability Results - Fixed Point Method: n Even Positive Integer

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.3) where n is an even positive integer in Banach space using fixed point method.

6.1 f IS AN ODD FUNCTION

Theorem 6.24. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a odd mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the condition*

$$\lim_{p\to\infty} \frac{\omega\left(\kappa_j^p v_1, \cdots, \kappa_j^p v_n\right)}{\kappa_j^p} = 0 \tag{6.1}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ where

$$\kappa_j = \begin{cases} 2 & if \quad j = 0; \\ \frac{1}{2} & if \quad j = 1, \end{cases} \tag{6.2}$$

such that the functional inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{6.3}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the property

$$\Psi(v, v, \cdots, -v, v) = \frac{L}{\kappa_j}\,\Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, \kappa_j v\right). \tag{6.4}$$

for all $v \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{A}(v) - a(v)\| = \|\mathcal{A}(v) - \{f(2v) - 8f(v)\}\| \leq \frac{L^{1-j}}{1-L}\Psi(v, v, \cdots, -v, v) \tag{6.5}$$

for all $v \in \mathcal{Y}$.

Corollary 6.22. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d fulfilling (4.6) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|a(v) - \mathcal{A}(v)\| \leq \begin{cases} 5b, \\ \frac{b||v||^d(5n-1+2^{rd})}{|2-2^d|}, \\ \frac{b||v||^d(4+2^{rnd})}{|2-2^{nd}|}, \\ \frac{b||v||^{nd}(5n-3+2^{rd}+2^{rnd})}{|2-2^{nd}|} \end{cases} \tag{6.6}$$

for all $v \in \mathcal{Y}$.

Theorem 6.25. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a odd mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the condition*

$$\lim_{p\to\infty} \frac{\omega\left(\kappa_j^p v_1, \cdots, \kappa_j^p v_n\right)}{\kappa_j^{3p}} = 0 \tag{6.7}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined in (6.2) such that the functional inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \le \omega\left(v_1, \cdots, v_n\right) \tag{6.8}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there existss $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the property

$$\Psi(v, v, \cdots, -v, v) = \frac{L}{\kappa_j^3}\,\Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, \kappa_j v\right). \tag{6.9}$$

for all $v \in \mathcal{Y}$. Then there exists a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{C}(v) - c(v)\| = \|\mathcal{C}(v) - \{f(2v) - 2f(v)\}\| \le \frac{L^{1-j}}{1-L}\Psi(v, v, \cdots, -v, v) \tag{6.10}$$

for all $v \in \mathcal{Y}$.

Corollary 6.23. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d fulfilling (4.12) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|c(v) - \mathcal{C}(v)\| \le \begin{cases} 5b, \\ \dfrac{b||v||^d(5n - 1 + 2^{rd})}{|8 - 2^d|}, \\ \dfrac{b||v||^d(4 + 2^{rnd})}{|8 - 2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n + 3 + 2^{rd} + 2^{rnd})}{|8 - 2^{nd}|} \end{cases} \tag{6.11}$$

for all $v \in \mathcal{Y}$.

Theorem 6.26. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a odd mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the conditions (6.1) and (6.7) for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined in (6.2) such that the functional inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \le \omega\left(v_1, \cdots, v_n\right) \tag{6.12}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the properties (6.4) and (6.9) for all $v \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{A}(v) - \mathcal{C}(v)\| \le \frac{L^{1-j}}{6(1-L)}\Psi(v, v, \cdots, -v, v) \tag{6.13}$$

for all $v \in \mathcal{Y}$.

Corollary 6.24. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an odd mapping. If there exists real numbers b and d fulfilling (4.16) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|a(v) - \mathcal{A}(v)\| \le \begin{cases} \dfrac{5b}{6} + \dfrac{5b}{7.6}, \\ \dfrac{b||v||^d(5n - 1 + 2^{rd})}{6|2 - 2^d|} + \dfrac{b||v||^d(5n - 1 + 2^{rd})}{6|8 - 2^d|}, \\ \dfrac{b||v||^d(4 + 2^{rnd})}{6|2 - 2^{nd}|} + \dfrac{b||v||^d(4 + 2^{rnd})}{6|8 - 2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n + 3 + 2^{rd} + 2^{rnd})}{6|2 - 2^{nd}|} + \dfrac{b||v||^{nd}(5n + 3 + 2^{rd} + 2^{rnd})}{6|8 - 2^{nd}|} \end{cases} \tag{6.14}$$

for all $v \in \mathcal{Y}$.

6.2 f IS AN EVEN FUNCTION

Theorem 6.27. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a even mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the condition*

$$\lim_{p\to\infty} \frac{\omega\left(\kappa_j^p v_1, \cdots, \kappa_j^p v_n\right)}{\kappa_j^{2p}} = 0 \tag{6.15}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined in (6.2) such that the functional inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{6.16}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the property

$$\Psi(v, v, \cdots, -v, v) = \frac{L}{\kappa_j^2}\, \Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, \kappa_j v\right). \tag{6.17}$$

for all $v \in \mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{Q}_2(v) - q_2(v)\| = \|\mathcal{Q}_2(v) - \{f(2v) - 16f(v)\}\| \leq \frac{L^{1-j}}{1-L}\Psi(v, v, \cdots, -v, v) \tag{6.18}$$

for all $v \in \mathcal{Y}$.

Corollary 6.25. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d fulfilling (4.22) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|q_2(v) - \mathcal{Q}_2(v)\| \leq \begin{cases} 5b, \\ \dfrac{b||v||^d(5n - 1 + 2^{rd})}{|4 - 2^d|}, \\ \dfrac{b||v||^d(4 + 2^{rnd})}{|4 - 2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n + 3 + 2^{rd} + 2^{rnd})}{|4 - 2^{nd}|} \end{cases} \tag{6.19}$$

for all $v \in \mathcal{Y}$.

Theorem 6.28. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a even mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the condition*

$$\lim_{p\to\infty} \frac{\omega\left(\kappa_j^p v_1, \cdots, \kappa_j^p v_n\right)}{\kappa_j^{4p}} = 0 \tag{6.20}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined in (6.2) such that the functional inequality

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \leq \omega\left(v_1, \cdots, v_n\right) \tag{6.21}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the property

$$\Psi(v, v, \cdots, -v, v) = \frac{L}{\kappa_j^4}\, \Psi\left(\kappa_j v, \kappa_j v, \cdots, -\kappa_j v, \kappa_j v\right). \tag{6.22}$$

for all $v \in \mathcal{Y}$. Then there exists a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|\mathcal{Q}_4(v) - q_4(v)\| = \|\mathcal{Q}_4(v) - \{f(2v) - 4f(v)\}\| \leq \frac{L^{1-j}}{1-L}\Psi(v, v, \cdots, -v, v) \tag{6.23}$$

for all $v \in \mathcal{Y}$.

Corollary 6.26. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d fulfilling (4.28) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|q_4(v) - \mathcal{Q}_4(v)\| \le \begin{cases} 5b, \\ \dfrac{b||v||^d(5n-1+2^{rd})}{|16-2^d|}, \\ \dfrac{b||v||^d(4+2^{rnd})}{|16-2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{|16-2^{nd}|} \end{cases} \tag{6.24}$$

for all $v \in \mathcal{Y}$.

Theorem 6.29. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a even mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the conditions (6.15) and (6.20) for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined in (6.2) such that the functional inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \le \omega\left(v_1, \cdots, v_n\right) \tag{6.25}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there existss $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the properties (6.17) and (6.22) for all $v \in \mathcal{Y}$. Then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \le \frac{L^{1-j}}{12(1-L)}\Psi(v, v, \cdots, -v, v) \tag{6.26}$$

for all $v \in \mathcal{Y}$.

Corollary 6.27. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be an even mapping. If there exists real numbers b and d fulfilling (4.32) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|f(v) - \mathcal{Q}_2(v) - \mathcal{Q}_4(v)\| \le \begin{cases} 5b, \\ \dfrac{b||v||^d(5n-1+2^{rd})}{12|4-2^d|} + \dfrac{b||v||^d(5n-1+2^{rd})}{12|16-2^d|}, \\ \dfrac{b||v||^d(4+2^{rnd})}{12|4-2^{nd}|} + \dfrac{b||v||^d(4+2^{rnd})}{12|16-2^{nd}|}, \\ \dfrac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{12|4-2^{nd}|} + \dfrac{b||v||^{nd}(5n+3+2^{rd}+2^{rnd})}{12|16-2^{nd}|} \end{cases} \tag{6.27}$$

for all $v \in \mathcal{Y}$.

6.3 f IS AN ODD - EVEN FUNCTION

Theorem 6.30. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a mapping for which there exists a function $\omega, \Omega : \mathcal{Y}^n \to [0, \infty)$ with the conditions (6.1), (6.15), (6.7) and (6.20) for all $v_1, \cdots, v_n \in \mathcal{Y}$ where κ_j is defined in (6.2) such that the functional inequality*

$$\left\|Df_{aqcq}(v_1, \cdots, v_n)\right\| \le \omega\left(v_1, \cdots, v_n\right) \tag{6.28}$$

for all $v_1, \cdots, v_n \in \mathcal{Y}$. If there exists $L = L(i)$ such that the function

$$\Psi(v, v, \cdots, -v, v) = \Omega\left(\frac{v}{2}, \frac{v}{2}, \cdots, -\frac{v}{2}, \frac{v}{2}\right),$$

has the properties (6.4), (6.17), (6.9) and (6.22) for all $v \in \mathcal{Y}$. Then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ which satisfies (1.3) and

$$\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\| \le \frac{L^{1-j}}{(1-L)}\left(\frac{1}{6} + \frac{1}{12}\right)\Psi(v, v, \cdots, -v, v) \tag{6.29}$$

for all $v \in \mathcal{Y}$.

Corollary 6.28. *Let $Df_{aqcq} : \mathcal{Y} \to \mathcal{Z}$ be a mapping and if there exists real numbers b and d fulfilling (4.36) for all $v_1, \cdots, v_n \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ a unique quadratic function $\mathcal{Q}_2 : \mathcal{Y} \to \mathcal{Z}$ a unique cubic function $\mathcal{C} : \mathcal{Y} \to \mathcal{Z}$ a unique quartic function $\mathcal{Q}_4 : \mathcal{Y} \to \mathcal{Z}$ such that*

$$\|f(v) - \mathcal{A}(v) - \mathcal{Q}_2(v) - \mathcal{C}(v) - \mathcal{Q}_4(v)\|$$
$$\leq \begin{cases} \frac{5b}{2}\left\{\frac{1}{12}\left[\frac{1}{|3|}+\frac{1}{|15|}\right]+\frac{1}{6}\left[1+\frac{1}{|7|}\right]\right\}, \\ \frac{b||v||^d(5n-1+2^{rd})}{2}\left\{\frac{1}{12}\left[\frac{1}{|4-2^d|}+\frac{1}{|16-2^d|}\right]+\frac{1}{6}\left[\frac{1}{|2-2^d|}+\frac{1}{|8-2^d|}\right]\right\}, \\ \frac{b||v||^d(4+2^{rnd})}{2}\left\{\frac{1}{12}\left[\frac{1}{|4-2^{nd}|}+\frac{1}{|16-2^{nd}|}\right]+\frac{1}{6}\left[\frac{1}{|2-2^{nd}|}+\frac{1}{|8-2^{nd}|}\right]\right\}, \\ \frac{b||v||^{nd}(5n-3+2^{rd}+2^{rnd})}{2}\left\{\frac{1}{12}\left[\frac{1}{|4-2^{nd}|}+\frac{1}{|16-2^{nd}|}\right]+\frac{1}{6}\left[\frac{1}{|2-2^{nd}|}+\frac{1}{|8-2^{nd}|}\right]\right\}, \end{cases} \tag{6.30}$$

for all $v \in \mathcal{Y}$.

7 Counter Examples For Non Stable Cases

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = 1$ in condition (ii) of Corollaries 3.1, 4.8, 5.15 and 6.22.

Example 7.1. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v, & \text{if } |v| < 1 \\ \rho, & \text{otherwise} \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$a(v) = f(2v) - 8f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^r}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \leq 52\rho \sum_{i=1}^{n} |v_i| \tag{7.1}$$

for all $v \in \mathbb{R}$. Then there do not exist a additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{A}(v) - \{f(2v) - 8f(v)\}| \leq \eta |v|, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.2}$$

Proof. Now

$$|a(v)| = |f(2v) - 8f(v)| \leq \sum_{r=0}^{\infty} \frac{|\omega(2^r v)|}{|2^r|} = \sum_{n=0}^{\infty} \frac{\rho}{2^r} = 2\,\rho.$$

Therefore, we see that a is bounded. We are going to prove that a satisfies (7.1).

If $v = 0$ then (7.1) is trivial. If $\sum_{i=1}^{n} |v_i| \geq \frac{1}{2}$ then the left hand side of (7.1) is less than 52ρ. Now suppose that $0 < \sum_{i=1}^{n} |v_i| < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^k} \leq \sum_{i=1}^{n} |v_i| < \frac{1}{2^{k-1}}, \tag{7.3}$$

so that $2^{k-1} v_i (i = 1, 2, \cdots, n) < \frac{1}{2}$ and consequently

$$2^{k-1}\left(\sum_{i=1}^{n-1} v_i + 2v_n\right), 2^{k-1}\left(\sum_{i=1}^{n-1} v_i - 2v_n\right), 2^{k-1}\left(\sum_{i=1}^{n} v_i\right), 2^{k-1}\left(\sum_{i=1}^{n-1} v_i - v_n\right),$$
$$2^{k-1} f\left(\sum_{i=1}^{n-1} v_i\right), 2^{k-1}(2v_n), 2^{k-1}(-2v_n), 2^{k-1}(v_n), 2^{k-1}(-v_n) \in (-1, 1).$$

Therefore for each $r = 0, 1, \dots, k-1$, we have

$$2^r\left(\sum_{i=1}^{n-1} v_i + 2v_n\right), 2^r\left(\sum_{i=1}^{n-1} v_i - 2v_n\right), 2^r\left(\sum_{i=1}^{n} v_i\right), 2^r\left(\sum_{i=1}^{n-1} v_i - v_n\right),$$
$$2^r f\left(\sum_{i=1}^{n-1} v_i\right), 2^r\left(2v_n\right), 2^r\left(-2v_n\right), 2^r\left(v_n\right), 2^r\left(-v_n\right) \in (-1,1)$$

and

$$\omega\left(2^r\sum_{i=1}^{n-1} v_i + 2^r\cdot 2v_n\right) + \omega\left(2^r\sum_{i=1}^{n-1} v_i - 2^r\cdot 2v_n\right) - 4\omega\left(2^r\sum_{i=1}^{n} v_i\right) - 4\omega\left(2^r\sum_{i=1}^{n-1} v_i - 2^r v_n\right)$$
$$+ 6\omega\left(2^r\sum_{i=1}^{n-1} v_i\right) - \omega\left(2^r\cdot 2v_n\right) - \omega\left(-2^r\cdot 2v_n\right) + 4\omega\left(2^r v_n\right) + 4\omega f\left(-2^r v_n\right) = 0$$

for $r = 0, 1, \dots, k-1$. From the definition of a and (7.3), we obtain that

$$\begin{aligned}
&\Bigg| a\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + a\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4a\left(\sum_{i=1}^{n} v_i\right) - 4a\left(\sum_{i=1}^{n-1} v_i - v_n\right) \\
&\qquad +6a\left(\sum_{i=1}^{n-1} v_i\right) - a\left(2v_n\right) - a\left(-2v_n\right) + 4a\left(v_n\right) + 4a\left(-v_n\right)\Bigg| \\
&\leq \sum_{r=0}^{\infty} \frac{1}{2^r}\Bigg|\omega\left(2^r\sum_{i=1}^{n-1} v_i + 2^r\cdot 2v_n\right) + \omega\left(2^r\sum_{i=1}^{n-1} v_i - 2^r\cdot 2v_n\right) - 4\omega\left(2^r\sum_{i=1}^{n} v_i\right) \\
&\qquad - 4\omega\left(2^r\sum_{i=1}^{n-1} v_i - 2^r v_n\right) + 6\omega\left(2^r\sum_{i=1}^{n-1} v_i\right) - \omega\left(2^r\cdot 2v_n\right) - \omega\left(-2^r\cdot 2v_n\right) \\
&\qquad\qquad + 4\omega\left(2^r v_n\right) + 4\omega f\left(-2^r v_n\right)\Bigg| \\
&\leq \sum_{r=k}^{\infty} \frac{1}{2^r}\Bigg|\omega\left(2^r\sum_{i=1}^{n-1} v_i + 2^r\cdot 2v_n\right) + \omega\left(2^r\sum_{i=1}^{n-1} v_i - 2^r\cdot 2v_n\right) - 4\omega\left(2^r\sum_{i=1}^{n} v_i\right) \\
&\qquad - 4\omega\left(2^r\sum_{i=1}^{n-1} v_i - 2^r v_n\right) + 6\omega\left(2^r\sum_{i=1}^{n-1} v_i\right) - \omega\left(2^r\cdot 2v_n\right) - \omega\left(-2^r\cdot 2v_n\right) \\
&\qquad\qquad + 4\omega\left(2^r v_n\right) + 4\omega f\left(-2^r v_n\right)\Bigg| \\
&= \sum_{r=k}^{\infty} \frac{1}{2^r} \times 26\rho = 26\rho \times \frac{2}{2^k} \leq 52\rho \sum_{i=1}^{n} |v_i|
\end{aligned}$$

Thus a satisfies (7.1) for all $x \in \mathbb{R}$ with $0 < \sum_{i=1}^{n} |v_i| < \frac{1}{2}$.

We claim that the additive functional equation (1.3) is not stable for $r = 1$ in condition (ii) of Corollary 3.1. Suppose on the contrary that there exist a additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ satisfying (7.2). Since a is bounded and continuous for all $x \in \mathbb{R}$, $\mathcal{A}$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.3, $\mathcal{A}$ must have the form $\mathcal{A}(v) = cv$ for any v in $\mathbb{R}$. Thus, we obtain that

$$|a(v)| \leq (\eta + |c|)\,|v|. \tag{7.4}$$

But we can choose a positive integer s with $s\rho > \eta + |c|$.

If $v \in \left(0, \frac{1}{2^{s-1}}\right)$, then $2^r v \in (0,1)$ for all $r = 0, 1, \dots, s-1$. For this v, we get

$$a(v) = \sum_{n=0}^{\infty} \frac{\omega(2^r v)}{2^r} \geq \sum_{r=0}^{s-1} \frac{\rho(2^r v)}{2^r} = s\rho v > (\eta + |c|)\, v$$

which contradicts (7.4). Therefore the additive functional equation (1.3) is not stable in sense of Ulam, Hyers and Rassias if $r = 1$, assumed in the inequality condition (ii). $\square$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = \frac{1}{n}$ in condition (iii) of Corollaries 3.1,5.15 and condition (iv) of Corollaries 4.8 and 6.22.

Example 7.2. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v, & \text{if } |v| < \frac{1}{n} \\ \frac{\rho}{n}, & \text{otherwise} \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$a(v) = f(2v) - 8f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^r}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \leq \frac{52\rho}{n} \sum_{i=1}^{n} |v_i|^{\frac{1}{n}} \tag{7.5}$$

for all $v \in \mathbb{R}$. Then there do not exist a additive mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{A}(v) - \{f(2v) - 8f(v)\}| \leq \eta |v|^{\frac{1}{n}}, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.6}$$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = 2$ in condition (ii) of Corollaries 3.1, 4.8, 5.15 and 6.22.

Example 7.3. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v^2, & \text{if } |v| < 1 \\ \rho, & \text{otherwise} \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$q_2(v) = f(2v) - 3f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^{2r}}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \leq \frac{26\rho \times 4^2}{3} \sum_{i=1}^{n} |v_i|^2 \tag{7.7}$$

for all $v \in \mathbb{R}$. Then there do not exist a quadratic mapping $\mathcal{Q}_2 : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{Q}_2(v) - \{f(2v) - 16f(v)\}| \leq \eta |v|^2, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.8}$$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = \frac{2}{n}$ in condition (iii) of Corollaries 3.1,5.15 and condition (iv) of Corollaries 4.8 and 6.22.

Example 7.4. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v, & \text{if } |v| < \frac{2}{n} \\ \frac{2\rho}{n}, & \text{otherwise} \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$q_2(v) = f(2v) - 16f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^{2r}}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \le \frac{52\rho \times 4^2}{3n} \sum_{i=1}^{n} |v_i|^{\frac{2}{n}} \tag{7.9}$$

for all $v \in \mathbb{R}$. Then there do not exist a quadratic mapping $\mathcal{Q}_2 : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{Q}_2(v) - \{f(2v) - 16f(v)\}| \le \eta |v|^{\frac{2}{n}}, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.10}$$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = 3$ in condition (ii) of Corollaries 3.1, 4.8, 5.15 and 6.22.

Example 7.5. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v^3, & if\ |v| < 1 \\ \rho, & otherwise \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$c(v) = f(2v) - 3f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^{3r}}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \le \frac{26\rho \times 8^3}{7} \sum_{i=1}^{n} |v_i|^{3} \tag{7.11}$$

for all $v \in \mathbb{R}$. Then there do not exist a cubic mapping $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{C}(v) - \{f(2v) - 2f(v)\}| \le \eta |v|^{3}, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.12}$$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = \frac{3}{n}$ in condition (iii) of Corollaries 3.1,5.15 and condition (iv) of Corollaries 4.8 and 6.22.

Example 7.6. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v, & if\ |v| < \frac{3}{n} \\ \frac{3\rho}{n}, & otherwise \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$c(v) = f(2v) - 2f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^{3r}}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \le \frac{78\rho \times 8^3}{7n} \sum_{i=1}^{n} |v_i|^{\frac{3}{n}} \tag{7.13}$$

for all $v \in \mathbb{R}$. Then there do not exist a cubic mapping $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{C}(v) - \{f(2v) - 2f(v)\}| \le \eta |v|^{\frac{3}{n}}, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.14}$$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = 2$ in condition (ii) of Corollaries 3.1, 4.8, 5.15 and 6.22.

Example 7.7. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v^4, & if\ |v| < 1 \\ \rho, & otherwise \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$q_4(v) = f(2v) - 4f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^{4r}}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \le \frac{26\rho \times 4^2}{15} \sum_{i=1}^{n} |v_i|^4 \tag{7.15}$$

for all $v \in \mathbb{R}$. Then there do not exist a quartic mapping $\mathcal{Q}_4 : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{Q}_4(v) - \{f(2v) - 4f(v)\}| \le \eta |v|^4, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.16}$$

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $d = \frac{4}{n}$ in condition (iii) of Corollaries 3.1,5.15 and condition (iv) of Corollaries 4.8 and 6.22.

Example 7.8. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\omega(v) = \begin{cases} \rho v, & if\ |v| < \frac{4}{n} \\ \frac{4\rho}{n}, & otherwise \end{cases}$$

where $\rho > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$q_2(v) = f(2v) - 4f(v) = \sum_{r=0}^{\infty} \frac{\omega(2^r v)}{2^{4r}}, \qquad for\ all \quad v \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\left| f\left(\sum_{i=1}^{n-1} v_i + 2v_n\right) + f\left(\sum_{i=1}^{n-1} v_i - 2v_n\right) - 4f\left(\sum_{i=1}^{n} v_i\right) - 4f\left(\sum_{i=1}^{n-1} v_i - v_n\right) \right.$$
$$\left. +6f\left(\sum_{i=1}^{n-1} v_i\right) - f(2v_n) - f(-2v_n) + 4f(v_n) + 4f(-v_n) \right| \le \frac{144\rho \times 4^2}{15n} \sum_{i=1}^{n} |v_i|^{\frac{4}{n}} \tag{7.17}$$

for all $v \in \mathbb{R}$. Then there do not exist a quartic mapping $\mathcal{Q}_4 : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|\mathcal{Q}_4(v) - \{f(2v) - 4f(v)\}| \le \eta |v|^{\frac{4}{n}}, \qquad for\ all \quad v \in \mathbb{R}. \tag{7.18}$$

References

[1] **J. Aczel and J. Dhombres**, *Functional Equations in Several Variables,* Cambridge Univ, Press, 1989.

[2] **C. Alsina**, *On the stability of a functional equation arising in probabilistic normed spaces*, General Inequal., Oberwolfach **5**, 263–271 (1986). Birkhuser, Basel (1987).

[3] **T. Aoki**, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64–66.

[4] **M. Arunkumar, Matina J. Rassias, Y. Zhang**, *Ulam-Hyers stability of a 2-variable AC-mixed type functional equation: direct and fixed point methods*, J. Mod. Math. Front., **1** (3) (2012), 10–26.

[5] **M. Arunkumar**, *Solution and stability of modified additive and quadratic functional equation in generalized 2-normed spaces*, International J. Math. Sci. Eng. Appl., **7**, No. I (2013), 383–391.

[6] **M. Arunkumar, G. Britto Antony Xavier**, *Functional equation Originating from sum of higher Powers of Arithmetic Progression using Difference Operator is stable in Banach space: Direct and Fixed point Methods*, Malaya J. Mat., **1**. Iss. 1 (2014), 49–60.

[7] **M. Arunkumar, S. Karthikeyan**, *Solution and Intuitionistic Fuzzy stability of n-dimensional quadratic functional equation: Direct and Fixed Point Methods*, Int. J. Adv. Math. Sci., **2** (1) (2014), 21–33.

[8] **M. Arunkumar**, *Perturbation of n Dimensional AQ - mixed type Functional Equation via Banach Spaces and Banach Algebra: Hyers Direct and Alternative Fixed Point Methods*, International Journal of Advanced Mathematical Sciences (IJAMS), Vol. 2 (1), (2014), 34-56.

[9] **M. Arunkumar, T. Namachivayam**, *Stability of a n-dimensional additive functional equation in random normed space*, Int. J. Math. . Anal., **4**, No. 2 (2012), 179–186.

[10] **M. Arunkumar, T. Namachivayam**, *Stability of a n-dimensional additive functional equation in random normed space: A fixed point approach*, International Journal of Physical, Chemical and Mathematical Sciences (IJPCMS) - ISSN 2278 - 683X, Vol. 2; No. 1, (2013), 100 - 108.

[11] **M. Arunkumar, S. Hemalatha, E. Sathya**, *2-Variable AQCQ - functional equation*, Int. J. Adv. Math. Sci., **3** (1) (2015), 65–86 .

[12] **M. Arunkumar, S. Hema latha**, *Additive -Quartic Functional Equations Are Stable in Random Normed Space*, Jamal Academic Research Journal an Interdisciplinary, (2015), 31-38.

[13] **M. Arunkumar, S. Murthy, S. Hema latha, M. Arulselvan**, *Additive - quartic functional equations are stable in random normed space: a fixed point method*, Malaya Journal of Mathematics, (2015), 215 - 227.

[14] **A. Bodaghi**, *Stability of a quartic functional equation*, The Scientific World Journal. **2014**, Art. ID 752146, 9 pages, doi:10.1155/2014/752146.

[15] **A. Bodaghi, I. A. Alias and M. H. Ghahramani**, *Ulam stability of a quartic functional equation*, Abs. Appl. Anal., **2012**, Art. ID 232630 (2012).

[16] **D. G. Bourgin**, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc., **57** (1951), 223–237.

[17] **L. Cădariu, V. Radu**, *Fixed points and the stability of quadratic functional equations*,An. Univ. Timisoara, Ser. Mat. Inform. **41** (2003), 25–48.

[18] **L. Cădariu, V. Radu**, *On the stability of the Cauchy functional equation: A fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.

[19] **E. Castillo, A. Iglesias and R. Ruiz-coho**, *Functional Equations in Applied Sciences*, Elsevier, B.V.Amslerdam, 2005.

[20] **Y. J. Cho, M. Eshaghi Gordji and S. Zolfaghari**, *Solutions and Stability of Generalized Mixed Type QC Functional Equations in Random Normed Spaces*, J. Ineq. Appl, doi:10.1155/2010/403101

[21] **S. Czerwik**, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[22] **G. L. Forti**, *Comments on the core of the direct method for proving Hyers-Ulam stability of functional equations*, J. Math. Anal. Appl., **295** (2004), 127-133.

[23] **Z. Gajda**, *On the stability of additive mappings*, Inter. J. Math. Math. Sci., **14** (1991), 431–434.

[24] **P. Găvruta**, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings* , J. Math. Anal. Appl., **184** (1994), 431–436.

[25] **M. Eshaghi Gordji, H. Khodaie**, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces*, arxiv: 0812. 2939v1 Math FA, 15 Dec 2008.

[26] **M. Eshaghi Gordji and M. B. Savadkouhi**, *Stability of Mixed Type Cubic and Quartic Functional Equations in Random Normed Spaces*, J. Ineq. Appl., doi:10.1155/2009/527462.

[27] **M. Eshaghi Gordji, M. Bavand Savadkouhi, Choonkil Park**, *Quadratic-Quartic Functional Equations in RN-Spaces*, Journal of Inequalities and Applications, doi:10.1155/2009/868423.

[28] **M. Eshaghi Gordji, S. Zolfaghari , J. M. Rassias and M. B. Savadkouhi**, *Solution and Stability of a Mixed type Cubic and Quartic functional equation in Quasi-Banach spaces*, Abstract and Applied Analysis, Volume 2009, Art. ID 417473, 1-14, Doi:10.1155/2009/417473.

[29] **H. Azadi Kenary, S. Y. Jang and C. Park**, *A fixed point approach to the Hyers-Ulam stability of a functional equation in various normed spaces*, Fixed Point The. and Appl., doi:10.1186/1687-1812-2011-67.

[30] **D. H. Hyers**, *On the stability of the linear functional equation*, Proc.Nat. Acad.Sci.,U.S.A., **27** (1941) 222-224.

[31] **D. H. Hyers, G. Isac and Th.M. Rassias**, *Stability of functional equations in several variables*,Birkhauser, Basel, 1998.

[32] **K. W. Jun and H. M. Kim**, *On the stability of an n-dimensional quadratic and additive type functional equation*, Math. Ineq. Appl 9(1) (2006), 153-165.

[33] **S. M. Jung**, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[34] **P. Kannappan**, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, 2009.

[35] **L. Maligranda**, *A result of Tosio Aoki about a generalization of Hyers-Ulam stability of additive functions-a question of priority*, Aequ. Math., **75** (2008), 289–296.

[36] **B. Margolis and J. B. Diaz**, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **126** (1968), 305–309.

[37] **D. Mihet and V. Radu**, *On the stability of the additive Cauchy functional equation in random normed spaces*, J Math Anal Appl. **343** (2008), 567–572 .

[38] **D. Mihet**, *The probabilistic stability for a functional equation in a single variable*, Acta Math Hungar. **123**, 249256 (2009). doi:10.1007/s10474-008-8101-y

[39] **M. Mohamadi, Y. Cho, C. Park, P. Vetro, R. Saadati**, *Random stability of an additive-quadratic-quartic functional equation* J Inequal Appl 2010, 18 (2010). Article ID 754210.

[40] **S. Murthy, M. Arunkumar, V. Govindan**, *General Solution and Generalized Ulam-Hyers Stability of a Generalized n- Type Additive Quadratic Functional Equation in Banach Space and Banach Algebra: Direct and Fixed Point Methods*, International Journal of Advanced Mathematical Sciences, 3 (1) (2015), 25-64.

[41] **C. Park and A. Bodaghi**, *On the stability of $*$-derivations on Banach $*$-algebras*, Adv. Difference Equ. **2012**, Article No. 138, 10 pages (2012).

[42] **C. Park, J. R. Lee**, *An AQCQ-functional equation in paranormed spaces*, Advances in Difference Equations, doi: 10.1186/1687-1847-2012-63.

[43] **J. Matina, Rassias, M. Arunkumar, S. Ramamoorthi**, *Stability of the Leibniz additive-quadratic functional equation in quasi- β normed spaces: direct and fixed point methods*, Journal of Concrete and Applicable Mathematics, **14** (2014), 22–46.

[44] **J. M. Rassias**, *On approximately of approximately linear mappings by linear mappings*, J. Funct. Anal. USA, **46** (1982) 126–130.

[45] **J. M. Rassias**, *Solution of the Ulam problem for cubic mappings*, An. Univ. Timisoara Ser. Mat. Inform. **38** (2000), 121–132.

[46] **J. M. Rassias**, *Solution of the Ulam stability problem for quartic mappings*, Glasnik Mathematicki, **34**(54) no.2, (1999), 243–252.

[47] **J. M. Rassias and H. M. Kim**, *Generalized Hyers-Ulam stability for general additive functional equations in quasi-β-normed spaces* J. Math. Anal. Appl. **356** (2009), no. 1, 302–309.

[48] **J. M. Rassias, K.Ravi, M.Arunkumar and B.V.Senthil Kumar**, *Ulam Stability of Mixed type Cubic and Additive functional equation*, Functional Ulam Notions (F.U.N) Nova Science Publishers, 2010, Chapter 13, 149 - 175.

[49] **J.M. Rassias, R. Saadati, G. Sadeghi, J. Vahidi**, *On nonlinear stability in various random normed spaces* Journal of Inequalities and Applications 2011 doi:10.1186/1029-242X-2011-62.

[50] **John M. Rassias, Matina J. Rassias, M.Arunkumar, T. Namachivayam**, *Ulam - Hyers stability of a 2-variable AC - mixed Type functional equation in Felbin's type spaces: Fixed point methods*, International Mathematical Forum, Vol. 8, No. 27, (2013), 1307 - 1322.

[51] **J. M. Rassias, M. Arunkumar, E. Sathya, N. Mahesh Kumar** *Solution And Stability Of A ACQ Functional Equation In Generalized 2-Normed Spaces*, Intern. J. Fuzzy Math. Arch., **7**, No. 2, (2015), 213–224.

[52] **J. M. Rassias, M. Arunkumar and T. Namachivayam**, *Stability Of The Leibniz Additive-Quadratic Functional Equation In Felbin's And Random Normed Spaces: A Direct Method*, Jamal Academic Research Journal an Interdisciplinary, (2015), 102–110.

[53] **Th. M. Rassias**, *On the stability of the linear mapping in Banach spaces*, Proc.Amer.Math. Soc., **72** (1978), 297–300.

[54] **Th. M. Rassias**, *On a modified Hyers-Ulam sequence*, J. Math. Anal. Appl. **158** no. 1, (1991), 106–113.

[55] **T. M. Rassias and P. Semrl**, *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. **114** no. 4, (1992), 989–993.

[56] **Th. M. Rassias**, *The problem of S. M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl.,**246** (2000), 352–378.

[57] **Th.M. Rassias**, *Functional Equations, Inequalities and Applications*, Kluwer Acedamic Publishers, Dordrecht, Bostan London, 2003.

[58] **K. Ravi, M. Arunkumar and J.M. Rassias**, *On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, International Journal of Mathematical Sciences, **3**, No. 08 (2008), 36–47.

[59] **K. Ravi, J.M. Rassias, M. Arunkumar and R. Kodandan**, *Stability of a generalized mixed type additive, quadratic, cubic and quartic functional equation*, J. Inequal. Pure Appl. Math. **10** (2009), no. 4, Art. 114, 29 pages.

[60] **F. Skof**, *Proprieta locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano, **53** (1983), 113–129.

[61] **S. M. Ulam**, *Problems in Modern Mathematics*, Science Editions, Wiley, NewYork, 1964.

[62] **T.Z. Xu, J.M. Rassias, W.X Xu**, *Generalized Ulam-Hyers stability of a general mixed AQCQ-functional equation in multi-Banach spaces: a fixed point approach*, Eur. J. Pure Appl. Math. **3** (2010), 1032-1047.

[63] **T.Z. Xu, J. M. Rassias, W.X. Xu**, *A fixed point approach to the stability of a general mixed AQCQ-functional equation in non-Archimedean normed spaces*, Discrete Dyn. Nat. Soc. 2010, Art. ID 812545, 24 pages.

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