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Chaos and bifurcation of the Logistic discontinuous dynamical systems with piecewise constant arguments

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Abstract

In this paper we are concerned with the definition and some properties of the discontinuous dynamical systems generated by piecewise constant arguments. Then we study two discontinuous dynamical system of the Logistic equation as an example. The local stability at the fixed points is studied. The bifurcation analysis and chaos are discussed. In addition, we compare our results with the discrete dynamical systems of the Logistic equation.

Keywords: Discontinuous dynamical systems, piecewise constant argument, Logistic equation, fixed points, bifurcation, chaos.

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1 Introduction

The discontinuous dynamical systems generated by the retarded functional equations have been defined in [1]-[4]. The dynamical systems with piecewise constant arguments have been studied in [5]-[8] and the references therein. In this work we define the discontinuous dynamical systems generated by functional equations with piecewise constant arguments. The dynamic properties of two discontinuous dynamical systems of the Logistic equation will be discussed. Comparison with the corresponding discrete dynamical systems of the Logistic equation

$$x_n = \rho x_{n-1}(1 - x_{n-1}), \quad n = 1, 2, 3, ...,$$

and

$$x_{n+1} = \rho x_n(1 - x_{n-1}), \quad n = 1, 2, 3, ...,$$

will be given.

1.1 Piecewise constant arguments

Consider the problem of functional equation with piecewise constant arguments

$$x(t) = f(x(r[\frac{t}{r}])), \quad t > 0, r > 0. \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

where [.] denotes the greatest integer function.
Let $n = 1, 2, 3, ...$ and $t \in [nr, (n+1)r)$, then

$$x(t) = f(x(nr)), \quad t \in [nr, (n+1)r).$$

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Let $r = 1$ and take the limit as $t \to n+1$, we get

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, ...$$

This shows that the discrete dynamical system

$$x_n = f(n, x_{n-1}), \quad n = 1, 2, 3, ..., T.$$

$$x(0) = x_o,$$

is a special case of the problem of functional equation with piecewise constant arguments (1.1)-(1.2). Now let $t \in [0, r)$, then $\frac{t}{r} \in [0, 1)$, $x(r[\frac{t}{r}]) = x(0)$ and the solution of (1.1)-(1.2) is given by

$$x(t) = x_1(r) = f(x(0)), \quad t \in [0, r),$$

with

$$x_1(r) = \lim_{t \to r^-} x(t) = f(x(0)).$$

For $t \in [r, 2r)$, then $\frac{t}{r} \in [1, 2)$, $x(r[\frac{t}{r}]) = x(r)$ and the solution of (1.1)-(1.2) is given by

$$x(t) = x_2(t) = f(x_1(r)), \quad t \in [r, 2r).$$

Repeating the process we can easily deduce that the solution of (1.1)-(1.2) is given by

$$x(t) = x_{(n+1)}(t) = f(x_n(nr)), \quad t \in [nr, (n+1)r),$$

which is continuous on each subinterval $[k, (k+1))$, $k = 1, 2, 3, ..., n$, but

$$\lim_{t \to kr^+} x_{(k+1)}(t) = f(x_k(kr)) \neq x_k(kr).$$

Hence the problem (1.1)-(1.2) is piecewise continuous which we call it "discontinuous" and we have proved the following theorem

Theorem 1.1. *The solution of the problem of functional equation with piecewise constant arguments (1.1)-(1.2) is discontinuous (sectionally continuous) even if the function f is continuous.*

Now let $f : [0, T] \times \mathbb{R}^{n+1} \to \mathbb{R}$ and $r \in \mathbb{R}^+$. Then, the following definition can be given

Definition 1.1. *The discontinuous dynamical system generated by piecewise constant arguments is the problem*

$$x(t) = f(t, x(r[\frac{t}{r}]), x(r[\frac{t-1}{r}]), ..., x(r[\frac{t-n}{r}])), \quad t \in [0, T], \tag{1.3}$$

$$x(t) = x_0, \quad t \leq 0. \tag{1.4}$$

Definition 1.2. *The fixed points of the discontinuous dynamical system (1.3) and (1.4) are the solution of the equation*

$$x(t) = f(t, x, x, ..., x).$$

2 Main problems

Consider the discontinuous dynamical systems generated by piecewise constant arguments of Logistic equation

$$x(t) = \rho x(r[\frac{t}{r}])(1 - x(r[\frac{t}{r}])), \quad t, r > 0, \text{ and } x(0) = x_0. \tag{2.1}$$

and

$$x(t) = \rho x(r[\frac{t}{r}])(1 - x(r[\frac{t-r}{r}])), \quad t, r > 0, \text{ and } x(0) = x_0. \tag{2.2}$$

Here we study the stability at the fixed points. In order to study bifurcation and chaos we take firstly $r = 1$ and we compare the results with the results of the discrete dynamical systems of Logistic difference equation

$$x_{n+1} = \rho x_n(1 - x_n), \quad n = 1, 2, 3, ..., \text{ and } x_0 = x_o. \tag{2.3}$$

and

$$x_{n+1} = \rho x_n(1 - x_{n-1}) \quad n = 1, 2, 3, ..., \text{ and } x_0 = x_o. \tag{2.4}$$

Secondly, we take some other values of r and T and study some examples.

2.1 Fixed points and stability

As in the case of discrete dynamical systems, the fixed points of the dynamical systems (2.1) and (2.2) are the solution to the equation $f(x) = x$. Thus there are two fixed points which are

$$(x_{fixed})_1 = 0,$$

$$(x_{fixed})_2 = 1 - \frac{1}{\rho}.$$

To study the stability of these fixed points we take into account the following theorem.

Theorem 2.1. *[9] Let f be a smooth map on* $\mathbb{R}$*, and assume that* x_0 *is a fixed point of f.*
1. If $|f^{'}(x_0)| < 1$*, then* x_0 *is stable.*
2. If $|f^{'}(x_0)| > 1$*, then* x_0 *is unstable.*

Now since in our case $f(x) = \rho x(1 - x)$, the first fixed point $(x_{fixed})_1 = 0$ is stable if

$$|\rho| < 1,$$

that is, $-1 < \rho < 1$.
The second fixed point $(x_{fixed})_2 = 1 - \frac{1}{\rho}$ is stable if

$$|2 - \rho| < 1,$$

that is, $1 < \rho < 3$.
Figures (1) and (2) show the trajectories of (2.1) and (2.2) when $r = 1$ respectively, while Figures (3) and (4) show the trajectories of (2.3) and (2.4), respectively.

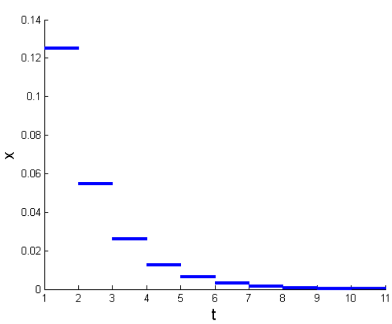

Figure 1: Trajectories of (2.1), r=1.

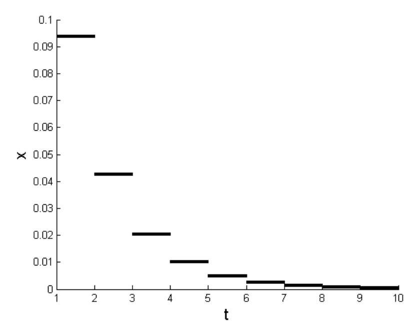

Figure 2: Trajectories of (2.2), r=1.

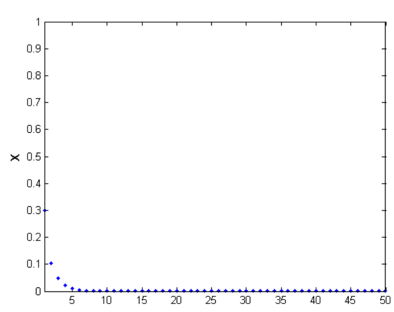

Figure 3: Trajectories of (2.3).

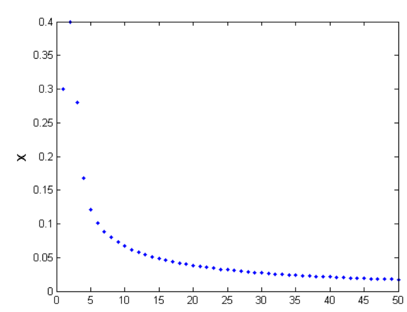

Figure 4: Trajectories of (2.4).

3 Bifurcation and Chaos

In this section, the numerical experiments show that the dynamical behaviors of the discontinuous dynamical systems (2.1) and (2.2) change when we change both r and T as follows

1. Take $r = 1$ and $t \in [0, 30]$, in this case the dynamical behaviors of the two dynamical systems (2.1 and (2.3) are identical (Figure 5).

2. Take $r = 1$ and $t \in [0, 30]$, in this case the dynamical behaviors of the two dynamical systems (2.2) and (2.4) are identical (Figure 6).

3. Take $r = 0.25$ and $t \in [0, 2]$ in the dynamical system (2.1) (Figure 7).

4. Take $r = 0.5$ and $t \in [0, 2]$ in the dynamical system (2.1) (Figure 8).

5. Take $r = 0.25$ and $t \in [0, 3]$ in the dynamical system (2.2) (Figure 9).

6. Take $r = 0.5$ and $t \in [0, 3]$ in the dynamical system (2.2) (Figure 10).

7. Take $r = 0.25$ and $T = N = 13$ in the dynamical system (2.1) (Figure 11).

8. Take $r = 0.5$ and $T = N = 35$ in the dynamical system (2.1) (Figure 12).

9. Take $r = 0.25$ and $T = N = 13$ in the dynamical system (2.2) (Figure 13).

10. Take $r = 0.5$ and $T = N = 35$ in the dynamical system (2.2) (Figure 14).

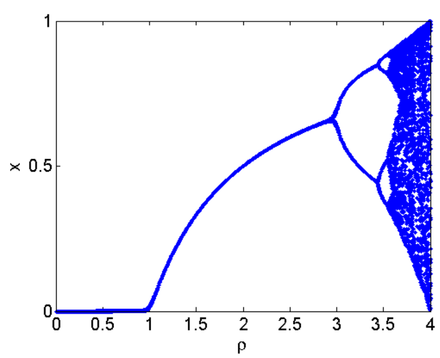

Figure 5: Bifurcation diagram of the dynamical systems (2.1) with $r = 1$ and (2.3) where $N = T = 70$.

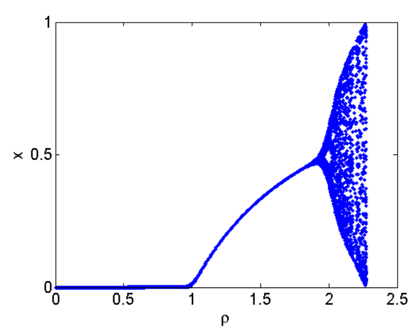

Figure 6: Bifurcation diagram of the dynamical systems (2.2) with $r = 1$ and (2.4) where $N = T = 70$.

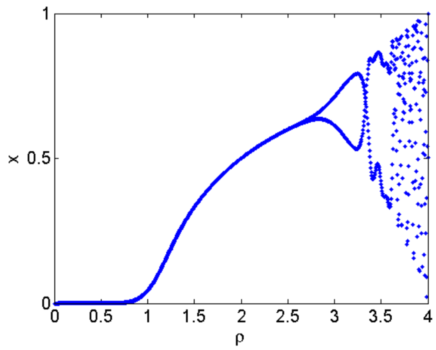

Figure 7: Bifurcation diagram for (2.1), $r = 0.25$, $t = [0, 3]$.

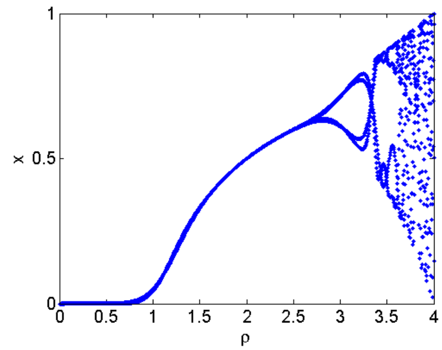

Figure 8: Bifurcation diagram for (2.1), $r = 0.5$, $t = [0, 3]$.

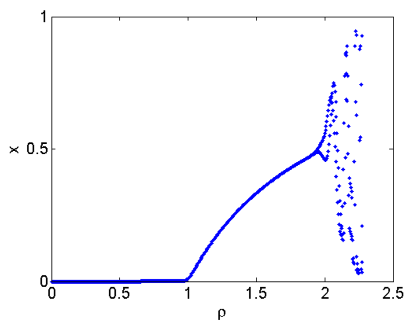

Figure 9: Bifurcation diagram for (2.2), $r = 0.25$, $t = [0, 3]$.

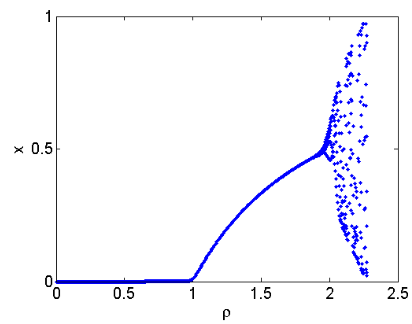

Figure 10: Bifurcation diagram for (2.2), $r = 0.5$, $t = [0, 3]$.

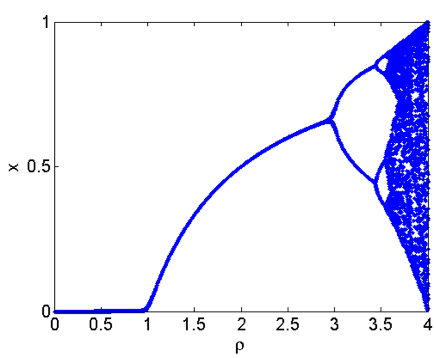

Figure 11: Bifurcation diagram for (2.1), $r = 0.25$, $T = N = 13$.

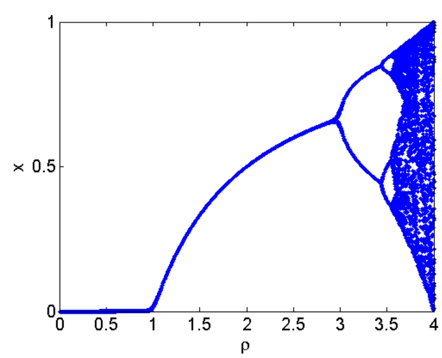

Figure 12: Bifurcation diagram for (2.1), $r = 0.5$, $T = N = 35$.

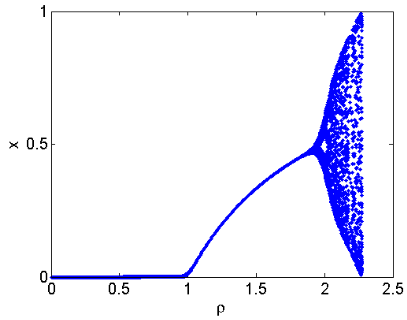

Figure 13: Bifurcation diagram for (2.2), $r = 0.25$, $T = N = 13$.

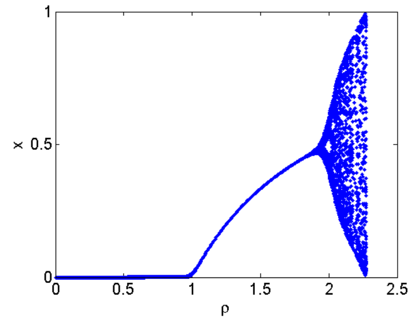

Figure 14: Bifurcation diagram for (2.2), $r = 0.5$, $T = N = 35$.

4 Conclusion

The discontinuous dynamical system models generated by piecewise constant arguments have the same behavior as its discrete version when $r = 1$.
On the other hand, changing the parameter r together with the time $t \in [0, T]$ affects the chaotic behavior of the dynamical system generated by the piecewise constant arguments model as it is shown clearly in the above figures.

References

[1] A. El-Sayed, A. El-Mesiry and H. EL-Saka, On the fractional-order logistic equation, *Applied Mathematics Letters*, 20(2007), 817-823.

[2] A. M. A. El-Sayed and M. E. Nasr, Existence of uniformly stable solutions of nonau- tonomous discontinuous dynamical systems, *J. Egypt Math. Soc.,* 19(1-2)(2011), 91-94.

[3] A. M. A. El-Sayed and M. E. Nasr, On some dynamical properties of discontinuous dynamical systems, *American Academic and Scholarly Research Journal*, 2(1)(2012), 28-32.

[4] A. M. A. El-Sayed and M. E. Nasr, On some dynamical properties of the discontinuous dynamical system represents the Logistic equation with different delays, *I-manager's Journal on Mathematics*, 'accepted'.

[5] D. Altintan, Extension of the Logistic equation with piecewise constant arguments and population dynamics, Master dissertation, Turkey 2006 .

[6] X. Fu, K. Mei, Approximate controllability of semilinear partial functional differential systems, *Journal of Dynamical and Control Systems*, 15(2009), 425-443.

[7] M. U. Akhmet, Stability of differential equations with picewise constant arguments of generalized type, *Nonlinear Anal.,* 68(4)(2008), 794-803.

[8] M. U. Akhmet, D. Altntana, T. Ergenc, Chaos of the logistic equation with piecewise constant arguments, arXiv:1006.4753, (2010), 2-5.

[9] N. A. Bohai, Continuous solutions of systems of nonlinear difference equations with continuous arguments and their properties, *Journal of Nonlinear oscillations*, 10(2)2007, 177-183.

[10] S.N. Elaidy, *An introduction to Difference Equations*, Third Edition, Undergraduate Texts in Mathematics, Springer, New York, 2005.

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