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Synchronization of dynamical systems of different orders and different dimensions

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Abstract
A method of tracking control is proposed to achieve synchronization between the systems of fractional order and integer order. This article presents two cases of synchronization, in first case synchronization for three dimensional integer order Cai system and a four dimensional fractional order hyperchaotic Gao system is achieved and in second case synchronization for three dimensional fractional order Newton-Leipnik system and four dimensional hyperchaotic Pang-Liu system is achieved by tracking control method. Computational results shows that the controllers designed are useful to synchronize the considered master and slave systems in both the cases. In order to execute numerical results Matlab software is used.

Keywords
Synchronization, fractional order system, integer order system, tracking Control.

AMS Subject Classification
37B25, 37D45, 37N35, 70K99.

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Contents

1. Introduction

Chaotic phenomenon has been present in the nature since a long time but when the Lorenz system [1] was found in 1963, this field arose as a new attraction in non linear sciences. After that pioneer invention of Lorenz a lot of literature is available on chaotic and hyperchaotic systems. Chaos is the property of typical dynamics and highly unpredictable behavior of a system. Having a 300 years old history fractional calculus has always been a branch of intrinsic interest for researchers. Chaos in fractional order systems have also been the field of new and application based researches. Several chaotic systems with fractional order are discovered and studied by the researchers. Some of the well known fractional order systems are Lorenz, Rossler, Chen, Liu, Volta fractional order systems [2]-[6]. Importance of fractional order can be seen as there are so many systems which can be modeled in a better and more accurate manner as a fractional order system in comparison of integer order. Different methods of solving fractional order systems and stability of these methods have been analyzed by some researchers [7, 8]. Several other significant analysis have also been done on fractional differential equations [9]-[11].

In several different fields applications of fractional calculus can be easily seen like biology, chemistry and bioengineering etc [12]-[14]. Three decades ago when Pecora and Carroll [15] presented a new finding in the field of chaos which is known as synchronization, inclination of researchers towards this branch has always been immense. A large number of synchronization schemes have been studied analytically and experimentally [16]-[18]. In recent years, use of chaos syn-

chronization can be seen with significant developments in the field of secure communication [19, 20]. In secure communication, although the information signal is always masked by a chaotic signal but when it comes to the security of transmitted message, role of chaos synchronization becomes vital. Synchronization has been an integral part of chaotic lasers [21] due to its applicability in optical secure communication. Several methods have been found for security of transmitted signal in secure communication including chaos masking. But it is found that information signal masked with chaotic signal is not much secure, therefore to enhance the grade of security of information signal several types of synchronizations were suggested among which projective synchronization was found much effective for secure communication because of its unpredictable multiple factor. Several methods like active control [22], adaptive control [23], feedback control [24], pinning control [25], sliding mode control [26], fuzzy control [27] etc have been studied and found effective to obtain the desired synchronization. Some work have also been done on synchronization of fractional order and integer order system [28, 29], but still a very few work have been done on different order and different dimensional systems. A very particular case was considered by Ping et. al [28] to achieve complete synchronization which is extended in this paper in a more generalized way to achieve different kinds of synchronizations.
This paper is categorized in five sections. Section 2 presents synchronization scheme, fractional order derivatives and stability of fractional order systems. Section 3 presents the brief discussion of chaotic and hyperchaotic nature of all four systems considered. In section 4 construction of controllers and numerical results are presented to justify the theoretical approach and the last section 5 is conclusion where the main findings of this paper are highlighted.

2. Fractional order derivatives, stability of fractional order system and synchronization scheme

2.1 Fractional order derivative [8]

For differentiation and integration, in fractional calculus the operator ${}_bD_t^\beta$ is defined where b and t are the limits of the operator and β is the non integer order. This operator is defined in the following manner

$$ {}_bD_t^\beta = \begin{cases} \frac{d^\beta}{dt^\beta} & if \beta > 0 \\ 0 & if \beta = 0 \\ \int_b^t (d\tau)^\beta & if \beta < 0 \end{cases} \tag{2.1} $$

Three definitions for fractional integro differential operator are Riemann-Liouville definition, Grunwald Letnikov definition and Caputo's definition.

The Riemann-Liouville definition is defined by

$$ {}_bD_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_b^t (t-\tau)^{n-\beta-1} f(\tau) d(\tau), \tag{2.2} $$

for $n-1 < \beta < n$. The Caputo's fractional derivative is given by

$$ {}_bD_t^\beta = \frac{1}{\Gamma(n-\beta)} \int_b^t \frac{f^n(\tau)}{(t-\tau)^{\beta+1-n}} d\tau, \quad : n-1 < \beta < n \tag{2.3} $$

where $\Gamma(.)$ is the Gamma function. The Grunwald Letnikov definition is

$$ {}_bD_t^\beta f(t) = \lim_{h\to 0} \frac{1}{h^\beta} \sum_{j=0}^{[\frac{(t-\beta)}{h}]} (-1)^j \begin{pmatrix} \beta \\ j \end{pmatrix} f(t-jh) \tag{2.4} $$

where $[.]$ denotes greatest integer part.

2.2 Stability of fractional order system[28]

Theorem: Suppose a fractional order system is considered as

$$ D^\beta x = Bx, \quad x(0) = x_0 $$

Here $B \in R^{p\times p}$, and $\beta = (\beta_1, \beta_2, ..., \beta_p), (0 < \beta_i \leq 1)$. The system is asymptotically stable if and only if $| arg(\nu_i) | > \beta\pi/2$ is satisfied for all eigenvalues ν_i of the matrix B. Furthermore this system is stable if and only if $|arg(\nu_i)| \geq \beta\pi/2$ is satisfied for all eigenvalues ν_i of the matrix B and the critical eigenvalues for which the condition $|arg(\nu_i)| = \beta\pi/2$ have geometric multiplicity one. The geometric multiplicity of an eigenvalue is defined as the dimension of the associated eigenspace, i.e., number of linearly independent eigenvectors with that eigenvalue.

2.3 Synchronization Scheme

A brief discussion is presented in this subsection to achieve synchronization between two different order systems. Suppose a p-dimensional system is considered as master system

$$ \frac{d^\eta m}{dt^\eta} = h(m) \tag{2.5} $$

where h is a differentiable function from R^p to R^p, $m \in R^p$, $\eta = (\eta_1, \eta_2, ..., \eta_p)$ is the order and $0 < \eta_i \leq 1$ for all $i = 1, 2....p$. Suppose the slave system with controller is

$$ \frac{d^\kappa s}{dt^\kappa} = l(s) + U \tag{2.6} $$

where l is a differentiable function from R^p to R^p, $s \in R^p$, $\kappa = (\kappa_1, \kappa_2, ..., \kappa_p)$ is the order and $0 < \eta_i \leq 1$ for all $i = 1, 2....p$. U is the controller which will be decomposed in two subcontrollers U_1, U_2 as $U = U_1 + U_2$ and

$$ U_1 = \chi \frac{d^\kappa m}{dt^\kappa} - l(\chi m) \tag{2.7} $$

where $\chi = (\chi_1, \chi_2, ..., \chi_p)$ is a $1 \times p$ vector of scaling factors by choosing of which different types of synchronization could be achieved and $m = (m_1, m_2, ..., m_p)^t$ is a $p \times 1$ vector. Here U_1 is a compensation controller and U_2 is feedback controller. If errors are defined as

$$e = s - \chi m \tag{2.8}$$

The goal is to design an effective controller so that error will tend to zero and the desired synchronization could be achieved. As the slave systems considered in this paper is of greater dimension than master system so the last dimension of slave system will be synchronized to zero.

3. Description of chaotic and hyperchaotic systems

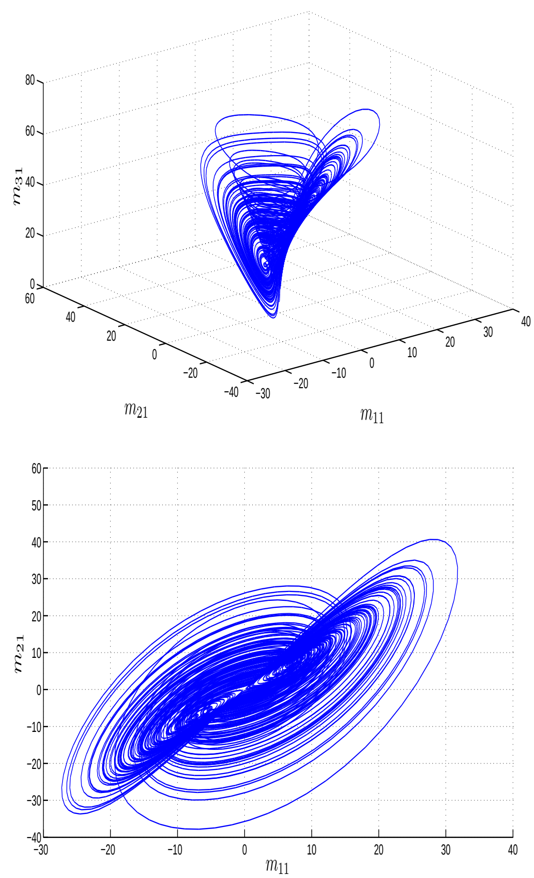

Figure 1. Chaotic attractor of Cai system in three dimensional space and in $m_{11} - m_{31}$ space

Dynamics of integer order Cai [30] system is given below

$$\begin{aligned} m_{11} =& \lambda_1 (m_{21} - m_{11}) \\ m_{21} =& \mu_1 m_{11} + \rho_1 m_{21} - m_{11} m_{31} \\ m_{31} =& m_{11}^2 - \sigma_1 m_{31} \end{aligned} \tag{3.1}$$

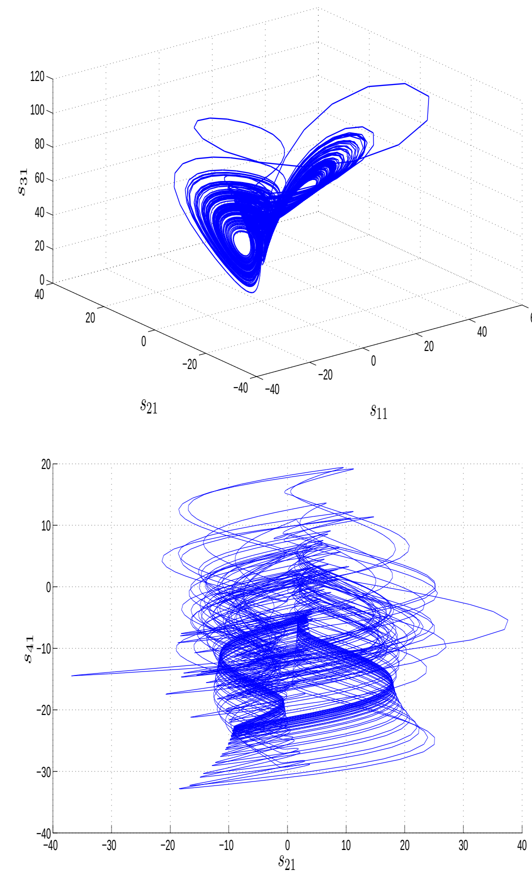

Figure 2. Hyperchaotic attractor of Gao system in three dimensional space and in $s_{11} - s_{41}$ space

which shows chaotic behavior for $\lambda_1 = 20, \mu_1 = 14, \rho_1 = 10.6, \sigma_1 = 2.8$ as shown in Figure (1). Dynamics of fractional order Gao system [31] is

$$\begin{aligned} \frac{d^{\kappa_1} s_{11}}{dt^{\kappa_1}} =& -\lambda_2 s_{11} + \mu_2 s_{21} \\ \frac{d^{\kappa_2} s_{21}}{dt^{\kappa_2}} =& \rho_2 s_{11} - s_{11} s_{31} - s_{21} + s_{41} \\ \frac{d^{\kappa_3} s_{31}}{dt^{\kappa_3}} =& s_{11}^2 - \sigma_2 (s_{11} + s_{31}) \\ \frac{d^{\kappa_4} s_{41}}{dt^{\kappa_4}} =& -k_2 s_{11} \end{aligned} \tag{3.2}$$

which shows hyperchaotic behavior for parameter values $\lambda_2 = 25, \mu_2 = 60, \rho_2 = 40, \sigma_2 = 4, k_2 = 5$ and order $\kappa_1 = 0.95, \kappa_2 = 0.95, \kappa_3 = 0.95, \kappa_4 = 0.95$. Hyperchaotic attractors of Gao system are shown in Figure (2) and Figure (3) . Fractional order Newton-Leipnik system was studied by Sheu et.al [32]

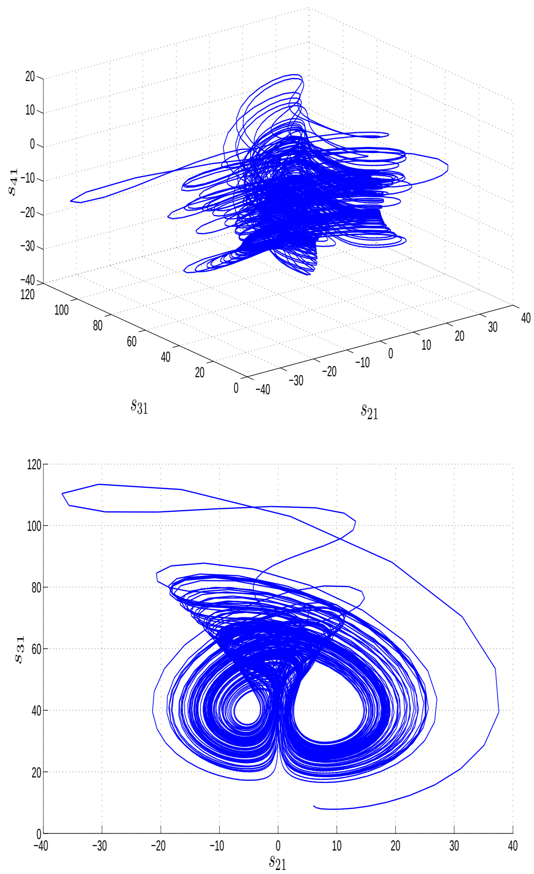

Figure 3. Hyperchaotic attractor of Gao system in three dimensional space and in $s_{21}-s_{31}$ space

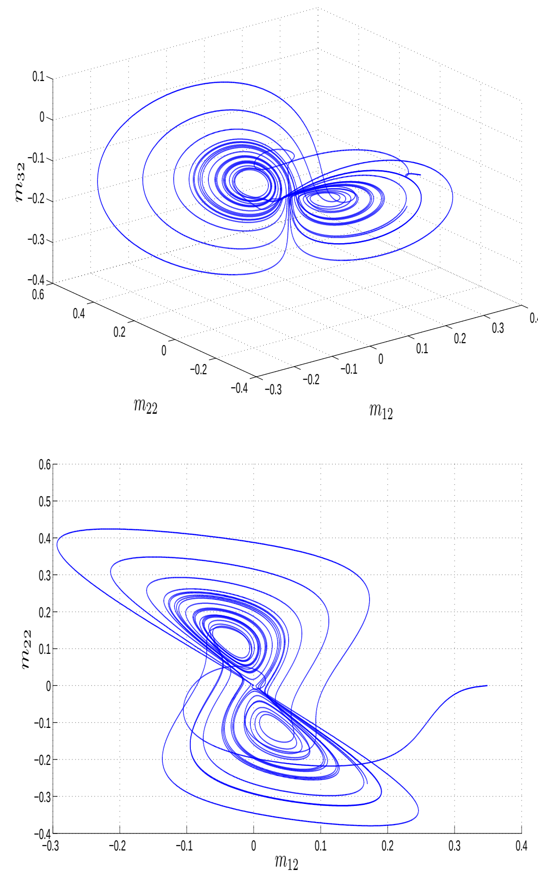

Figure 4. Chaotic attractor of fractional order Newton Leipnik system in three dimensional space and in $m_{12}-m_{32}$ space

which is given by

$$\begin{aligned}\frac{d^{\kappa_1}m_{12}}{dt^{\kappa_1}} &= -\lambda_3 m_{12}+m_{22}+10m_{22}m_{32}\\ \frac{d^{\kappa_2}m_{22}}{dt^{\kappa_2}} &= -m_{12}-0.4m_{22}+5m_{12}m_{32}\\ \frac{d^{\kappa_3}m_{32}}{dt^{\kappa_3}} &= \mu_3 m_{32}-5m_{12}m_{22}\end{aligned} \tag{3.3}$$

which shows chaotic behavior for parameter values $\lambda_3 = 0.4, \mu_3 = 0.175$ and order $\kappa_1 = 0.95, \kappa_2 = 0.95, \kappa_3 = 0.95$. Chaotic attractors of fractional order Newton-Leipnik system are shown in Figure (4).

Integer order hyperchaotic Pang-Liu system [33] is given by

$$\begin{aligned}s_{12} &= \lambda_4(s_{22}-s_{12})\\ s_{22} &= \rho_4 s_{22}-s_{12}s_{32}+s_{42}\\ s_{32} &= -\mu_4 s_{32}+s_{12}s_{22}\\ s_{42} &= -h_4 s_{12}-k_4 s_{22}\end{aligned} \tag{3.4}$$

which exhibits hyperchaotic behavior for the parameter values $\lambda_4 = 36, \mu_4 = 3, \rho_4 = 20, h_4 = 2, k_4 = 2$. Hyperchaotic attractors of Pang-Liu system are shown in Figure (5) and Figure (6) .

4. Synchronization of systems of different order and different dimension and numerical results

Two cases are considered in this paper. In first case Cai system is considered as master system and fractional order Gao system is considered as slave system and in second case fractional order Newton-Leipnik system is considered as master system and integer order Pang Liu system is considered as slave system.

Case 1 Suppose scaling vector is taken as $\chi = (\chi_1, \chi_2, \chi_3, \chi_4)$. Then errors are

$$\begin{aligned}e_{11} &= s_{11}-\chi_1 m_{11}\\ e_{21} &= s_{21}-\chi_2 m_{21}\\ e_{31} &= s_{31}-\chi_3 m_{31}\\ e_{41} &= s_{41}-\chi_4 m_{41}\end{aligned} \tag{4.1}$$

To synchronize three dimensional Cai system and the four dimensional fractional order hyperchaotic Gao system via tracking control, s_{41} has to be synchronized to 0. So the fourth dimension of master system is assumed to be zero which is used in calculations as state variable m_{14}.

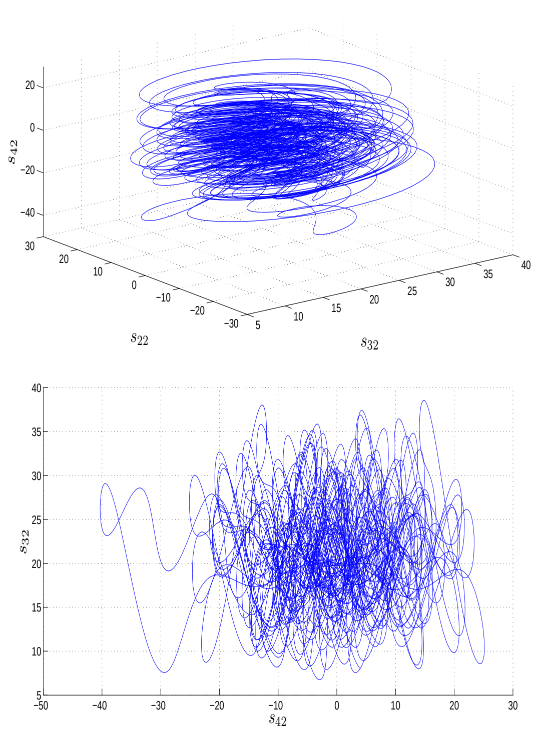

Figure 5. Hyperchaotic attractor of Pang-Liu system in three dimensional space and in $s_{32}-s_{42}$ space

Theorem 1 Hyperchaotic fractional order Gao system (3.2) will attain synchronization with chaotic Cai system (3.1) for the errors defined by (4.1), if the compensation controller V_1 and feedback controller V_2 are

$$V_1 = \chi \begin{pmatrix} \frac{d^{\kappa_1} m_{11}}{dt^{\kappa_1}} \\ \frac{d^{\kappa_2} m_{21}}{dt^{\kappa_2}} \\ \frac{d^{\kappa_3} m_{31}}{dt^{\kappa_3}} \\ \frac{d^{\kappa_4} m_{41}}{dt^{\kappa_4}} \end{pmatrix} - \begin{pmatrix} -\lambda_2\chi_1 m_{11} + \mu_2\chi_2 m_{21} \\ \rho_2\chi_1 m_{11} - \chi_1 m_{11}\chi_3 m_{31} - \chi_2 m_{21} + \chi_4 m_{41} \\ \chi_1^2 m_{11}^2 - \sigma_2(\chi_1 m_{11} + \chi_3 m_{31}) \\ -k_2\chi_1 m_{11} \end{pmatrix} \tag{4.2}$$

and

$$V_2 = \begin{pmatrix} Q\bar{e_1} - G_1(m', \bar{e_1}, \bar{e_2}) \\ S\bar{e_2} - G_3(m', \bar{e_1}, \bar{e_2}) \end{pmatrix} \tag{4.3}$$

where Q, S are suitably chosen matrices, $\bar{e_1} = e_{11}, \bar{e_2} = (e_{21}, e_{31}, e_{41})$ and m' represents terms containing master system's state variables.

Proof Suppose fractional order Gao system with controller is

$$\begin{pmatrix} \frac{d^{\kappa_1} s_{11}}{dt^{\kappa_1}} \\ \frac{d^{\kappa_2} s_{21}}{dt^{\kappa_2}} \\ \frac{d^{\kappa_3} s_{31}}{dt^{\kappa_3}} \\ \frac{d^{\kappa_4} s_{41}}{dt^{\kappa_4}} \end{pmatrix} = \begin{pmatrix} -\lambda_2 s_{11} + \mu_2 s_{21} \\ \rho_2 s_{11} - s_{11}s_{31} - s_{21} + s_{41} \\ s_{11}^2 - \sigma_2(s_{11} + s_{31}) \\ -k_2 s_{11} \end{pmatrix} + V \tag{4.4}$$

where V is the controller. Suppose $V = V_1 + V_2$ where V_1 is compensation and V_2 is the feedback controller. Then by using equation (4.2) in the previous equation

$$\begin{pmatrix} \frac{d^{\kappa_1} s_{11}}{dt^{\kappa_1}} \\ \frac{d^{\kappa_2} s_{21}}{dt^{\kappa_2}} \\ \frac{d^{\kappa_3} s_{31}}{dt^{\kappa_3}} \\ \frac{d^{\kappa_4} s_{41}}{dt^{\kappa_4}} \end{pmatrix} = \begin{pmatrix} -\lambda_2 s_{11} + \mu_2 s_{21} \\ (\rho_2 - s_{31})s_{11} - s_{21} + s_{41} \\ s_{11}^2 - \sigma_2(s_{11} + s_{31}) \\ -k_2 s_{11} \end{pmatrix} + V_2 + \chi \begin{pmatrix} \frac{d^{\kappa_1} m_{11}}{dt^{\kappa_1}} \\ \frac{d^{\kappa_2} m_{21}}{dt^{\kappa_2}} \\ \frac{d^{\kappa_3} m_{31}}{dt^{\kappa_3}} \\ \frac{d^{\kappa_4} m_{41}}{dt^{\kappa_4}} \end{pmatrix} - \begin{pmatrix} -\lambda_2\chi_1 m_{11} + \mu_2\chi_2 m_{21} \\ (\rho_2\chi_1 - \chi_1\chi_3 m_{31})m_{11} - \chi_2 m_{21} + \chi_4 m_{41} \\ \chi_1^2 m_{11}^2 - \sigma_2(\chi_1 m_{11} + \chi_3 m_{31}) \\ -k_2\chi_1 m_{11} \end{pmatrix} \tag{4.5}$$

Then error dynamical system will be

$$\begin{pmatrix} \frac{d^{\kappa_1} e_{11}}{dt^{\kappa_1}} \\ \frac{d^{\kappa_2} e_{21}}{dt^{\kappa_2}} \\ \frac{d^{\kappa_3} e_{31}}{dt^{\kappa_3}} \\ \frac{d^{\kappa_4} e_{41}}{dt^{\kappa_4}} \end{pmatrix} = \begin{pmatrix} -\lambda_2 s_{11} + \mu_2 s_{21} \\ \rho_2 s_{11} - s_{11}s_{31} - s_{21} + s_{41} \\ s_{11}^2 - \sigma_2(s_{11} + s_{31}) \\ -k_2 s_{11} \end{pmatrix} + V_2 - \begin{pmatrix} -\lambda_2\chi_1 m_{11} + \mu_2\chi_2 m_{21} \\ \rho_2\chi_1 m_{11} - \chi_1 m_{11}\chi_3 m_{31} - \chi_2 m_{21} + \chi_4 m_{41} \\ \chi_1^2 m_{11}^2 - \sigma_2(\chi_1 m_{11} + \chi_3 m_{31}) \\ -k_2\chi_1 m_{11} \end{pmatrix} \tag{4.6}$$

Hence

$$\begin{pmatrix} \frac{d^{\kappa_1} e_{11}}{dt^{\kappa_1}} \\ \frac{d^{\kappa_2} e_{21}}{dt^{\kappa_2}} \\ \frac{d^{\kappa_3} e_{31}}{dt^{\kappa_3}} \\ \frac{d^{\kappa_4} e_{41}}{dt^{\kappa_4}} \end{pmatrix} = V_2 + \begin{pmatrix} -\lambda_2 e_{11} + \mu_2 e_{21} \\ (\rho_2 - e_{31} - \chi_3 m_{31})e_{11} - e_{31}\chi_1 m_{11} - e_{21} + e_{41} \\ e_{11}(e_{11} + 2\chi_1 m_{11}) - \sigma_2(e_{11} + e_{31}) \\ -k_2 e_{11} \end{pmatrix} \tag{4.7}$$

Our aim is to design controller V_2 so that a stable error dynamical system can be achieved.

$$\begin{pmatrix} \frac{d^{\kappa_1} e_{\bar{1}1}}{dt^{\kappa_1}} \\ \frac{d^{\kappa_2} e_{\bar{2}1}}{dt^{\kappa_2}} \end{pmatrix} = \begin{pmatrix} P e_{\bar{1}1} + G_1(m', e_{\bar{1}1}, e_{\bar{2}1}) \\ R e_{\bar{2}1} + G_2(m', e_{\bar{1}1}, e_{\bar{2}1}) + G_3(m', e_{\bar{1}1}, e_{\bar{2}1}) \end{pmatrix} + V_2 \tag{4.8}$$

where, $P = (-\lambda_2)$, $R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $e_{\bar{1}1} = e_{11}$, $e_{\bar{2}1} = (e_{21}, e_{31}, e_{41})$ and m' represents terms containing master system's state variables.

$$\begin{cases} G_1(m', e_{\bar{1}1}, e_{\bar{2}1}) = \mu_2 e_{21} \\ G_2(m', e_{\bar{1}1}, e_{\bar{2}1}) = \begin{pmatrix} \rho_2 e_{11} - e_{11} e_{31} - e_{11} \chi_3 m_{31} \\ e_{11}(e_{11} + 2\chi_1 m_{11}) - \sigma_2 e_{11} \\ -k_2 e_{11} \end{pmatrix} \\ G_3(m', e_{\bar{1}1}, e_{\bar{2}1}) = \begin{pmatrix} -e_{31} \chi_1 m_{11} + e_{41} \\ -\sigma_2 e_{31} \\ 0 \end{pmatrix} \end{cases} \tag{4.9}$$

Hence by defining

$$V_2 = \begin{pmatrix} Q e_{\bar{1}1} - G_1(m', e_{\bar{1}1}, e_{\bar{2}1}) \\ S e_{\bar{2}1} - G_3(m', e_{\bar{1}1}, e_{\bar{2}1}) \end{pmatrix} \tag{4.10}$$

The error dynamical system will be

$$\begin{pmatrix} \frac{d^{\kappa_1} e_{\bar{1}1}}{dt^{\kappa_1}} \\ \frac{d^{\kappa_2} e_{\bar{2}1}}{dt^{\kappa_2}} \end{pmatrix} = \begin{pmatrix} (P+Q) e_{\bar{1}1} \\ (R+S) e_{\bar{2}1} + G_2(m', e_{\bar{1}1}, e_{\bar{2}1}) \end{pmatrix} \tag{4.11}$$

where $\lim_{e_{\bar{1}1} \to 0} G_2(m', e_{\bar{1}1}, e_{\bar{2}1}) = 0$. By choosing suitable Q, S matrices condition of eigenvalues of the system can be made suitable so that asymptotical stability condition will be satisfied.

For numerical simulation of first case to achieve anti-synchronization scaling vector is chosen as $(\chi_1, \chi_2, \chi_3, \chi_4) = (-1, -1, -1, -1)$. Parameters of master and slave system are chosen as $\lambda_1 = 20, \mu_1 = 14, \rho_1 = 10.6, \sigma_1 = 2.8, \lambda_2 = 36, \mu_2 = 3, \rho_2 = 20, h_4 = 2, k_4 = 2$. Fractional order of slave system is $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = 0.95$. Initial values for master system and slave systems are $(-3, -8, -4)$ and $(2, 6, 9, 2)$. Matrices Q and S are chosen in such a way that eigenvalues of $P+Q$ is -1 and eigenvalue of $Q+S$ are $-3, -2, -4$. Hence the condition $|arg(\nu_i)| \geq \beta\pi/2$ will be satisfied. Initial conditions for the error systems will be $(-1, -2, 5, 2)$ and order of error dynamical system is $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = 0.95$. Errors converging to zero are shown in Figure (7).

Case2

In this case fractional order Newton Leipnik system is considered as master system and integer order Pang Liu system is considered as slave system. Errors are defined as

$$\begin{aligned} e_{12} &= s_{12} - \chi_1 m_{12} \\ e_{22} &= s_{22} - \chi_2 m_{22} \\ e_{32} &= s_{32} - \chi_3 m_{32} \\ e_{42} &= s_{42} - \chi_4 m_{42} \end{aligned} \tag{4.12}$$

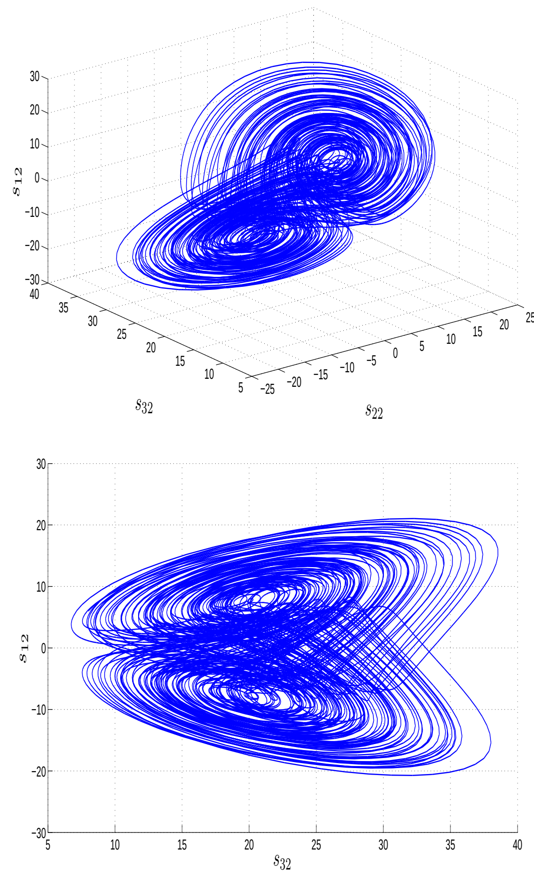

Figure 6. Hyperchaotic attractor of Pang-Liu system in three dimensional space and in $s_{12} - s_{32}$ space

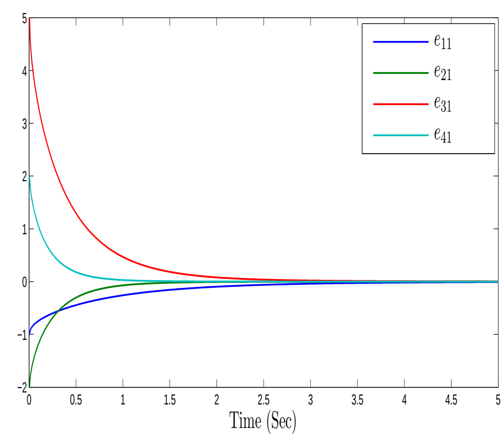

Figure 7. Convergence of anti-synchronization errors to zero for integer order Cai system and fractional order Gao system

Theorem 2 Hyperchaotic integer order Pang-Liu system (3.4) will attain synchronization with chaotic fractional order Newton-Leipnik system (3.3) for the errors defined by (4.12), if the compensation controller V_1 and feedback controller V_2 are

$$V_1 = \chi \begin{pmatrix} \frac{dm_{12}}{dt} \\ \frac{dm_{22}}{dt} \\ \frac{dm_{32}}{dt} \\ \frac{dm_{42}}{dt} \end{pmatrix} - \begin{pmatrix} \lambda_4(\chi_2 m_{22} - \chi_1 m_{12}) \\ \rho_4 \chi_2 m_{22} - \chi_1 \chi_3 m_{12} m_{32} + \chi_4 m_{42} \\ -\mu_4 \chi_3 m_{32} + m_{12}\chi_2 m_{22} \\ -h_4 \chi_1 m_{12} - k_4 \chi_2 m_{22} \end{pmatrix} \tag{4.13}$$

and

$$V_2 = \begin{pmatrix} Q' \bar{e_{12}} - H_1(m', \bar{e_{12}}, \bar{e_{22}}) \\ S' \bar{e_{22}} - H_3(m', \bar{e_{12}}, \bar{e_{22}}) \end{pmatrix} \tag{4.14}$$

where Q', S' are suitably chosen matrices, $\bar{e_{12}} = e_{12}, \bar{e_{22}} = (e_{22}, e_{32}, e_{42})$ and m' represents terms containing master system's state variables.

Proof On applying the same procedure which is described in previous theorem an error dynamical system will be obtained in the following form

$$\begin{pmatrix} \frac{de_{12}}{dt} \\ \frac{de_{22}}{dt} \\ \frac{de_{32}}{dt} \\ \frac{de_{42}}{dt} \end{pmatrix} = \begin{pmatrix} \lambda_4(e_{22} - e_{12}) \\ \rho_4 e_{22} - e_{12}e_{32} - e_{12}\chi_3 m_{32} - e_{32}\chi_1 m_{12} - e_{42} \\ -\mu_4 e_{32} + e_{12}e_{22} + e_{12}\chi_2 m_{22} + e_{22}\chi_1 m_{12} \\ -h_4 e_{12} - k_4 e_{22} \end{pmatrix} + V_2 \tag{4.15}$$

$$\begin{pmatrix} \frac{d\bar{e_{12}}}{dt} \\ \frac{d\bar{e_{22}}}{dt} \end{pmatrix} = \begin{pmatrix} P' \bar{e_{12}} + H_1(m', \bar{e_{12}}, \bar{e_{22}}) \\ R' \bar{e_{22}} + H_2(m', \bar{e_{12}}, \bar{e_{22}}) + H_3(m', \bar{e_{12}}, \bar{e_{22}}) \end{pmatrix} + V_2 \tag{4.16}$$

where $P' = (-\lambda_4)$, $R' = \begin{pmatrix} \rho_4 & 0 & 0 \\ 0 & -\mu_4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\bar{e_{12}} = e_{12}, \bar{e_{22}} = (e_{22}, e_{32}, e_{42})$ and m represents terms containing master system's state variables.

$$\begin{cases} H_1(m', \bar{e_{12}}, \bar{e_{22}}) = \lambda_4 e_{22} \\ H_2(m', \bar{e_{12}}, \bar{e_{22}}) = \begin{pmatrix} -e_{12}e_{32} - e_{12}\chi_3 m_{32} \\ e_{12}e_{22} + e_{12}\chi_2 e_{22} \\ -k_4 e_{12} \end{pmatrix} \\ H_3(m', \bar{e_{12}}, \bar{e_{22}}) = \begin{pmatrix} -e_{32}\chi_1 m_{12} - e_{42} \\ e_{22}\chi_1 m_{12} \\ -k_4 e_{22} \end{pmatrix} \end{cases} \tag{4.17}$$

Hence by defining

$$V_2 = \begin{pmatrix} Q' \bar{e_{12}} - H_1(m', \bar{e_{12}}, \bar{e_{22}}) \\ S' \bar{e_{22}} - H_3(m', \bar{e_{12}}, \bar{e_{22}}) \end{pmatrix} \tag{4.18}$$

Hence error dynamical system will be

$$\begin{pmatrix} \frac{d\bar{e_{12}}}{dt} \\ \frac{d\bar{e_{22}}}{dt} \end{pmatrix} = \begin{pmatrix} (P' + Q')\bar{e_{12}} \\ (R' + S')\bar{e_{22}} + H_2(m', \bar{e_{12}}, \bar{e_{22}}) \end{pmatrix} \tag{4.19}$$

where $\lim_{\bar{e_{12}} \to 0} H_2(s, \bar{e_{12}}, \bar{e_{22}}) = 0$. Similarly, by choosing suitable Q', S' an asymptotically stable error dynamical system can be obtained.

For the second case to achieve hybrid projective synchronization scaling vector is chosen as $(\chi_1, \chi_2, \chi_3, \chi_4) = (1, -1, 1, -1)$. Parameters of master and slave system are chosen as $\lambda_3 = 0.4, \mu_3 = 0.175, \lambda_4 = 36, \mu_4 = 3, \rho_4 = 20, h_4 = 2, k_4 = 2$. Fractional order of master system is $\kappa_1 = \kappa_2 = \kappa_3 = 0.95$. Initial values for master system and slave systems are $(0.349, 0, -0.16)$ and $(6, 7, 8, 4)$. Matrices Q' and S' are chosen in such a way that eigenvalues of $P + Q$ is -1 and eigenvalue of $Q + S$ are $-1, -2, -3, -4$. Initial conditions for error systems will be $(5.651, 7, 8.16, 4)$. Errors converging to zero are shown in Figure (8).

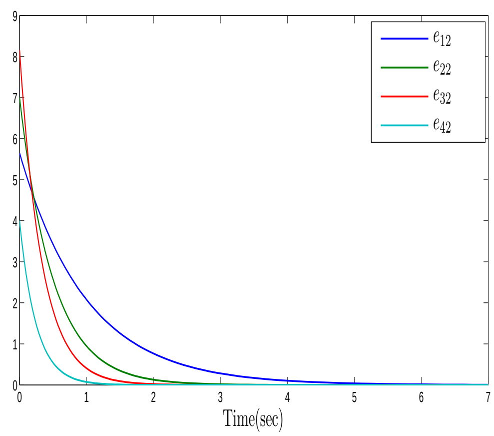

Figure 8. Convergence of hybrid projective synchronization errors to zero for fractional order Newton Leipnik system and integer order Pang Liu system

5. Conclusion

Synchronization between fractional and integer order systems are achieved in both the cases, where master system is chaotic and slave system is hyperchaotic and dimensions are also different for both the systems. Numerical results and the analysis done in theoretical approach are in excellent agreement. So many new developments are still possible as the proposed approach can be applied on different chaotic systems or new methods can be developed to achieve the desired synchronization. This scheme is very important for secure communication as different fractional and integer order makes it very difficult to decode the information associated with the transmitted signal.

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