Malaya Journal of Matematik, Vol. 6, No. 1, 69-72, 2018
https://doi.org/10.26637/MJM0601/0010

Linear fractional differential equation of incomplete hypergeometric function

Dharmendra Kumar Singh[1] and Geeta Yadav[2]

Abstract
The object of this paper is to find out the solution of linear fractional differential equation of the incomplete hypergeometric function by using Caputo derivative.

Keywords
Incomplete hypergeometric function, incomplete pochhammer symbol, Caputo derivative.

AMS Subject Classification
26A33, 33Cxx, 34A08.

[1,2] *Department of Mathematics, University Institute of Engineering and Technology, CSJM University, Kanpur, India*
*Corresponding author: 1 drdksinghabp@gmail.com
Article History: Received **6** June **2017**; Accepted **7** August **2017**

Contents

1. Introduction

Caputo derivative
The fractional derivative [1, 4, 5, 6] of $f(x)$ in the Caputo sense is defined as

$$\begin{aligned} D^{\alpha} f(x) &= I^{m-\alpha} D^{m} f(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-t)^{m-\alpha-1} f^{m}(t)\,dt \end{aligned} \tag{1.1}$$

for $m-1<\alpha \leq m, \quad m \in N, \quad x>0$

For the Caputo derivative, we have $D^{\alpha} C=0$, C is constant

$$D^{\alpha} t^{n}=\begin{cases} 0 & n \leq \alpha-1 \\ \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}, & n>\alpha-1 . \end{cases} \tag{1.2}$$

Incomplete hypergeometric function
The incomplete hypergeometric functions was introduced and studied by H.M. Srivastava and Agarwal [3, p. 675 equation (4.1) and (4.2)], is defined by

$${}_{p}\gamma_{q}\left[\begin{array}{r} (a_{1};x), a_{2}, \ldots a_{p}; \\ b_{1}, \ldots b_{q}; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{(a_{1};x)_{n}(a_{2})_{n} \ldots (a_{p})_{n}}{(b_{1})_{n} \ldots (b_{q})_{n}} \frac{z^{n}}{n!} \tag{1.3}$$

and

$${}_{p}\Gamma_{q}\left[\begin{array}{r} [a_{1};x], a_{2}, \ldots a_{p}; \\ b_{1}, \ldots b_{q}; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{[a_{1};x]_{n}(a_{2})_{n} \ldots (a_{p})_{n}}{(b_{1})_{n} \ldots (b_{q})_{n}} \frac{z^{n}}{n!}, \tag{1.4}$$

in terms of incomplete gamma functions $\gamma(z,x)$ and $\Gamma(z,x)$. In (1.3) and (1.4)$(\lambda;x)_{v}$ and $[\lambda;x]_{v}$ are represent incomplete pochhammer symbol and defined as follows

$$(\lambda;x)_{v}=\frac{\gamma(\lambda+v,x)}{\Gamma(\lambda)} \tag{1.5}$$

and

$$(\lambda;x)_{v}=\frac{\Gamma(\lambda+v,x)}{\Gamma(\lambda)} . \tag{1.6}$$

and these incomplete Pochhammer symbol hold the following decomposition relation

$$(\lambda;x)_{v}+[\lambda;x]_{v}=(\lambda)_{v} \quad (\lambda, v \in C;\ x \geq 0), \tag{1.7}$$

Mittag-Leffler Function
The well known Mittag-Leffler function $E_{\alpha}(z)$ [2] named

after its originator, the Swedish mathematician Gosta Mittag-Leffler (1846-1927), is defined by

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \tag{1.8}$$

where z is a complex variable and $\Re(\alpha)\geq 0$.

2. Numerical Application

Example 2.1. *Consider the following fractional differential equation [7]*

$$\frac{d^{\alpha}y}{dx^{\alpha}}=A'y \tag{2.1}$$

If (1.4) suggests that the linear terms r(x) is decomposed by an infinite series of components

$$r(x)=F([a;z],b,c;Ax^{\alpha})=\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{(Ax^{\alpha})^n}{n!} \tag{2.2}$$

From (1.1), we have

$$D^{\alpha}r=\frac{1}{\Gamma(m-\alpha)}\int_0^x(x-t)^{m-\alpha-1}$$
$$\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}(D^m t^{n\alpha})dt$$
$$=\frac{1}{\Gamma(m-\alpha)}\int_0^x(x-t)^{m-\alpha-1}$$
$$\sum_{n=1}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)t^{n\alpha-m}}{\Gamma(n\alpha-m+1)}dt$$
$$D^{\alpha}r=\sum_{n=1}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma(\alpha(n-1)+1)}x^{\alpha(n-1)}. \tag{2.3}$$

With (2.1) and (2.2), we have

$$\sum_{n=1}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma(\alpha(n-1)+1)}x^{\alpha(n-1)}-A'$$
$$\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0$$

Replacing n by $n+1$ in the first summation, we get

$$\sum_{n=0}^{\infty}\frac{[a;z]_{n+1}(b)_{n+1}}{(c)_{n+1}}\frac{A^{n+1}}{(n+1)!}\frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}x^{\alpha(n+1-1)}$$
$$\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0$$
$$\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0$$
$$\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{x^{n\alpha}}{n!}$$
$$\left[\frac{(a+n)(b+n)}{(c+n)}\frac{\Gamma(n+1)\alpha+1)}{\Gamma(n\alpha+1)}\frac{A^{n+1}}{(n+1)}-A'A^n\right]=0.$$

With the coefficients of equal to zero and identifying the coefficients, we obtain

$$\frac{(a+n)(b+n)}{(c+n)}\Gamma\frac{A^{n+1}}{(n+1)}=A'A^n$$

put $n=0$

$$\frac{ab}{c}\Gamma(\alpha+1)A^1=A'$$
$$A^1=\frac{cA'}{ab}\frac{1}{\Gamma(\alpha+1)}$$

put $n=1$

$$\frac{(a+1)(b+1)}{(c+1)}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\frac{A^2}{2}=A'A^1$$
$$A^2=(A')^2\frac{c(c+1)}{a(a+1)b(b+1)}\frac{2}{\Gamma(2\alpha+1)}$$

and so on.

From equation (2.2)

$$r=\frac{\Gamma[a;z]}{\Gamma(a)}+\frac{[a;z]b}{c}A^1x^{\alpha}$$
$$+\frac{[a;z]_2(b)_2}{(c)_2}A^2\frac{x^{2\alpha}}{2!}+\dots$$
$$r=\frac{\Gamma[a;z]}{\Gamma(a)}+\frac{[a;z]}{a}A'\frac{1}{\Gamma(\alpha+1)}x^{\alpha}$$
$$+\frac{[a;z]_2}{a(a+1)}\frac{(A')^2x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{[a;z]_3(A')^3x^{3\alpha}}{a(a+1)(a+2)\Gamma(3\alpha+1)}+\dots. \tag{2.4}$$

Similarly for $(\lambda;z)_n$

$$s=\frac{\gamma(a;z)}{\gamma(a)}+\frac{(a;z)}{a}A'\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{(a;z)_2}{a(a+1)}\frac{x^{2\alpha}(A')^2}{\Gamma(2\alpha+1)}$$
$$+\frac{(a;z)_3}{a(a+1)(a+2)}\frac{x^{3\alpha}(A')^3}{\Gamma(3\alpha+1)}+\dots. \tag{2.5}$$

With (1.7), (2.4) and (2.5) together can be represent as

$$y=1+A'\frac{x^{\alpha}}{\Gamma(\alpha+1)}+(A')^2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+(A')^3\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\dots \tag{2.6}$$

This is the required result.

Example 2.2. *Consider the fractional differential equation [7]*

$$D^{2\alpha}y-y=0. \tag{2.7}$$

with (1.1) and (2.2), the above expression (2.7) can be express as

$$\sum_{n=2}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma((n-2)\alpha+1)}x^{\alpha(n-2)}$$

$$-\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0.$$

Replacing n by $(n+2)$ in the first summation, we get

$$\sum_{n=0}^{\infty}\frac{[a;z]_{n+2}(b)_{n+2}}{(c)_{n+2}}\frac{A^{n+2}}{(n+2)!}\frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}x^{n\alpha}$$

$$\sum_{n=0}^{\infty}\frac{[a;z]_{n+2}(b)_{n+2}}{(c)_{n+2}}\frac{A^{n+2}}{(n+2)!}\frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}x^{n\alpha}$$

$$\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0.$$

$$\sum_{n=0}^{\infty}\frac{[a;z]_n(a+n)(a+n+1)\Gamma(b+n+2)\Gamma(c)}{\Gamma(c+n+2)\Gamma(b)}$$

$$\frac{A^{n+2}}{(n+2)(n+1)n!}$$

$$\times\frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}x^{n\alpha}-\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0$$

$$\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{x^{n\alpha}}{n!}$$

$$\times\left[\frac{[(a+n)(a+n+1)(b+n+1)(b+n)A^{n+2}\Gamma((n+2)\alpha+1)}{(c+n+1)(c+n)(n+2)(n+1)\Gamma(n\alpha+1)}-A^n\right]=0.$$

With the coefficients of $x^{n\alpha}$ equal to zero and identifying the coefficients, we obtain

$$\frac{(a+n)(a+n+1)(b+n+1)(b+n)}{(c+n+1)(c+n)}$$

$$\frac{A^{n+2}}{(n+2)(n+1)}\frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}=A^n$$

put $n=0$

$$A^2=\frac{c(c+1)}{a(a+1)b(b+1)}\frac{2}{\Gamma(2\alpha+1)}$$

put $n=1$

$$A^3=\frac{(c+1)(c+2)}{(a+1)(a+2)(b+2)(b+1)}\frac{3.2\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}A^1$$

put $n=2$

$$\frac{(a+2(a+3)(b+2)(b+3)}{(c+2)(c+3)}\frac{A^4}{4.3}\frac{\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)}=A^2$$

and so on.

From equation (2.2)

$$r=\frac{\Gamma[a;z]}{\Gamma(a)}+\frac{[a;z]b}{c}Ax^{\alpha}+\frac{[a;z]_2}{a(a+1)}\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{[a;z]_3}{(a+1)(a+2)}\frac{b}{c}\frac{A\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}x^{3\alpha}+\dots. \tag{2.8}$$

Similarly for $(\lambda;x)_n$

$$s=\frac{\gamma(a;z)}{\Gamma(a)}+\frac{(a;z)b}{c}Ax^{\alpha}+\frac{(a;z)_2}{a(a+1)}\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(a;z)_3}{(a+1)(a+2)}\frac{b}{c}\frac{A\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}x^{3\alpha}+\dots. \tag{2.9}$$

With (1.7), (2.8) and (2.9) together can be represent as

$$y=1+\frac{ab}{c}Ax^{\alpha}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{ab}{c}\frac{A\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}x^{3\alpha}+\dots \tag{2.10}$$

This is the required result.

Example 2.3. *Consider the fractional differential equation [2]*

$$D^{2\alpha}y+D^{\alpha}y-2y=0. \tag{2.11}$$

With (1.1) and (2.2) above expression (2.11) can be express as

$$\sum_{n=2}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma(\alpha(n-2)+1)}x^{\alpha(n-2)}$$

$$+\sum_{n=2}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma(\alpha(n-1)+1)}x^{\alpha(n-1)}$$

$$-2\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0$$

Replacing n by $(n+2)$ and n by $(n+1)$ in the first and second summation respectively.

$$\sum_{n=0}^{\infty}\frac{[a;z]_{n+2}(b)_{n+2}}{(c)_{n+2}}\frac{A^{n+2}}{(n+2)!}\frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}x^{n\alpha}$$

$$+\sum_{n=0}^{\infty}\frac{[a;z]_{n+1}(b)_{n+1}}{(c)_{n+1}}\frac{A^{n+1}}{(n+1)!}\frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}x^{n\alpha}$$

$$-2\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0$$

$$\sum_{n=0}^{\infty}\frac{[a;z]_n(b)_n}{(c)_n}\frac{x^{n\alpha}}{n!}$$

$$\times\left[\begin{array}{l}\frac{(a+n)(a+n+1)(b+n+1)(b+n)\Gamma((n+2)\alpha+1)A^{n+2}}{(c+n+1)(c+n)\Gamma(n\alpha+1)(n+2)(n+1)}\\+\frac{(a+n)(b+n)}{(c+n)}\frac{A^{n+1}}{(n+1)}\frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}-2A^{n}\end{array}\right]=0.$$

With the coefficient of $x^{n\alpha}$ equal to zero and identifying the coefficients, we obtain

$$\frac{(a+n)(a+n+1)(b+n+1)(b+n)\Gamma((n+2)\alpha+1)}{(c+n+1)(c+n)\Gamma(n\alpha+1)(n+2)(n+1)}A^{n+2}$$
$$+\frac{(a+n)(b+n)}{(c+n)}\frac{A^{n+1}}{(n+1)}\frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}-2A^{n}=0$$

put $n=0$

$$A^{2}=\{2-\frac{ab}{c}A\Gamma(\alpha+1)\}\frac{c(c+1)}{a(a+1)b(b+1)}\frac{2.1}{\Gamma(2\alpha+1)}$$

put $n=1$

$$\frac{(a+1)(a+2)(b+1)(b+2)}{(c+1)(c+2)}\frac{A^{3}}{3.2}\frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)}$$
$$+\frac{(a+1)(b+1)}{(c+1)}\frac{A^{2}}{2}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}=2A^{1}$$

$$A^{3}=\left[2A-\frac{(a+1)(b+1)}{(c+1)}\frac{A^{2}}{2}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\right]$$

$$\frac{(c+1)(c+2)3.2}{(a+1)(a+2)(b+1)(b+2)}\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}$$

$$=\left[2A-\frac{(a+1)(b+1)}{(c+1).2}\left\{\left(2-\frac{ab}{c}A\Gamma(\alpha+1)\right)\right\}\right.$$

$$\left.\frac{c(c+1)}{a(a+1)b(b+1)}\frac{2.1}{\Gamma(2\alpha+1)}\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\right]$$

$$\times\frac{(c+1)(c+2)3!}{(a+1)(a+2)(b+1)(b+2)}\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}$$

and so on.

From equation (2.2)

$$r=\frac{\Gamma[a;z]}{\Gamma(a)}+\frac{[a;z]b}{c}Ax^{\alpha}+\frac{[a;z]_{2}}{a(a+1)}\left[\frac{2c-abA\Gamma(\alpha+1)}{c\Gamma(2\alpha+1)}\right]x^{2\alpha}$$

$$+\frac{[a;z]_{3}}{a(a+1)(a+2)}\left[\frac{3ab\Gamma(\alpha+1)A-2c}{c\Gamma(3\alpha+1)}\right]x^{3\alpha}+\ldots \quad (2.12)$$

Similarly for $(\lambda;x)_n$

$$s=\frac{\gamma(a;z)}{\Gamma(a)}+\frac{(a;z)b}{c}Ax^{\alpha}+\frac{(a;z)_{2}}{a(a+1)}\left[\frac{2c-abA\Gamma(\alpha+1)}{c\Gamma(2\alpha+1)}\right]x^{2\alpha}$$

$$+\frac{(a;z)_{3}}{a(a+1)(a+2)}\left[\frac{3ab\Gamma(\alpha+1)A-2c}{c\Gamma(3\alpha+1)}\right]x^{3\alpha}+\ldots \quad (2.13)$$

With (1.7), (2.12) and (2.13) together can be represent as

$$y=1+\frac{ab}{c}Ax^{\alpha}+\left[\frac{2c-abA\Gamma(\alpha+1)}{c\Gamma(2\alpha+1)}\right]x^{2\alpha}$$

$$+\frac{3ab\Gamma(\alpha+1)A-2c}{c\Gamma(3\alpha+1)}x^{3\alpha}+\ldots \quad (2.14)$$

which is the required result.

Acknowledgments

The authors would like to express sincere gratitude to the reviewers for his/her valuable suggestions.

References

[1] A.A. Kilbas, H.M. Srivastava, and and J.J. Trujillo, *Theory and applications of Fractional Differential Equations*, Vol. 204 of North Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, The Netherlands, (2006).

[2] G. Mittag-Leffler, *Sur la nouvelle function $E_\alpha(x)$ C R* Acad Sci Paris 137, 554-558.

[3] H.M. Srivastava, M.A. Chaudhary, R.P. Agarwal. *The incomplete pochhammer symbols and their applications to hypergeometric and related functions*, integral Transforms spec. function, (23), 659-683 (2012).

[4] I. Podlubny, *Fractional Differential Equation*, Vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, (1999).

[5] K.S. Miller and B.Ross, *An introduction to the fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, John Wiley & Sons, New York, NY, USA, (1993).

[6] M. Caputo, *Elasticita e Dissipazione*, Zani-Chelli, Bologna, 1969.

[7] Z.M. Odibat and N.T. Shawagfeh, *Generalized Taylor's formula*, Applied Mathematics and computation, Vol. 186, No. 1, pp. 286-293, (2007).

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆