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Stability of 2-Variable Additive-Quadratic-Cubic-Quartic Functional Equation Using Fixed Point Method

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Abstract

In this paper, the authors proved the generalized Ulam-Hyers stability of 2-variable Additive-Quadratic-Cubic-Quartic functional equation

$$\begin{aligned} f(x+2y,u+2v)+f(x-2y,u-2v) =& 4f(x+y,u+v)-4f(x-y,u-v)-6f(x,u)+f(2y,2v)\\ &+f(-2y,-2v)-4f(y,v)-4f(-y,-v) \end{aligned}$$

using fixed point method.

Keywords: Additive-quadratic-cubic-quartic functional equations, generalized Ulam-Hyers stability, Banach space, fixed point.

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1 Introduction and Preliminaries

Under what condition is there a homomorphism near an approximately homomorphism between a group and a metric group? This is called the stability problem of functional equations which was first raised by S. M. Ulam [49] in 1940. In next year, D. H. Hyers [24] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by T. Aoki [2] and Th. M. Rassias [44], respectively. The terminology Hyers- Ulam-Rassias stability originates from this historical background. Since then, a great deal of work has been done by a number of authors (for instance, [11, 13, 25, 40–42, 45, 47]).

The stability of mixed type functional equations have been extensively investigated by a number of mathematicians in referenes (see [3–5, 14–22, 28–31, 33–38, 43, 48, 50–52]). In 2003, V. Radu [39] introduced a new method, successively developed in ([7–10]), to obtaining the existence of the exact solutions and the error estimations, based on the fixed point alternative.

Now we will recall the fundamental results in fixed point theory.

Theorem 1.1. *(Banach's contraction principle) Let (X,d) be a complete metric space and consider a mapping $T: X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx,Ty) \le Ld(x,y)$ for some (Lipschitz constant) $L<1$. Then,
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is

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(A2) $lim_{n\to\infty}T^n x = x^*$, *for any starting point* $x \in X$;
(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ x \in X$;

(A4) $d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X$.

Theorem 1.2. *[12](The alternative of fixed point) Suppose that for a complete generalized metric space* (X, d) *and a strictly contractive mapping* $T : X \to X$ *with Lipschitz constant L. Then, for each given element* $x \in X$, *either*
$(B1)$ $d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \geq 0,$
or
$(B2)$ *there exists a natural number* n_0 *such that:*
(i) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(ii)*The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T
(iii) y^* *is the unique fixed point of* T *in the set* $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;
(iv) $d(y^*, y) \leq \frac{1}{1-L}\ d(y, Ty)$ *for all* $y \in Y$.

Recently, M. Arunkumar et al. introduced and investigated the general solution and generalized Ulam-Hyers stability of 2-Variable AQCQ Functional equation

$$f(x+2y, u+2v) + f(x-2y, u-2v) = 4f(x+y, u+v) - 4f(x-y, u-v) - 6f(x, u) + f(2y, 2v) + f(-2y, -2v) - 4f(y, v) - 4f(-y, -v) \tag{1.1}$$

using direct method.

In this paper, the authors proved the generalized Ulam-Hyers stability of 2-variable Additive-Quadratic-Cubic-Quartic functional equation (1.1) using fixed point method.

Through out this paper, let X be a normed space and Y be a Banach space respectively. Define a mapping $Df : X \to Y$ by

$$\begin{aligned} Df(x, y, u, v) = {} & f(x+2y, u+2v) + f(x-2y, u-2v) - 4f(x+y, u+v) + 4f(x-y, u-v) + 6f(x, u) \\ & - f(2y, 2v) - f(-2y, -2v) + 4f(y, v) + 4f(-y, -v) \end{aligned}$$

for all $x, y, u, v \in X$.

2 Stability Results: Odd Case

In this section, the authors presented the generalized Ulam-Hyers stability of the functional equation (1.1) for odd case using fixed point method.

2.1 Additive Stability Results

Theorem 2.1. *Let* $Df : X \to Y$ *be a mapping for which there exist a function* $\alpha : X^4 \to [0, \infty)$ *with the condition*

$$\lim_{k\to\infty} \frac{1}{\mu_i^k}\alpha\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right) = 0 \tag{2.1}$$

where $\mu_i = 2$ *if* $i = 0$ *and* $\mu_i = \frac{1}{2}$ *if* $i = 1$, *such that the functional inequality with*

$$\|D\ f(x, y, u, v)\| \leq \alpha(x, y, u, v) \tag{2.2}$$

for all $x, y, u, v \in X$. *If there exists* $L = L(i) < 1$ *such that the function*

$$u \to \Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right), \tag{2.3}$$

has the property

$$\Phi(u) = L\frac{1}{\mu_i}\Phi\left(\mu_i u\right) \tag{2.4}$$

where $\beta(u,u,u,u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)$ *for all* $x \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *satisfying the functional equation (1.1) and*

$$\| f(2u,2u) - 8f(u,u) - A(u,u) \| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{2.5}$$

for all $u \in X$.

Proof. Consider the set

$$\Omega = \{p/p : X \to Y,\ p(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(p,q) = \inf\{K \in (0,\infty) :\| p(u) - q(u) \| \leq K\Phi(u), u \in X\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega \to \Omega$ by

$$Tp(u) = \frac{1}{\mu_i} p(\mu_i u),$$

for all $u \in X$. Now $p, q \in \Omega$, we have

$$\begin{aligned} d(p,q) \leq K &\Rightarrow \| p(u) - q(u) \| \leq K\Phi(u), \forall u \in X. \\ &\Rightarrow \left\| \frac{1}{\mu_i} p(\mu_i u) - \frac{1}{\mu_i} q(\mu_i u) \right\| \leq \frac{1}{\mu_i} K\Phi(\mu_i u), \forall u \in X, \\ &\Rightarrow \left\| \frac{1}{\mu_i} p(\mu_i u) - \frac{1}{\mu_i} q(\mu_i u) \right\| \leq LK\Phi(u), \forall u \in X, \\ &\Rightarrow \| Tp(u) - Tq(u) \| \leq LK\Phi(u), \forall u \in X, \\ &\Rightarrow d(p,q) \leq LK. \end{aligned}$$

This implies $d(Tp, Tq) \leq Ld(p,q)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.
Replacing (x,y,u,v) by (u,u,u,u) in (2.2), we get

$$\|f(3u,3u) - 4f(2u,2u) + 5f(u,u)\| \leq \|\alpha(u,u,u,u)\| \tag{2.6}$$

for all $u \in X$. Replacing (x,y,u,v) by $(2u,u,2u,u)$ in (2.2), we obtain

$$\|f(4u,4u) - 4f(3u,3u) + 6f(2u,2u) - 4f(u,u)\| \leq \|\alpha(2u,u,2u,u)\| \tag{2.7}$$

for all $u \in X$. Now, from (2.6) and (2.7), we have

$$\begin{aligned} &\|f(4u,4u) - 10f(2u,2u) + 16f(u,u)\| \\ &\quad \leq 4\,\|f(3u,3u) - 4f(2u,2u) + 5f(u,u)\| + \|f(4u,4u) - 4f(3u,3u) + 6f(2u,2u) - 4f(u,u)\| \\ &\quad \leq 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u) \end{aligned} \tag{2.8}$$

for all $u \in X$. From (2.8), we arrive

$$\|f(4u,4u) - 10f(2u,2u) + 16f(u,u)\| \leq \beta(u,u,u,u) \tag{2.9}$$

where $\beta(u,u,u,u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)$ for all $u \in X$. It is easy from (2.9) that

$$\|f(4u,4u) - 8f(2u,2u) - 2\left(f(2u,2u) - 8f(u,u)\right)\| \leq \beta(u,u,u,u) \tag{2.10}$$

for all $u \in X$. Let $a : X \to Y$ be a mapping defined by $a(u,u) = f(2u,2u) - 8f(u,u)$. From (2.10), we conclude that

$$\|a(2u,2u) - 2a(u,u)\| \leq \beta(u,u,u,u) \tag{2.11}$$

for all $u \in X$. From (2.11), we arrive

$$\left\| \frac{a(2u, 2u)}{2} - a(u, u) \right\| \leq \frac{1}{2}\beta(u, u, u, u) \tag{2.12}$$

for all $u \in X$. Using (2.3) and (2.4) for the case $i = 0$ it reduces to

$$\left\| \frac{a(2u, 2u)}{2} - a(u, u) \right\| \leq L\Phi(u)$$

for all $u \in X$,

$$\text{i.e.,} \quad d(f, Tf) \leq L \leq L^1 < \infty.$$

Again replacing $u = \frac{u}{2}$ in (2.11), we get

$$\left\| a(u, u) - 2a\left(\frac{u}{2}, \frac{u}{2}\right) \right\| \leq \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right) \tag{2.13}$$

for all $u \in X$. Using (2.3) and (2.4) for the case $i = 1$ it reduces to

$$\left\| a(u, u) - 2a\left(\frac{u}{2}, \frac{u}{2}\right) \right\| \leq \Phi(u)$$

for all $u \in X$,

$$\text{i.e.,} \quad d(f, Tf) \leq 1 \leq L^0 < \infty.$$

In both cases, we arrive

$$d(f, Tf) \leq L^{1-i}.$$

Therefore, $(A1)$ holds. By $(A2)$, it follows that there exists a fixed point A of T in Ω such that

$$A(u, u) = \lim_{k\to\infty} \frac{1}{\mu_i^k} \left(f(\mu_i^{k+1}u, \mu_i^{k+1}u) - 8f(\mu_i^k u, \mu_i^k u) \right) \tag{2.14}$$

for all $u \in X$.

To prove $A : X \to Y$ is additive. Replacing (x, y, u, v) by $\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right)$ in (2.2) and dividing by μ_i^k, it follows from (2.1) that

$$\begin{aligned} \|A(x, y, u, v)\| &= \lim_{k\to\infty} \frac{\left\| D\, f(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v) \right\|}{\mu_i^k} \\ &\leq \lim_{k\to\infty} \frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v)}{\mu_i^k} = 0 \end{aligned}$$

for all $x, y, u, v \in X$. i.e., A satisfies the functional equation (1.1).

By $(A3)$, A is the unique fixed point of T in the set $\Delta = \{A \in \Omega : d(f, A) < \infty\}$, A is the unique function such that

$$\|f(2u, 2u) - 8f(u, u) - A(u, u)\| \leq K\Phi(u)$$

for all $u \in X$ and $K > 0$. Finally by $(A4)$, we obtain

$$d(f, A) \leq \frac{1}{1 - L} d(f, Tf)$$

this implies

$$d(f, A) \leq \frac{L^{1-i}}{1 - L}$$

which yields

$$\| f(2u, 2u) - 8f(u, u) - A(u, u) \| \leq \frac{L^{1-i}}{1 - L}\Phi(u)$$

for all $u \in X$. This completes the proof. □

The following corollary is an immediate consequence of Theorem 2.1 concerning the stability of (1.1).

Corollary 2.1. *Let $Df : X \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x,y,u,v)\| \leq \begin{cases} \lambda, & \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right), & s \neq 1; \\ \lambda\left(||x||^s ||y||^s ||u||^s ||v||^s\right), & s \neq \frac{1}{4}; \\ \lambda\left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}, & s \neq \frac{1}{4}; \end{cases} \tag{2.15}$$

for all $x, y, u, v \in X$, then there exists a unique additive function $A : X \to Y$ such that

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \begin{cases} 5\lambda, \\ \dfrac{\left(18 + 2^{s+1}\right)\lambda||u||^s}{|2 - 2^s|}, \\ \dfrac{(4 + 4^s)\lambda||u||^{4s}}{|2 - 2^{4s}|}, \\ \dfrac{\left(22 + 4^s + 2^{4s+1}\right)\lambda||u||^{4s}}{|2 - 2^{4s}|}. \end{cases} \tag{2.16}$$

for all $u \in X$.

Proof. Setting

$$\alpha(x,y,u,v) = \begin{cases} \lambda; \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right); \\ \lambda\left(||x||^s ||y||^s ||u||^s ||v||^s\right); \\ \lambda\left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}; \end{cases}$$

for all $x, y, u, v \in X$. Now,

$$\frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k u)}{\mu_i^k} = \begin{cases} \lambda\mu_i^{-k}; \\ \lambda\mu_i^{k(s-1)}\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right); \\ \lambda\mu_i^{k(4s-1)}\left(||x||^s ||y||^s ||u||^s ||v||^s\right); \\ \lambda\mu_i^{k(4s-1)}\left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}; \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty. \end{cases}$$

Thus, (2.1) is holds.

But we have $\Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right)$ has the property $\Phi(u) = L \cdot \frac{1}{\mu_i}\Phi(\mu_i u)$ for all $u \in X$. Hence,

$$\Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right) = \begin{cases} 5\lambda, \\ \dfrac{\left(18 + 2^{s+1}\right)\lambda}{2^s}||u||^s; \\ \dfrac{(4 + 4^s)\lambda}{2^{4s}}||u||^{4s}; \\ \dfrac{\left(22 + 4^s + 2^{4s+1}\right)\lambda}{2^{4s}}||u||^{4s}. \end{cases}$$

Now,

$$\frac{1}{\mu_i}\Phi(\mu_i u) = \begin{cases} \dfrac{5\lambda}{\mu_i}; \\ \dfrac{\left(18 + 2^{s+1}\right)\lambda}{\mu_i 2^s}\left(||\mu_i u||^s\right); \\ \dfrac{(4 + 4^s)\lambda}{\mu_i 2^{4s}}\left(||\mu_i u||^{4s}\right); \\ \dfrac{\left(22 + 4^s + 2^{4s+1}\right)\lambda}{\mu_i 2^{4s}}\left(||\mu_i u||^{4s}\right); \end{cases} = \begin{cases} \mu_i^{-1}\Phi(u); \\ \mu_i^{s-1}\Phi(u); \\ \mu_i^{4s-1}\Phi(u); \\ \mu_i^{4s-1}\Phi(u). \end{cases}$$

From (2.5), we prove the following cases:
Case:1 $L = 2^{-1}$ if $i = 0$;

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \lambda \left(\frac{(2^{-1})^{1-0}}{1 - 2^{(-1)}} \right) = 5\lambda.$$

Case:2 $L = 2^{1}$ if $i = 1$,

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \lambda \left(\frac{(2^{1})^{1-1}}{1 - 2^{1}} \right) = -5\lambda.$$

Case:3 $L = 2^{s-1}$ for $s < 1$ if $i = 0$,

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \frac{(18+2^{s+1})\,\lambda}{2^{s}} \left(\frac{\left(2^{(s-1)}\right)^{1-0}}{1 - 2^{(s-1)}} \right) ||u||^{s} = \frac{(18+2^{s+1})\,\lambda}{2 - 2^{s}} ||u||^{s}.$$

Case:4 $L = 2^{1-s}$ for $s > 1$ if $i = 1$,

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \frac{(18+2^{s+1})\,\lambda}{2^{s}} \left(\frac{\left(2^{(1-s)}\right)^{1-1}}{1 - 2^{(1-s)}} \right) ||u||^{s} = \frac{(18+2^{s+1})\,\lambda}{2^{s} - 2} ||u||^{s}.$$

Case:5 $L = 2^{4s-1}$ for $s < \frac{1}{4}$ if $i = 0$,

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \frac{(4+4^{s})\,\lambda}{2^{4s}} \left(\frac{\left(2^{(4s-1)}\right)^{1-0}}{1 - 2^{(4s-1)}} \right) ||u||^{4s} = \frac{(4+4^{s})\,\lambda}{2 - 2^{4s}} ||u||^{4s}.$$

Case:6 $L = 2^{1-4s}$ for $s > \frac{1}{4}$ if $i = 1$,

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \frac{(4+4^{s})\,\lambda}{2^{4s}} \left(\frac{\left(2^{(1-4s)}\right)^{1-0}}{1 - 2^{(1-4s)}} \right) ||u||^{4s} = \frac{(4+4^{s})\,\lambda}{2^{4s} - 2} ||u||^{4s}.$$

This completes the proof. □

2.2 Cubic Stability Results

Theorem 2.2. *Let $Df : X \to Y$ be a mapping for which there exist a function $\alpha : X^4 \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\mu_i^{3k}} \alpha\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right) = 0 \tag{2.17}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$, such that the functional inequality with

$$\|D\, f(x,y,u,v)\| \leq \alpha(x,y,u,v) \tag{2.18}$$

for all $x,y,u,v \in X$. If there exists $L = L(i) < 1$ such that the function

$$u \to \Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right), \tag{2.19}$$

has the property

$$\Phi(u) = L \frac{1}{\mu_i^3} \Phi(\mu_i u) \tag{2.20}$$

where $\beta(u,u,u,u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)$ for all $x \in X$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equation (1.1) and

$$\| f(2u,2u) - 2f(u,u) - C(u,u) \| \leq \frac{L^{1-i}}{1-L} \Phi(u) \tag{2.21}$$

for all $u \in X$.

Proof. Consider the set

$$\Omega = \{p/p : X \to Y,\ p(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(p,q) = \inf\{K \in (0,\infty) :\| p(u) - q(u) \|\leq K\Phi(u), u \in X\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega \to \Omega$ by

$$Tp(u) = \frac{1}{\mu_i^3} p(\mu_i u),$$

for all $u \in X$. Now $p,q \in \Omega$, we have

$$\begin{aligned}
d(p,q) \leq K \Rightarrow & \| p(u) - q(u) \|\leq K\Phi(u), \forall u \in X. \\
& \Rightarrow \left\| \frac{1}{\mu_i^3} p(\mu_i u) - \frac{1}{\mu_i^3} q(\mu_i u) \right\| \leq \frac{1}{\mu_i^3} K\Phi(\mu_i u), \forall u \in X, \\
& \Rightarrow \left\| \frac{1}{\mu_i^3} p(\mu_i u) - \frac{1}{\mu_i^3} q(\mu_i u) \right\| \leq LK\Phi(u), \forall u \in X, \\
& \Rightarrow \| Tp(u) - Tq(u) \|\leq LK\Phi(u), \forall u \in X, \\
& \Rightarrow d(p,q) \leq LK.
\end{aligned}$$

This implies $d(Tp, Tq) \leq Ld(p,q)$, for all $p,q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

It is easy from (2.9) that

$$\|f(4u,4u) - 2f(2u,2u) - 8\left(f(2u,2u) - 2f(u,u)\right)\| \leq \beta(u,u,u,u) \tag{2.22}$$

for all $u \in X$. Let $c : X \to Y$ be a mapping defined by $c(u,u) = f(2u,2u) - 2f(u,u)$. From (2.22), we conclude that

$$\|c(2u,2u) - 8c(u,u)\| \leq \beta(u,u,u,u) \tag{2.23}$$

for all $u \in X$. From (2.23), we arrive

$$\left\| \frac{c(2u,2u)}{8} - c(u,u) \right\| \leq \frac{1}{8}\beta(u,u,u,u) \tag{2.24}$$

for all $u \in X$. Using (2.19) and (2.20) for the case $i = 0$ it reduces to

$$\left\| \frac{c(2u,2u)}{8} - c(u,u) \right\| \leq\ L\Phi(u)$$

for all $u \in X$,

$$\text{i.e.,}\quad d(f, Tf) \leq L \leq L^1 < \infty.$$

Again replacing $u = \frac{u}{2}$ in (2.23), we get

$$\left\| c(u,u) - 8c\left(\frac{u}{2}, \frac{u}{2}\right) \right\| \leq \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right) \tag{2.25}$$

for all $u \in X$. Using (2.19) and (2.20) for the case $i = 1$ it reduces to

$$\left\| c(u,u) - 8c\left(\frac{u}{2}, \frac{u}{2}\right) \right\| \leq\ \Phi(u)$$

for all $u \in X$,

$$\text{i.e.,}\quad d(f, Tf) \leq 1 \leq L^0 < \infty.$$

In both cases, we arrive

$$d(f, Tf) \leq L^{1-i}.$$

Therefore, $(A1)$ holds. By $(A2)$, it follows that there exists a fixed point C of T in Ω such that

$$C(u,u) = \lim_{k\to\infty} \frac{1}{\mu_i^{3k}} \left(f(\mu_i^{k+1}u, \mu_i^{k+1}u) - 2f(\mu_i^k u, \mu_i^k u) \right) \tag{2.26}$$

for all $u \in X$.

To prove $C : X \to Y$ is cubic. Replacing (x,y,u,v) by $\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right)$ in (2.18) and dividing by μ_i^{3k}, it follows from (2.17) that

$$\begin{aligned} \|C(x,y,u,v)\| &= \lim_{k\to\infty} \frac{\left\|D\, f(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v)\right\|}{\mu_i^{3k}} \\ &\leq \lim_{k\to\infty} \frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v)}{\mu_i^{3k}} = 0 \end{aligned}$$

for all $x, y, u, v \in X$. i.e., C satisfies the functional equation (1.1).

By $(A3)$, C is the unique fixed point of T in the set $\Delta = \{C \in \Omega : d(f,C) < \infty\}$, C is the unique function such that

$$\|f(2u,2u) - 2f(u,u) - C(u,u)\| \leq K\Phi(u)$$

for all $u \in X$ and $K > 0$. Finally by $(A4)$, we obtain

$$d(f,C) \leq \frac{1}{1-L} d(f,Tf)$$

this implies

$$d(f,C) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\| \, f(2u,2u) - 2f(u,u) - C(u,u) \, \| \leq \frac{L^{1-i}}{1-L}\Phi(u)$$

for all $u \in X$. This completes the proof. □

The following corollary is an immediate consequence of Theorem 2.2 concerning the stability of (1.1).

Corollary 2.2. *Let $Df : X \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x,y,u,v)\| \leq \begin{cases} \lambda, & \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right), & s \neq 3; \\ \lambda\left(||x||^s ||y||^s ||u||^s ||v||^s\right), & s \neq \frac{3}{4}; \\ \lambda\left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}, & s \neq \frac{3}{4}; \end{cases} \tag{2.27}$$

for all $x, y, u, v \in X$, then there exists a unique cubic function $A : X \to Y$ such that

$$\|f(2u,2u) - 8f(u,u) - A(u,u)\| \leq \begin{cases} \dfrac{5\lambda}{7}, \\ \dfrac{\left(18 + 2^{s+1}\right)\lambda||u||^s}{|2^3 - 2^s|}, \\ \dfrac{\left(4 + 4^s\right)\lambda||u||^{4s}}{|2^3 - 2^{4s}|}, \\ \dfrac{\left(22 + 4^s + 2^{4s+1}\right)\lambda||u||^{4s}}{|2^3 - 2^{4s}|}. \end{cases} \tag{2.28}$$

for all $u \in X$.

Proof. Setting

$$\alpha(x,y,u,v) = \begin{cases} \lambda; \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right); \\ \lambda\left(||x||^s ||y||^s ||u||^s ||v||^s\right); \\ \lambda\left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}; \end{cases}$$

for all $x, y, u, v \in X$. Now,

$$\frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k u)}{\mu_i^{3k}} = \begin{cases} \lambda \mu_i^{-3k}; \\ \lambda \mu_i^{k(s-3)} \left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right); \\ \lambda \mu_i^{k(4s-3)} \left(||x||^s ||y||^s ||u||^s ||v||^s\right); \\ \lambda \mu_i^{k(4s-3)} \left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}; \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty. \end{cases}$$

Thus, (2.17) is holds.

But we have $\Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right)$ has the property $\Phi(u) = L \cdot \frac{1}{\mu_i^3} \Phi(\mu_i u)$ for all $u \in X$. Hence

$$\Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right) = \begin{cases} 5\lambda, \\ \frac{\left(18 + 2^{s+1}\right)\lambda}{2^s} ||u||^s; \\ \frac{\left(4 + 4^s\right)\lambda}{2^{4s}} ||u||^{4s}; \\ \frac{\left(22 + 4^s + 2^{4s+1}\right)\lambda}{2^{4s}} ||u||^{4s}. \end{cases}$$

Now,

$$\frac{1}{\mu_i^3}\Phi(\mu_i u) = \begin{cases} \frac{5\lambda}{\mu_i^3}; \\ \frac{\left(18 + 2^{s+1}\right)\lambda}{\mu_i^3 2^s} \left(||\mu_i u||^s\right); \\ \frac{\left(4 + 4^s\right)\lambda}{\mu_i^3 2^{4s}} \left(||\mu_i u||^{4s}\right); \\ \frac{\left(22 + 4^s + 2^{4s+1}\right)\lambda}{\mu_i^3 2^{4s}} \left(||\mu_i u||^{4s}\right); \end{cases} = \begin{cases} \mu_i^{-3}\Phi(u); \\ \mu_i^{s-3}\Phi(u); \\ \mu_i^{4s-3}\Phi(u); \\ \mu_i^{4s-3}\Phi(u). \end{cases}$$

From (2.21), we prove the following cases:

Case:1 $L = 2^{-3}$ if $i = 0$;

$$\|f(2u, 2u) - 2f(u, u) - C(u, u)\| \leq 5\lambda \left(\frac{\left(2^{-3}\right)^{1-0}}{1 - 2^{(-3)}}\right) = \frac{5\lambda}{7}.$$

Case:2 $L = 2^3$ if $i = 1$,

$$\|f(2u, 2u) - 2f(u, u) - C(u, u)\| \leq 5\lambda \left(\frac{\left(2^3\right)^{1-1}}{1 - 2^3}\right) = \frac{-5\lambda}{7}.$$

Case:3 $L = 2^{s-3}$ for $s < 3$ if $i = 0$,

$$\|f(2u, 2u) - 2f(u, u) - C(u, u)\| \leq \frac{\left(18 + 2^{s+1}\right)\lambda}{2^s} \left(\frac{\left(2^{(s-3)}\right)^{1-0}}{1 - 2^{(s-3)}}\right) ||u||^s = \frac{\left(18 + 2^{s+1}\right)\lambda}{2^3 - 2^s} ||u||^s.$$

Case:4 $L = 2^{3-s}$ for $s > 3$ if $i = 1$,

$$\|f(2u, 2u) - 2f(u, u) - C(u, u)\| \leq \frac{\left(18 + 2^{s+1}\right)\lambda}{2^s} \left(\frac{\left(2^{(3-s)}\right)^{1-1}}{1 - 2^{(3-s)}}\right) ||u||^s = \frac{\left(18 + 2^{s+1}\right)\lambda}{2^s - 2^3} ||u||^s.$$

Case:5 $L = 2^{4s-3}$ for $s < \frac{3}{4}$ if $i = 0$,

$$\|f(2u,2u) - 2f(u,u) - C(u,u)\| \leq \frac{(4+4^s)\lambda}{2^{4s}} \left(\frac{\left(2^{(4s-3)}\right)^{1-0}}{1-2^{(4s-3)}} \right) ||u||^{4s} = \frac{(4+4^s)\lambda}{2^3 - 2^{4s}} ||u||^{4s}.$$

Case:6 $L = 2^{3-4s}$ for $s > \frac{3}{4}$ if $i = 1$,

$$\|f(2u,2u) - 2f(u,u) - C(u,u)\| \leq \frac{(4+4^s)\lambda}{2^{4s}} \left(\frac{\left(2^{(3-4s)}\right)^{1-0}}{1-2^{(3-4s)}} \right) ||u||^{4s} = \frac{(4+4^s)\lambda}{2^{4s} - 2^3} ||u||^{4s}.$$

This finishes the proof. □

2.3 Additive-Cubic Mixed Stability Results

Theorem 2.3. *Let $Df : X \to Y$ be a mapping for which there exist a function $\alpha : X^4 \to [0, \infty)$ with the condition given in (2.1) and (2.17) respectively, such that the functional inequality*

$$\|D f(x,y,u,v)\| \leq \alpha(x,y,u,v) \tag{2.29}$$

for all $x, y, u, v \in X$. If there exists $L = L(i) < 1$ such that the function

$$u \to \Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right), \tag{2.30}$$

has the property (2.4) and (2.20), then there exists a unique 2-variable additive function $A : X \to Y$ and a unique 2-variable cubic function $C : X \to Y$ satisfying the functional equation (1.1) and

$$\| f(u,u) - A(u,u) - C(u,u) \| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{2.31}$$

for all $u \in X$.

Proof. By Theorems 2.1 and 2.2, there exists a unique 2-variable additive function $A_1 : X \to Y$ and a unique 2-variable cubic function $C_1 : X \to Y$ such that

$$\|f(2u,2u) - 8f(u,u) - A_1(u,u)\| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{2.32}$$

for all $u \in X$ and

$$\|f(2u,2u) - 2f(u,u) - C_1(u,u)\| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{2.33}$$

for all $u \in X$. Now, from (2.32) and (2.33) that

$$\begin{aligned}
\left\| f(u,u) + \frac{1}{6}A_1(u,u) - \frac{1}{6}C_1(u,u) \right\| &= \left\| \left\{ -\frac{f(2u,2u)}{6} + \frac{8}{6}f(u,u) + \frac{1}{6}A_1(u,u) \right\} \right. \\
&\qquad \left. + \left\{ \frac{f(2u,2u)}{6} - \frac{2}{6}f(u,u) - \frac{1}{6}C_1(u,u) \right\} \right\| \\
&\leq \frac{1}{6} \|\{-f(2u,2u) + 8f(u,u) + A_1(u,u)\} \\
&\qquad + \{f(2u,2u) - 2f(u,u) - C_1(u,u)\}\| \\
&\leq \frac{1}{6} \left\{ \frac{L^{1-i}}{1-L}\Phi(u) + \frac{L^{1-i}}{1-L}\Phi(u) \right\}
\end{aligned}$$

for all $u \in X$. Thus we obtain (2.31) by defining $A(u,u) = \frac{-1}{6}A_1(u,u)$ and $C(u,u) = \frac{1}{6}C_1(u,u)$, where $A(u,u)$ and $C(u,u)$ are defined in (2.14) and (2.26) respectively, for all $u \in X$. □

The following corollary is an immediate consequence of Theorem 2.3 concerning the stability of (1.1).

Corollary 2.3. *Let $Df : X \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x,y,u,v)\| \leq \begin{cases} \lambda, & \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right), & s \neq 1,3; \\ \lambda\left(||x||^s||y||^s||u||^s||v||^s\right), & s \neq \frac{1}{4}, \frac{3}{4}; \\ \lambda\left\{||x||^s||y||^s||u||^s||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}, & s \neq \frac{1}{4}, \frac{3}{4}; \end{cases} \tag{2.34}$$

for all $x,y,u,v \in X$, then there exists a unique 2-variable additive function $A : X \to Y$ and a unique 2-variable cubic function $C : X \to Y$ such that

$$\|f(u,u) - A(u,u) - C(u,u)\| \leq \begin{cases} \dfrac{20\lambda}{21}; \\ \dfrac{(18+2^{s+1})}{6}\left(\dfrac{1}{|2-2^s|} + \dfrac{1}{|2^3-2^s|}\right)\lambda||u||^s; \\ \dfrac{(4+4^s)}{6}\left(\dfrac{1}{|2-2^{4s}|} + \dfrac{1}{|2^3-2^{4s}|}\right)\lambda||u||^{4s}; \\ \dfrac{(22+4^s+2^{4s+1})}{6}\left(\dfrac{1}{|2-2^{4s}|} + \dfrac{1}{|2^3-2^{4s}|}\right)\lambda||u||^{4s}; \end{cases} \tag{2.35}$$

for all $u \in X$.

3 Stability Results: Even Case

In this section, the authors given the generalized Ulam-Hyers stability of the functional equation (1.1) for even case using fixed point method.

3.1 Quadratic Stability Results

Theorem 3.4. *Let $Df : X \to Y$ be a mapping for which there exist a function $\alpha : X^4 \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\mu_i^{2k}} \alpha\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right) = 0 \tag{3.1}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$, such that the functional inequality with

$$\|D\, f(x,y,u,v)\| \leq \alpha(x,y,u,v) \tag{3.2}$$

for all $x,y,u,v \in X$. If there exists $L = L(i) < 1$ such that the function

$$u \to \Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right), \tag{3.3}$$

has the property

$$\Phi(u) = L\frac{1}{\mu_i^2}\Phi\left(\mu_i u\right) \tag{3.4}$$

where $\beta(u,u,u,u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)$ for all $u \in X$. Then there exists a unique 2-variable quadratic mapping $Q_2 : X \to Y$ satisfying the functional equation (1.1) and

$$\|\ f(2u,2u) - 16f(u,u) - Q_2(u,u)\ \| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{3.5}$$

for all $u \in X$.

Proof. Consider the set

$$\Omega = \{p/p : X \to Y,\ p(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(p,q) = \inf\{K \in (0,\infty) : \|\ p(u) - q(u)\ \| \leq K\Phi(u), u \in X\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega \to \Omega$ by

$$Tp(u) = \frac{1}{\mu_i^2} p(\mu_i u),$$

for all $u \in X$. Now $p, q \in \Omega$, we have

$$\begin{aligned} d(p,q) \leq K \Rightarrow & \| \ p(u) - q(u) \ \| \leq K\Phi(u), \forall u \in X. \\ & \Rightarrow \left\| \frac{1}{\mu_i^2} p(\mu_i u) - \frac{1}{\mu_i^2} q(\mu_i u) \right\| \leq \frac{1}{\mu_i^2} K\Phi(\mu_i u), \forall u \in X, \\ & \Rightarrow \left\| \frac{1}{\mu_i^2} p(\mu_i u) - \frac{1}{\mu_i^2} q(\mu_i u) \right\| \leq LK\Phi(u), \forall u \in X, \\ & \Rightarrow \| \ Tp(u) - Tq(u) \ \| \leq LK\Phi(u), \forall u \in X, \\ & \Rightarrow d(p,q) \leq LK. \end{aligned}$$

This implies $d(Tp, Tq) \leq Ld(p,q)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

Replacing (x, y, u, v) by (u, u, u, u) in (3.2) and using evenness of f, we get

$$\|f(3u,3u) - 6f(2u,2u) + 15f(u,u)\| \leq \|\alpha(u,u,u,u)\| \tag{3.6}$$

for all $u \in X$. Replacing (x, y, u, v) by $(2u, u, 2u, u)$ in (3.2), we obtain

$$\|f(4u,4u) - 4f(3u,3u) + 4f(2u,2u) + 4f(u,u)\| \leq \|\alpha(2u,u,2u,u)\| \tag{3.7}$$

for all $u \in X$. Now, from (3.6) and (3.7), we have

$$\begin{aligned} & \|f(4u,4u) - 20f(2u,2u) + 64f(u,u)\| \\ & \quad \leq 4\,\|f(3u,3u) - 6f(2u,2u) + 15f(u,u)\| + \|f(4u,4u) - 4f(3u,3u) + 4f(2u,2u) + 4f(u,u)\| \\ & \quad \leq 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u) \end{aligned} \tag{3.8}$$

for all $u \in X$. From (3.8), we arrive

$$\|f(4u,4u) - 20f(2u,2u) + 64f(u,u)\| \leq \beta(u,u,u,u) \tag{3.9}$$

where $\beta(u,u,u,u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)$ for all $u \in X$. It is easy from (3.9) that

$$\|f(4u,4u) - 16f(2u,2u) - 4\left(f(2u,2u) - 16f(u,u)\right)\| \leq \beta(u,u,u,u) \tag{3.10}$$

for all $u \in X$. Let $q_2 : X \to Y$ be a mapping defined by $q_2(u,u) = f(2u,2u) - 16f(u,u)$. From (3.10), we conclude that

$$\|q_2(2u,2u) - 4q_2(u,u)\| \leq \beta(u,u,u,u) \tag{3.11}$$

for all $u \in X$. From (3.11), we arrive

$$\left\| \frac{q_2(2u,2u)}{4} - q_2(u,u) \right\| \leq \frac{1}{4}\beta(u,u,u,u) \tag{3.12}$$

for all $u \in X$. Using (3.3) and (3.4) for the case $i = 0$ it reduces to

$$\left\| \frac{q_2(2u,2u)}{4} - q_2(u,u) \right\| \leq \ L\Phi(u)$$

for all $u \in X$,

$$\text{i.e.,} \quad d(f, Tf) \leq L \leq L^1 < \infty.$$

Again replacing $u = \frac{u}{2}$ in (3.11), we get

$$\left\| q_2(u,u) - 4q_2\left(\frac{u}{2}, \frac{u}{2}\right) \right\| \leq \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right) \tag{3.13}$$

for all $u \in X$. Using (3.3) and (3.4) for the case $i = 1$ it reduces to

$$\left\| q_2(u,u) - 4q_2\left(\frac{u}{2}, \frac{u}{2}\right) \right\| \leq \Phi(u)$$

for all $u \in X$,

$$\text{i.e.,}\quad d(f, Tf) \leq 1 \leq L^0 < \infty.$$

In both cases, we arrive

$$d(f, Tf) \leq L^{1-i}.$$

Therefore, $(A1)$ holds. By $(A2)$, it follows that there exists a fixed point Q_2 of T in Ω such that

$$Q_2(u,u) = \lim_{k\to\infty} \frac{1}{\mu_i^{2k}} \left(f(\mu_i^{k+1}u, \mu_i^{k+1}u) - 16f(\mu_i^k u, \mu_i^k u) \right) \tag{3.14}$$

for all $u \in X$.

To prove $Q_2 : X \to Y$ is quadratic. Replacing (x, y, u, v) by $\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right)$ in (3.2) and dividing by μ_i^{2k}, it follows from (3.1) that

$$\begin{aligned}
\|Q_2(x,y,u,v)\| &= \lim_{k\to\infty} \frac{\left\| D\, f(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v) \right\|}{\mu_i^{2k}} \\
&\leq \lim_{k\to\infty} \frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v)}{\mu_i^{2k}} = 0
\end{aligned}$$

for all $x, y, u, v \in X$. i.e., Q_2 satisfies the functional equation (1.1).

By $(A3)$, Q_2 is the unique fixed point of T in the set $\Delta = \{Q_2 \in \Omega : d(f, Q_2) < \infty\}$, Q_2 is the unique function such that

$$\|f(2u, 2u) - 16f(u,u) - Q_2(u,u)\| \leq K\Phi(u)$$

for all $u \in X$ and $K > 0$. Finally by $(A4)$, we obtain

$$d(f, Q_2) \leq \frac{1}{1-L} d(f, Tf)$$

this implies

$$d(f, Q_2) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\| f(2u, 2u) - 16f(u,u) - Q_2(u,u) \| \leq \frac{L^{1-i}}{1-L}\Phi(u)$$

for all $u \in X$. This finishes the proof. □

The following corollary is an immediate consequence of Theorem 3.4 concerning the stability of (1.1).

Corollary 3.4. *Let $Df : X \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x,y,u,v)\| \leq \begin{cases} \lambda, & \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right), & s \neq 2; \\ \lambda\left(||x||^s ||y||^s ||u||^s ||v||^s\right), & s \neq \frac{1}{2}; \\ \lambda\left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}, & s \neq \frac{1}{2}; \end{cases} \tag{3.15}$$

for all $x, y, u, v \in X$, then there exists a unique 2-variable quadratic function $Q_2 : X \to Y$ such that

$$\|f(2u, 2u) - 16f(u,u) - Q_2(u,u)\| \leq \begin{cases} \dfrac{5\lambda}{3}, \\ \dfrac{\left(18 + 2^{s+1}\right)\lambda||u||^s}{|2^2 - 2^s|}, \\ \dfrac{\left(4 + 4^s\right)\lambda||u||^{4s}}{|2^2 - 2^{4s}|}, \\ \dfrac{\left(22 + 4^s + 2^{4s+1}\right)\lambda||u||^{4s}}{|2^2 - 2^{4s}|}. \end{cases} \tag{3.16}$$

for all $u \in X$.

Proof. Setting

$$\alpha(x,y,u,v)=\begin{cases}\lambda;\\ \lambda\left(||x||^s+||y||^s+||u||^s+||v||^s\right);\\ \lambda\left(||x||^s||y||^s||u||^s||v||^s\right);\\ \lambda\left\{||x||^s||y||^s||u||^s||v||^s+||x||^{4s}+||y||^{4s}+||u||^{4s}+||v||^{4s}\right\};\end{cases}$$

for all $x,y,u,v\in X$. Now,

$$\frac{\alpha(\mu_i^k x,\mu_i^k y,\mu_i^k u,\mu_i^k u)}{\mu_i^{2k}}=\begin{cases}\lambda\mu_i^{-2k};\\ \lambda\mu_i^{k(s-2)}\left(||x||^s+||y||^s+||u||^s+||v||^s\right);\\ \lambda\mu_i^{k(4s-2)}\left(||x||^s||y||^s||u||^s||v||^s\right);\\ \lambda\mu_i^{k(4s-2)}\left\{||x||^s||y||^s||u||^s||v||^s+||x||^{4s}+||y||^{4s}+||u||^{4s}+||v||^{4s}\right\};\end{cases}$$

$$=\begin{cases}\to 0 \text{ as } k\to\infty;\\ \to 0 \text{ as } k\to\infty;\\ \to 0 \text{ as } k\to\infty;\\ \to 0 \text{ as } k\to\infty.\end{cases}$$

Thus, (3.1) is holds.

But we have $\Phi(u)=\beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right)$ has the property $\Phi(u)=L\cdot\frac{1}{\mu_i^2}\Phi(\mu_i u)$ for all $u\in X$. Hence,

$$\Phi(u)=\beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right)=\begin{cases}5\lambda,\\ \dfrac{\left(18+2^{s+1}\right)\lambda}{2^s}||u||^s;\\ \dfrac{\left(4+4^s\right)\lambda}{2^{4s}}||u||^{4s};\\ \dfrac{\left(22+4^s+2^{4s+1}\right)\lambda}{2^{4s}}||u||^{4s}.\end{cases}$$

Now,

$$\frac{1}{\mu_i^2}\Phi(\mu_i u)=\begin{cases}\dfrac{5\lambda}{\mu_i};\\ \dfrac{\left(18+2^{s+1}\right)\lambda}{\mu_i^2 2^s}\left(||\mu_i u||^s\right);\\ \dfrac{\left(4+4^s\right)\lambda}{\mu_i^2 2^{4s}}\left(||\mu_i u||^{4s}\right);\\ \dfrac{\left(22+4^s+2^{4s+1}\right)\lambda}{\mu_i^2 2^{4s}}\left(||\mu_i u||^{4s}\right);\end{cases}=\begin{cases}\mu_i^{-2}\Phi(u);\\ \mu_i^{s-2}\Phi(u);\\ \mu_i^{4s-2}\Phi(u);\\ \mu_i^{4s-2}\Phi(u).\end{cases}$$

From (3.5), we prove the following cases:

Case:1 $L=2^{-2}$ if $i=0$;

$$\|f(2u,2u)-16f(u,u)-Q_2(u,u)\|\le\lambda\left(\frac{\left(2^{-2}\right)^{1-0}}{1-2^{(-2)}}\right)=\frac{5\lambda}{3}.$$

Case:2 $L=2^1$ if $i=1$,

$$\|f(2u,2u)-16f(u,u)-Q_2(u,u)\|\le\lambda\left(\frac{\left(2^{2}\right)^{1-1}}{1-2^{2}}\right)=\frac{-5\lambda}{3}.$$

Case:3 $L=2^{s-2}$ for $s<2$ if $i=0$,

$$\|f(2u,2u)-16f(u,u)-Q_2(u,u)\|\le\frac{\left(18+2^{s+1}\right)\lambda}{2^s}\left(\frac{\left(2^{(s-2)}\right)^{1-0}}{1-2^{(s-2)}}\right)||u||^s=\frac{\left(18+2^{s+1}\right)\lambda}{2^2-2^s}||u||^s.$$

Case:4 $L = 2^{2-s}$ for $s > 2$ if $i = 1$,

$$\|f(2u,2u) - 16f(u,u) - Q_2(u,u)\| \leq \frac{(18+2^{s+1})\,\lambda}{2^s}\left(\frac{\left(2^{(2-s)}\right)^{1-1}}{1-2^{(2-s)}}\right)||u||^s = \frac{(18+2^{s+1})\,\lambda}{2^s-2^2}||u||^s.$$

Case:5 $L = 2^{4s-2}$ for $s < \frac{1}{2}$ if $i = 0$,

$$\|f(2u,2u) - 16f(u,u) - Q_2(u,u)\| \leq \frac{(4+4^s)\,\lambda}{2^{4s}}\left(\frac{\left(2^{(4s-2)}\right)^{1-0}}{1-2^{(4s-2)}}\right)||u||^{4s} = \frac{(4+4^s)\,\lambda}{2^2-2^{4s}}||u||^{4s}.$$

Case:6 $L = 2^{2-4s}$ for $s > \frac{2}{4}$ if $i = 1$,

$$\|f(2u,2u) - 16f(u,u) - Q_2(u,u)\| \leq \frac{(4+4^s)\,\lambda}{2^{4s}}\left(\frac{\left(2^{(2-4s)}\right)^{1-0}}{2-2^{(1-4s)}}\right)||u||^{4s} = \frac{(4+4^s)\,\lambda}{2^{4s}-2^2}||u||^{4s}.$$

This completes the proof. □

3.2 Quartic Stability Results

Theorem 3.5. *Let $Df : X \to Y$ be a mapping for which there exist a function $\alpha : X^4 \to [0,\infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\mu_i^{4k}}\alpha\left(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k v\right) = 0 \tag{3.17}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$, such that the functional inequality with

$$\|D\, f(x,y,u,v)\| \leq \alpha(x,y,u,v) \tag{3.18}$$

for all $x,y,u,v \in X$. If there exists $L = L(i) < 1$ such that the function

$$u \to \Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right), \tag{3.19}$$

has the property

$$\Phi(u) = L\frac{1}{\mu_i^4}\Phi(\mu_i u) \tag{3.20}$$

where $\beta(u,u,u,u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)$ for all $x \in X$. Then there exists a unique 2-variable quartic mapping $Q_4 : X \to Y$ satisfying the functional equation (1.1) and

$$\| f(2u,2u) - 4f(u,u) - Q_4(u,u) \| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{3.21}$$

for all $u \in X$.

Proof. Consider the set

$$\Omega = \{p/p : X \to Y,\ p(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(p,q) = \inf\{K \in (0,\infty) : \| p(u) - q(u) \| \leq K\Phi(u), u \in X\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega \to \Omega$ by

$$Tp(u) = \frac{1}{\mu_i^4}p(\mu_i u),$$

for all $u \in X$. Now $p,q \in \Omega$, we have

$$\begin{aligned} d(p,q) \leq K &\Rightarrow \| p(u) - q(u) \| \leq K\Phi(u), \forall u \in X. \\ &\Rightarrow \left\| \frac{1}{\mu_i^4} p(\mu_i u) - \frac{1}{\mu_i^4} q(\mu_i u) \right\| \leq \frac{1}{\mu_i^4} K\Phi(\mu_i u), \forall u \in X, \\ &\Rightarrow \left\| \frac{1}{\mu_i^4} p(\mu_i u) - \frac{1}{\mu_i^4} q(\mu_i u) \right\| \leq LK\Phi(u), \forall u \in X, \\ &\Rightarrow \| Tp(u) - Tq(u) \| \leq LK\Phi(u), \forall u \in X, \\ &\Rightarrow d(p,q) \leq LK. \end{aligned}$$

This implies $d(Tp,Tq) \leq Ld(p,q)$, for all $p,q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.
It is easy from (3.9) that

$$\| f(4u,4u) - 4f(2u,2u) - 16\left(f(2u,2u) - 4f(u,u)\right)\| \leq \beta(u,u,u,u) \tag{3.22}$$

for all $u \in X$. Let $q_4 : X \to Y$ be a mapping defined by $q_4(u,u) = f(2u,2u) - 4f(u,u)$. From (3.22), we conclude that

$$\| q_4(2u,2u) - 16q_4(u,u)\| \leq \beta(u,u,u,u) \tag{3.23}$$

for all $u \in X$. From (3.23), we arrive

$$\left\| \frac{q_4(2u,2u)}{16} - q_4(u,u) \right\| \leq \frac{1}{16}\beta(u,u,u,u) \tag{3.24}$$

for all $u \in X$. Using (3.19) and (3.20) for the case $i = 0$ it reduces to

$$\left\| \frac{q_4(2u,2u)}{16} - q_4(u,u) \right\| \leq \ L\Phi(u)$$

for all $u \in X$,

$$\text{i.e.,} \quad d(f,Tf) \leq L \leq L^1 < \infty.$$

Again replacing $u = \frac{u}{2}$ in (3.23), we get

$$\left\| q_4(u,u) - 16q_4\left(\frac{u}{2},\frac{u}{2}\right) \right\| \leq \beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right) \tag{3.25}$$

for all $u \in X$. Using (3.19) and (3.20) for the case $i = 1$ it reduces to

$$\left\| q_4(u,u) - 8q_4\left(\frac{u}{2},\frac{u}{2}\right) \right\| \leq \ \Phi(u)$$

for all $u \in X$,

$$\text{i.e.,} \quad d(f,Tf) \leq 1 \leq L^0 < \infty.$$

In both cases, we arrive

$$d(f,Tf) \leq L^{1-i}.$$

Therefore, $(A1)$ holds. By $(A2)$, it follows that there exists a fixed point Q_4 of T in Ω such that

$$Q_4(u,u) = \lim_{k\to\infty} \frac{1}{\mu_i^{4k}} \left(f(\mu_i^{k+1}u,\mu_i^{k+1}u) - 4f(\mu_i^k u,\mu_i^k u)\right) \tag{3.26}$$

for all $u \in X$.

To prove $Q_4 : X \to Y$ is quartic. Replacing (x,y,u,v) by $\left(\mu_i^k x,\mu_i^k y,\mu_i^k u,\mu_i^k v\right)$ in (3.18) and dividing by μ_i^{4k}, it follows from (3.17) that

$$\begin{aligned} \|Q_4(x,y,u,v)\| &= \lim_{k\to\infty} \frac{\left\| D\, f(\mu_i^k x,\mu_i^k y,\mu_i^k u,\mu_i^k v)\right\|}{\mu_i^{4k}} \\ &\leq \lim_{k\to\infty} \frac{\alpha(\mu_i^k x,\mu_i^k y,\mu_i^k u,\mu_i^k v)}{\mu_i^{4k}} = 0 \end{aligned}$$

for all $x, y, u, v \in X$. i.e., Q_4 satisfies the functional equation (1.1).

By $(A3)$, Q_4 is the unique fixed point of T in the set $\Delta = \{Q_4 \in \Omega : d(f, Q_4) < \infty\}, Q_4$ is the unique function such that

$$\|f(2u, 2u) - 4f(u, u) - Q_4(u, u)\| \leq K\Phi(u)$$

for all $u \in X$ and $K > 0$. Finally by $(A4)$, we obtain

$$d(f, Q_4) \leq \frac{1}{1-L} d(f, Tf)$$

this implies

$$d(f, Q_4) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\| f(2u, 2u) - 4f(u, u) - Q_4(u, u) \| \leq \frac{L^{1-i}}{1-L} \Phi(u)$$

for all $u \in X$. This completes the proof. □

The following corollary is an immediate consequence of Theorem 3.5 concerning the stability of (1.1).

Corollary 3.5. *Let $Df : X \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x, y, u, v)\| \leq \begin{cases} \lambda, & \\ \lambda \left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right), & s \neq 4; \\ \lambda \left(||x||^s ||y||^s ||u||^s ||v||^s\right), & s \neq 1; \\ \lambda \left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}, & s \neq 1; \end{cases} \tag{3.27}$$

for all $x, y, u, v \in X$, then there exists a unique 2-variable quartic function $A : X \to Y$ such that

$$\|f(2u, 2u) - 4f(u, u) - Q_4(u, u)\| \leq \begin{cases} \dfrac{5\lambda}{15}, \\ \dfrac{\left(18 + 2^{s+1}\right) \lambda ||u||^s}{|2^4 - 2^s|}, \\ \dfrac{\left(4 + 4^s\right) \lambda ||u||^{4s}}{|2^4 - 2^{4s}|}, \\ \dfrac{\left(22 + 4^s + 2^{4s+1}\right) \lambda ||u||^{4s}}{|2^4 - 2^{4s}|}. \end{cases} \tag{3.28}$$

for all $u \in X$.

Proof. Setting

$$\alpha(x, y, u, v) = \begin{cases} \lambda; \\ \lambda \left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right); \\ \lambda \left(||x||^s ||y||^s ||u||^s ||v||^s\right); \\ \lambda \left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}; \end{cases}$$

for all $x, y, u, v \in X$. Now,

$$\begin{aligned} \frac{\alpha(\mu_i^k x, \mu_i^k y, \mu_i^k u, \mu_i^k u)}{\mu_i^{4k}} &= \begin{cases} \lambda \mu_i^{-4k}; \\ \lambda \mu_i^{k(s-4)} \left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right); \\ \lambda \mu_i^{k(4s-4)} \left(||x||^s ||y||^s ||u||^s ||v||^s\right); \\ \lambda \mu_i^{k(4s-4)} \left\{||x||^s ||y||^s ||u||^s ||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}; \end{cases} \\ &= \begin{cases} \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty; \\ \to 0 \text{ as } k \to \infty. \end{cases} \end{aligned}$$

Thus, (3.17) is holds.

But we have $\Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right)$ has the property $\Phi(u) = L \cdot \frac{1}{\mu_i^4}\Phi(\mu_i u)$ for all $u \in X$. Hence

$$\Phi(u) = \beta\left(\frac{u}{2}, \frac{u}{2}, \frac{u}{2}, \frac{u}{2}\right) = \begin{cases} 5\lambda, \\ \frac{(18+2^{s+1})\lambda}{2^s}||u||^s; \\ \frac{(4+4^s)\lambda}{2^{4s}}||u||^{4s}; \\ \frac{(22+4^s+2^{4s+1})\lambda}{2^{4s}}||u||^{4s}. \end{cases}$$

Now,

$$\frac{1}{\mu_i^4}\Phi(\mu_i u) = \begin{cases} \frac{5\lambda}{\mu_i^4}; \\ \frac{(18+2^{s+1})\lambda}{\mu_i^4 2^s}(||\mu_i u||^s); \\ \frac{(4+4^s)\lambda}{\mu_i^4 2^{4s}}\left(||\mu_i u||^{4s}\right); \\ \frac{(22+4^s+2^{4s+1})\lambda}{\mu_i^4 2^{4s}}\left(||\mu_i u||^{4s}\right); \end{cases} = \begin{cases} \mu_i^{-4}\Phi(u); \\ \mu_i^{s-4}\Phi(u); \\ \mu_i^{4s-4}\Phi(u); \\ \mu_i^{4s-4}\Phi(u). \end{cases}$$

From (3.21), we prove the following cases:
Case:1 $L = 2^{-4}$ if $i = 0$;

$$\|f(2u,2u) - 4f(u,u) - Q_4(u,u)\| \le 5\lambda\left(\frac{(2^{-4})^{1-0}}{1-2^{(-4)}}\right) = \frac{5\lambda}{16}.$$

Case:2 $L = 2^4$ if $i = 1$,

$$\|f(2u,2u) - 4f(u,u) - Q_4(u,u)\| \le 5\lambda\left(\frac{(2^4)^{1-1}}{1-2^4}\right) = \frac{-5\lambda}{16}.$$

Case:3 $L = 2^{s-4}$ for $s < 4$ if $i = 0$,

$$\|f(2u,2u) - 4f(u,u) - Q_4(u,u)\| \le \frac{(18+2^{s+1})\lambda}{2^s}\left(\frac{\left(2^{(s-4)}\right)^{1-0}}{1-2^{(s-4)}}\right)||u||^s = \frac{(18+2^{s+1})\lambda}{2^4-2^s}||u||^s.$$

Case:4 $L = 2^{4-s}$ for $s > 3$ if $i = 1$,

$$\|f(2u,2u) - 4f(u,u) - Q_4(u,u)\| \le \frac{(18+2^{s+1})\lambda}{2^s}\left(\frac{\left(2^{(4-s)}\right)^{1-1}}{1-2^{(4-s)}}\right)||u||^s = \frac{(18+2^{s+1})\lambda}{2^s-2^4}||u||^s.$$

Case:5 $L = 2^{4s-4}$ for $s < 1$ if $i = 0$,

$$\|f(2u,2u) - 4f(u,u) - Q_4(u,u)\| \le \frac{(4+4^s)\lambda}{2^{4s}}\left(\frac{\left(2^{(4s-4)}\right)^{1-0}}{1-2^{(4s-4)}}\right)||u||^{4s} = \frac{(4+4^s)\lambda}{2^4-2^{4s}}||u||^{4s}.$$

Case:6 $L = 2^{4-4s}$ for $s > 1$ if $i = 1$,

$$\|f(2u,2u) - 4f(u,u) - Q_4(u,u)\| \le \frac{(4+4^s)\lambda}{2^{4s}}\left(\frac{\left(2^{(4-4s)}\right)^{1-0}}{1-2^{(4-4s)}}\right)||u||^{4s} = \frac{(4+4^s)\lambda}{2^{4s}-2^4}||u||^{4s}.$$

This finishes the proof. □

3.3 Quadratic-Quartic Mixed Stability Results

Theorem 3.6. *Let $Df : X \to Y$ be a mapping for which there exist a function $\alpha : X^4 \to [0,\infty)$ with the condition given in (3.1) and (3.17) respectively, such that the functional inequality*

$$\|D\,f(x,y,u,v)\| \leq \alpha(x,y,u,v) \tag{3.29}$$

for all $x,y,u,v \in X$. If there exists $L = L(i) < 1$ such that the function

$$u \to \Phi(u) = \beta\left(\frac{u}{2},\frac{u}{2},\frac{u}{2},\frac{u}{2}\right), \tag{3.30}$$

has the property (3.4) and (3.20), then there exists a unique 2-variable quadratic function $Q_2 : X \to Y$ and a unique 2-variable quartic function $Q_4 : X \to Y$ satisfying the functional equation (1.1) and

$$\| f(u,u) - Q_2(u,u) - Q_4(u,u) \| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{3.31}$$

for all $u \in X$.

Proof. By Theorems 3.4 and 3.5, there exists a unique 2-variable quadratic function $Q_{2_1} : X \to Y$ and a unique 2-variable quartic function $Q_{4_1} : X \to Y$ such that

$$\left\|f(2u,2u) - 16f(u,u) - Q_{2_1}(u,u)\right\| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{3.32}$$

for all $u \in X$ and

$$\left\|f(2u,2u) - 4f(u,u) - Q_{4_1}(u,u)\right\| \leq \frac{L^{1-i}}{1-L}\Phi(u) \tag{3.33}$$

for all $u \in X$. Now, from (3.32) and (3.33) that

$$\begin{aligned}
\left\|f(u,u) + \frac{1}{12}Q_{2_1}(u,u) - \frac{1}{12}Q_{4_1}(u,u)\right\| &= \left\|\left\{-\frac{f(2u,2u)}{12} + \frac{16}{12}f(u,u) + \frac{1}{12}Q_{2_1}(u,u)\right\}\right. \\
&\qquad \left. + \left\{\frac{f(2u,2u)}{12} - \frac{4}{12}f(u,u) - \frac{1}{12}Q_{4_1}(u,u)\right\}\right\| \\
&\leq \frac{1}{12}\left\|\left\{f(2u,2u) - 16f(u,u) - Q_{2_1}(u,u)\right\}\right. \\
&\qquad \left. + \left\{f(2u,2u) - 4f(u,u) - Q_{4_1}(u,u)\right\}\right\| \\
&\leq \frac{1}{12}\left\{\frac{L^{1-i}}{1-L}\Phi(u) + \frac{L^{1-i}}{1-L}\Phi(u)\right\}
\end{aligned}$$

for all $u \in X$. Thus we obtain (3.31) by defining $Q_2(u,u) = \frac{-1}{12}Q_{2_1}(u,u)$ and $Q_4(u,u) = \frac{1}{12}Q_{4_1}(u,u)$, where $Q_2(u,u)$ and $Q_4(u,u)$ are defined in (3.14) and (3.26) respectively, for all $u \in X$. □

The following corollary is an immediate consequence of Theorem 3.6 concerning the stability of (1.1).

Corollary 3.6. *Let $Df : X \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x,y,u,v)\| \leq \begin{cases} \lambda, & \\ \lambda\left(||x||^s + ||y||^s + ||u||^s + ||v||^s\right), & s \neq 2,4; \\ \lambda\left(||x||^s||y||^s||u||^s||v||^s\right), & s \neq \frac{1}{2},1; \\ \lambda\left\{||x||^s||y||^s||u||^s||v||^s + ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}, & s \neq \frac{1}{2},1; \end{cases} \tag{3.34}$$

for all $x,y,u,v \in X$, then there exists a unique 2-variable quadratic function $Q_2 : X \to Y$ and a unique 2-variable quartic function $Q_4 : X \to Y$ such that

$$\|f(u,u) - Q_2(u,u) - Q_4(u,u)\| \leq \begin{cases} \frac{\lambda}{18}; \\ \frac{(18+2^{s+1})}{6}\left(\frac{1}{|2^2-2^s|} + \frac{1}{|2^4-2^s|}\right)\lambda||u||^s; \\ \frac{(4+4^s)}{12}\left(\frac{1}{|2^2-2^{4s}|} + \frac{1}{|2^4-2^{4s}|}\right)\lambda||u||^{4s}; \\ \frac{(22+4^s+2^{4s+1})}{12}\left(\frac{1}{|2^2-2^{4s}|} + \frac{1}{|2^4-2^{4s}|}\right)\lambda||u||^{4s}; \end{cases} \tag{3.35}$$

for all $u \in X$.

4 Additive-Quadratic-Cubic-Quartic Mixed Stability Results

In this section, the authors proved the additive-quadratic-cubic-quartic mixed stability of the functional equation (1.1) using fixed point method.

Theorem 4.7. *Let $j = \pm 1$. Let $Df : X^2 \to Y$ be a mapping for which there exist a function $\alpha : X^4 \to [0, \infty)$ with the condition given in (2.1), (2.17), (3.1) and (3.17) respectively, such that the functional inequality*

$$\|Df(x, y, u, v)\| \leq \alpha(x, y, u, v) \tag{4.1}$$

for all $x, y, u, v \in X$. Then there exists a unique 2-variable additive mapping $A(u, u) : X^2 \to Y$, a unique 2-variable quadratic mapping $Q_2(u, u) : X^2 \to Y$, a unique 2-variable cubic mapping $C(u, u) : X^2 \to Y$ and a unique 2-variable quartic mapping $Q_4(u, u) : X^2 \to Y$satisfying the functional equation (1.1) and

$$\|g(u, u) - A(u, u) - Q_2(u, u) - C(u, u) - Q_4(u, u)\| \frac{L^{1-i}}{1 - L} \left(\Phi_{AC}(u) + \Phi_{Q_2Q_4}(u)\right) \tag{4.2}$$

for all $u \in X$, where $\Phi_{AC}(u)$ and $\Phi_{Q_2Q_4}(u)$ are defined by

$$\Phi_{AC}(u) = \frac{1}{6} [\Phi(u) + \Phi(-u)] \tag{4.3}$$

$$\Phi_{Q_2Q_4}(u) = \frac{1}{12} [\Phi(u) + \Phi(-u)] \tag{4.4}$$

respectively, for all $u \in X$.

Proof. Let $f_o(u, u) = \dfrac{1}{2} (f(u, u) - f(-u, -u))$ for all $u \in X$. Then $f_o(0, 0) = 0$ and $f_o(-u, -u) = -f_o(u, u)$ for all $u \in X$. Hence

$$\|Df_o(x, y, u, v)\| \leq \frac{1}{2} \{\alpha(x, y, u, v) + \alpha(-x, -y, -u, -v)\} \tag{4.5}$$

for all $x, y, u, v \in X$. By Theorem 2.3, there exists a unique 2-variable additive function $A(u, u) : X^2 \to Y$ and a unique 2-variable cubic function $C(u, u) : X^2 \to Y$ such that

$$\begin{aligned} \|f_o(u, u) - A(u, u) - C(u, u)\| &\leq \frac{1}{2} \left\{ \frac{1}{3} \frac{L^{1-i}}{1 - L} \Phi(u) + \frac{1}{3} \frac{L^{1-i}}{1 - L} \Phi(-u) \right\} \\ &\leq \frac{1}{6} \frac{L^{1-i}}{1 - L} \{\Phi(u) + \Phi(-u)\}, \end{aligned} \tag{4.6}$$

for all $u \in X$. Also, let $f_e(u, u) = \dfrac{1}{2} (f(u, u) + f(-u, -u))$ for all $u \in X$. Then $f_e(0, 0) = 0$ and $f_e(-u, -u) = f_e(u, u)$ for all $u \in X$. Hence

$$\|Df_e(x, y, u, v)\| \leq \frac{1}{2} \{\alpha(x, y, u, v) + \alpha(-x, -y, -u, -v)\} \tag{4.7}$$

for all $x, y, u, v \in X$. By Theorem 3.6, there exists a unique 2-variable quadratic mapping $Q_2(u, u) : X^2 \to Y$ and a unique 2-variable quartic mapping $Q_4(u, u) : X^2 \to Y$ such that

$$\begin{aligned} \|f_e(u, u) - Q_2(u, u) - Q_4(u, u)\| &\leq\leq \frac{1}{2} \left\{ \frac{1}{6} \frac{L^{1-i}}{1 - L} \Phi(u) + \frac{1}{6} \frac{L^{1-i}}{1 - L} \Phi(u-) \right\} \\ &\leq \frac{1}{12} \frac{L^{1-i}}{1 - L} \{\Phi(u) + \Phi(-u)\}, \end{aligned} \tag{4.8}$$

for all $u \in X$. Define

$$f(u, u) = f_o(u, u) + f_e(u, u) \tag{4.9}$$

for all $u \in X$. Now from (4.6), (4.8) and (4.9)

$$\begin{aligned}
&\|f(u,u) - A(u,u) - Q_2(u,u) - C(u,u) - Q_4(u,u)\| \\
&\quad = \|f_o(u,u) + f_e(u,u) - A(u,u) - Q_2(u,u) - C(u,u) - Q_4(u,u)\| \\
&\quad \leq \|f_o(u,u) - A(u,u) - C(u,u)\| + \|f_e(u,u) - Q_2(u,u) - Q_4(u,u)\| \\
&\quad \leq \frac{1}{6}\frac{L^{1-i}}{1-L}\{\Phi(u) + \Phi(-u)\} + \frac{1}{12}\frac{L^{1-i}}{1-L}\{\Phi(u) + \Phi(-u)\} \\
&\quad \leq \frac{L^{1-i}}{1-L}\left\{\Phi_{AC}(u) + \Phi_{Q_2Q_4}(u)\right\}
\end{aligned} \tag{4.10}$$

for all $u \in X$. This finishes the proof. □

The following corollary is an immediate consequence of Theorem 4.7, using Corollaries 2.3 and 3.6 concerning stability of (1.1).

Corollary 4.7. *Let $Df : X^2 \to Y$ be a mapping and there exits real numbers λ and s such that*

$$\|Df(x,y,u,v)\| \leq \begin{cases} \lambda, & \\ \lambda\left\{||x||^s + ||y||^s + ||u||^s + ||v||^s\right\}, & s \neq 1,2,3,4; \\ \lambda ||x||^s ||y||^s ||u||^s ||v||^s, & s \neq \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1; \\ \lambda\left\{||x||^s ||y||^s ||u||^s ||v||^s + \left\{||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}\right\}, & s \neq \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1; \end{cases} \tag{4.11}$$

for all $x,y,u,v \in X$, then there exists a unique 2-variable additive mapping $A(u,u) : X^2 \to Y$, a unique 2-variable quadratic mapping $Q_2(u,u) : X^2 \to Y$, a unique 2-variable cubic mapping $C(u,u) : X^2 \to Y$ and a unique 2-variable quartic mapping $Q_4(u,u) : X^2 \to Y$ such that

$$\|f(u,u) - A(u,u) - Q_2(u,u) - C(u,u) - Q_4(u,u)\| \leq \begin{cases} \frac{5\rho}{6}\left(1 + \frac{1}{7} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 15}\right), \\ \frac{(18+2^{s+1})}{6}\left(\frac{1}{|2-2^s|} + \frac{1}{|8-2^s|} + \frac{1}{2|4-2^s|} + \frac{1}{2|16-2^s|}\right)\rho||u||^s, \\ \frac{(4+2^{2s})}{6}\left(\frac{1}{|2-2^{4s}|} + \frac{1}{|8-2^{4s}|} + \frac{1}{2|4-2^{4s}|} + \frac{1}{2|16-2^{4s}|}\right)\rho||u||^{4s} \\ \frac{(22+2^{2s}+2^{4s+1})}{6}\left(\frac{1}{|2-2^{4s}|} + \frac{1}{|8-2^{4s}|} + \frac{1}{2|4-2^{4s}|} + \frac{1}{2|16-2^{4s}|}\right)\rho||u||^{4s} \end{cases} \tag{4.12}$$

for all $u \in X$.

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