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On k-step Hamiltonian Bipartite and Tripartite Graphs

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Abstract

For integer $k \geq 1$, a (p,q)-graph $G = (V,E)$ is said to admit an $AL(k)$-traversal if there exist a sequence of vertices $(v_1, v_2, \ldots, v_p)$ such that for each $i = 1, 2, \ldots, p-1$, the distance between v_i and v_{i+1} is k. We call a graph k-step Hamiltonian (or admits a k-step Hamiltonian tour) if it admits an $AL(k)$-traversal and $d(v_1, v_p) = k$. In this paper we consider k-step Hamiltonicity of bipartite and tripartite graphs. As an application, we found that a 2-step Hamiltonian tour of a graph could sometimes induce a super-edge-magic labeling of the graph.

Keywords: Hamiltonian tour, 2-step Hamiltonian tour, bipartite & tripartite graphs, NP-complete problem, super-edge-magic labeling.

2010 MSC: 05C78, 05C25.

1 Introduction

In 1856, Kirkman wrote a paper [13] in which he considered graphs with a cycle which passes through every vertex exactly once. The dodecahedron (see Figure 1) is a graph with such property that Hamilton played cycle games. Hence, such a graph is said to be Hamiltonian. The Hamiltonicity of a graph is the problem of determining for a given graph whether it contains a path/cycle that visits every vertex exactly once.

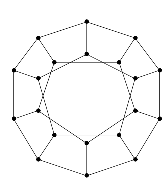

Figure 1: Dodecahedron.

There is no simple characterization on Hamiltonian graphs though they are related to the traveling salesman problem. So there are potential practical applications. In general we know very little about Hamiltonian graphs though their properties have been widely studied. A good reference for recent developments and open problems is [9].

In this paper we consider simple graphs with no loops. For integer $k \geq 1$, a (p,q)-graph $G = (V,E)$ is said to admit an $AL(k)$-traversal if there exist a sequence $(v_1, v_2, \ldots, v_p)$ such that for each $i = 1, 2, \ldots, p-1$, the

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distance between v_i and v_{i+1} is k. We call a graph k-step Hamiltonian (or admits a k-step Hamiltonian tour) if it admits an $AL(k)$-traversal and $d(v_1, v_p) = k$.

For example, the cubic graph in Figure 2 is 2-step Hamiltonian and two others admit an $AL(2)$-traversal but are not 2-step Hamiltonian.

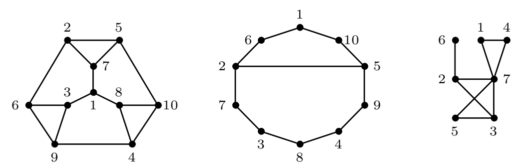


Figure 2: Example on 2-step Hamiltonicity.

There has been much research on Hamiltonicity of bipartite graphs [1, 2, 6, 10, 11, 14]. Clearly, 1-step Hamiltonian is Hamiltonian. In this paper we consider bipartite and tripartite graphs. As an application, we found that a 2-step Hamiltonian tour of a graph could sometimes induce a super-edge-magic labeling of the graph. For terms used but not defined, we refer to [3].

Definition 1.1. *For a graph G, let $D_k(G)$ denote the graph generated from G such that $V(D_k(G)) = V(G)$ and $E(D_k(G)) = \{uv|d(u,v) = k \text{ in } G.\}$.*

Lemma 1.1. *A graph G is k-step Hamiltonian or admits an $AL(k)$-traversal if and only if $D_k(G)$ is Hamiltonian or has a Hamiltonian path, respectively.*

Proof. It follows directly from Definition 1.1.

2 Main Results

We first give a sufficient condition for a graph to admit no k-step Hamiltonian tour.

Theorem 2.1. *Suppose G has a clique subgraph K_p. If $|V(G\backslash K_p)| < p$, then G is not k-step Hamiltonian for all $k \geq 2$.*

Proof. Observe that for any 2 vertices u, v in the clique subgraph K_p of G, $d(u,v) = 1$. Hence, K_p induces an empty graph in $D_k(G)$. If $D_k(G)$ is Hamiltonian, then these p vertices must be not adjacent in a Hamiltonian tour of $D_k(G)$. This implies that we need at least p more vertices to form such a Hamiltonian tour. Since $|V(G\backslash K_p)| < p$, it follows that no Hamiltonian tour exists in $D_k(G)$. By Lemma 1.1, the theorem follows. □

Theorem 2.2. *The vertex gluing of a graph G and an end-vertex of a path of length $n \geq k$ is not k-step Hamiltonian.*

Proof. Let $G(P_n)$ denote the graph such obtained. Observe that $D_k(G(P_n))$ has a cut-vertex and is not Hamiltonian. □

Theorem 2.3. *If graphs G and H are both k-step Hamiltonian, then so is $G \times H$.*

Proof. By Lemma 1.1, G is k-step Hamiltonian if and only if $D_k(G)$ is Hamiltonian. We show that $D_k(G) \times D_k(H)$ is a subgraph of $D_k(G \times H)$. Then any Hamiltonian cycle in $D_k(G) \times D_k(H)$ will also exist in $D_k(G \times H)$ and implies that $G \times H$ is also k-step Hamiltonian. Suppose that edge $e = (u,v)(u,w)$ is an edge in $D_k(G) \times D_k(H)$. Then (v,w) must be an edge in $D_k(H)$, so the distance between v and w in H is k. Let $v = v_0, v_1, v_2, \ldots, v_k = w$ be a length k path from v to w in H. Then (u,v), (u,v_1), (u,v_2),..., (u,w) is a length k path from (u,v) to (u,w) in $G \times H$, so the distance from (u,v) to (u,w) within $G \times H$ is no more than k. Suppose, however, that the distance from (u,v) to (u,w) is less than k in $G \times H$, and let $e_1, e_2, \ldots, e_m$, be a sequence of edges from (u,v) to (u,w) with $m < k$. All edges in this sequence will either be of the form $(z,x)(z,y)$ where xy is an edge in H, or $(x,z)(y,z)$ where xy is an edge in G. Consider the subsequence of edges which are of the first type, $(z,x)(z,y)$. This subsequence must be of the form $(z_1,x_0)(z_1,x_1), (z_2,x_1)(z_2,x_2), \ldots, (z_n,x_{n?})(z_n,x_n)$, where $x_0 = v, x_n = w$, and $n = m < k$. Furthermore, $x_0x_1, x_1x_2, \ldots, x_{n?}x_n$ must be a sequence of edges in H from $v = x_0$ to $w = x_n$, which has length n, where $n < k$. This contradicts the fact that the distance from v to w in H is actually k. Therefore the distance from

(u,v) to (u,w) in $G \times H$ is also k, and so $e = (u,v)(u,w)$ is also an edge of $D_k(G \times H)$; the argument for edges of the form $e = (u,v)(w,v)$ is identical. Since all edges and vertices of $D_k(G) \times D_k(H)$ are also in $D_k(G \times H)$, $D_k(G) \times D_k(H)$ is a subgraph of $D_k(G \times H)$. Since G and H are k-step Hamiltonian, $D_k(G) \times D_k(H)$ is Hamiltonian, and so is $D_k(G \times H)$, implying that $G \times H$ is k-step Hamiltonian. □

A Hamiltonian graph need not be 2-step Hamiltonian. The simplest example is the complete bipartite graph $K(2,2)$ that does not admit an $AL(2)$-Hamiltonian traversal, and hence cannot be 2-step Hamiltonian.

Theorem 2.4. *All bipartite graphs are not k-step Hamiltonian for even $k \geq 2$.*

Proof. Suppose $G = (V, E)$ is bipartite graph with bipartition (X, Y). If $k \geq 2$ is even, the vertex in X cannot connect with vertex in Y, vice versa, in $D_k(G)$. Thus $D_k(G)$ is a disconnected graph with two components X and Y. Hence $D_k(G)$ cannot have a Hamiltonian path. By Lemma 1.1, G is not k-step Hamiltonian. □

We now give a necessary and sufficient condition for cycles to admit a k-step Hamiltonian tour.

Theorem 2.5. *For integers $n \geq 3$ and $k \geq 2$, the cycle C_n is k-step Hamiltonian if and only if $n \geq 2k+1$ and $gcd(n,k) = 1$.*

Proof. If $n \leq 2k$, we have either $\operatorname{diam}(C_n) < k$ or $D_k(C_n)$ is disconnected. Hence, C_n is not k-step Hamiltonian. We may now assume that $n \geq 2k+1$.

Without loss of generality, we may assume that a k-step Hamiltonian tour of C_n is given by the sequence $u_1, u_{k+1}, u_{2k+1}, \ldots, u_{(n-1)k+1}$. Note that $\{1, k+1, 2k+1, 3k+1, \ldots, (n-1)k+1\} \pmod{n}$ is a set of distinct integers if and only if $ik+1 \not\equiv jk+1 \pmod{n}$ for $0 \leq i < j \leq n-1$ if and only if $(j-i)k \not\equiv 0 \pmod{n}$ if and only if $k/n \neq r/(j-i)$ for some integer r if and only if $gcd(n,k) = 1$. Hence, the theorem holds and the k-step Hamiltonian tour of C_n is obtained. □

Theorem 2.6. *The cylinder graph $C_n \times P_m$ is 2-step Hamiltonian for odd $n \geq 3$ and all $m \geq 3$.*

Proof. Case 1. $n = 3$. This case is handled separately since the three vertices in any 3-cycle within $C_3 \times P_m$ are distance 1 from each other. Figure 3 shows 2-step Hamiltonian tours for $C_3 \times P_2$ and $C_3 \times P_3$. Figure 4 shows 2-step Hamiltonian tours for $C_3 \times P_{4k}$. It is based on the 2-step Hamiltonian tour for $C_3 \times P_2$. Here vertices are labeled (a,b) and we may denote edges by listing their vertices: $(a,b)(c,d)$. Then to modify the 2-step Hamiltonian tour for $C_3 \times P_{4k}$ to one for $C_3 \times P_{4k-2}$, replace edge $(1,4k-2)(2,4k-1)$ by $(1,4k-3)(2,4k-2)$, shown as a dotted line, and also remove edge $(2,4k-2)(2,4k)$. The cases in which $n = 3$ and m is odd are handled with similar constructions, based instead on the 2-step Hamiltonian tour for $C_3 \times P_3$ also as shown below. The diagram shows the 2-step Hamiltonian tour for $C_3 \times P_{4k+3}$ which may be modified for the cases $C_3 \times P_{4k-1}$ by adding edge $(1,4k)(2,4k-1)$, again shown shown as a dotted line.

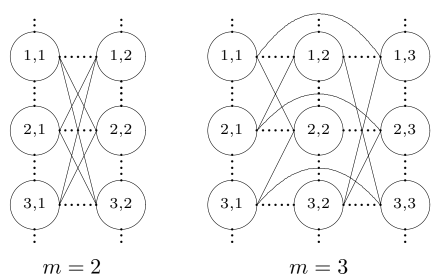
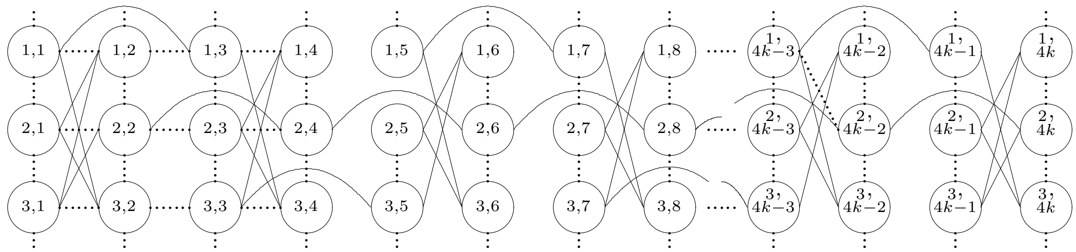


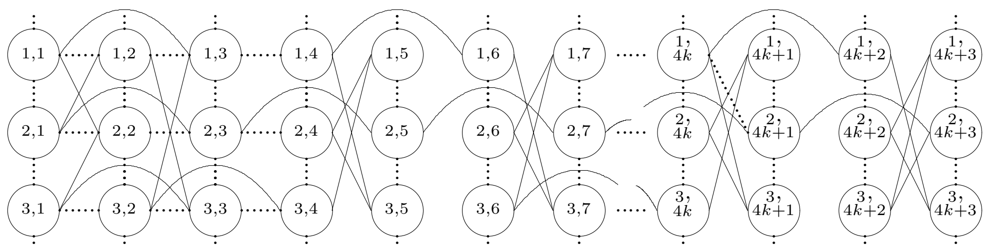
Figure 3: A 2-step Hamiltonian tour in $C_3 \times P_m, m = 2, 3$.

Case 2. $n = 2k+1 \geq 5$. In this case, we consider two subcases.

Subcase 2.1. $m = 2j+1 \geq 3$. Figure 5 gives a 2-step Hamiltonian cycle of this subcase. Note that we have partitioned the vertices in a checkerboard pattern so those whose coordinates have even sum are shown by circular vertices, and those whose coordinates have odd sum are shown by square vertices. The circular vertices compose one cycle and the square vertices compose another, except that the two cycles cross over and connect through the edges $(2j+1, 2k+1)(2j, 1)$ and $(2j+1, 2k)(2j+1, 1)$.



$C_3 \times P_{4k}$



$C_3 \times P_{4k+3}$

Figure 4: A 2-Hamiltnoian tour in $C_3 \times P_m$.

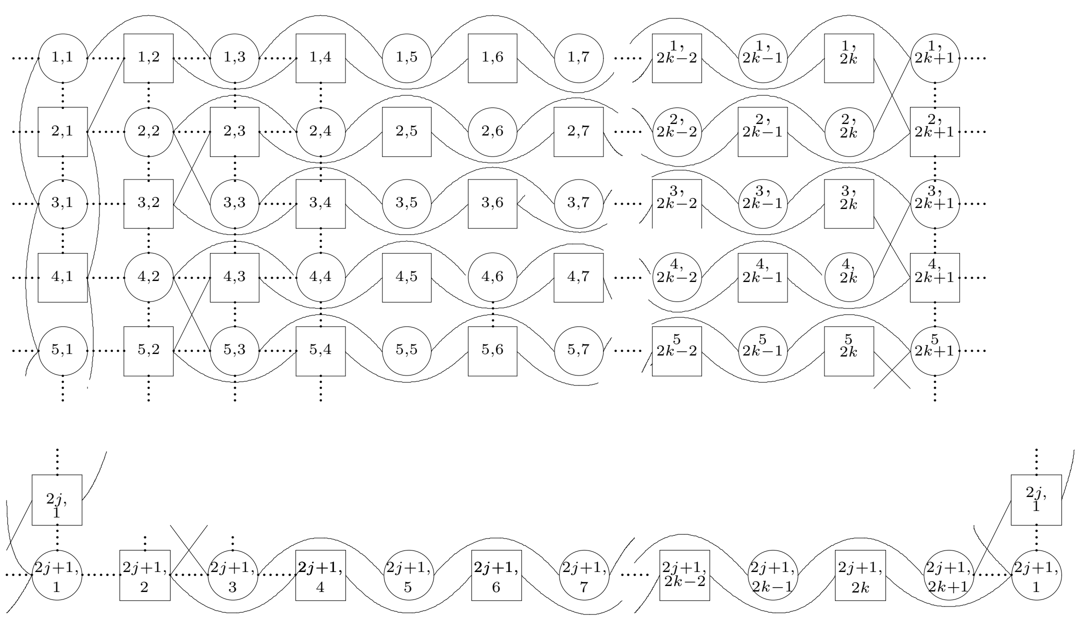


Figure 5: A 2-step Hamiltonian tour in $C_{2k+1} \times P_{2j+1}$.

Subcase 2.2. $m = 2j \geq 2$. Figure 6 gives a 2-step Hamiltonian cycle of this subcase. We have again partitioned the vertices in a checkerboard pattern so those whose coordinates have even sum are shown by circular vertices, and those whose coordinates have odd sum are shown by square vertices. The circular vertices compose one cycle and the square vertices compose another, except in this case the two cycles cross over and connect through the edges $(2j, 2k)(2j, 1)$ and $(2j-1, 2k+1)(2j-2, 1)$. Note that the vertices shown in the rightmost column are identical to those at the bottom of the first column.

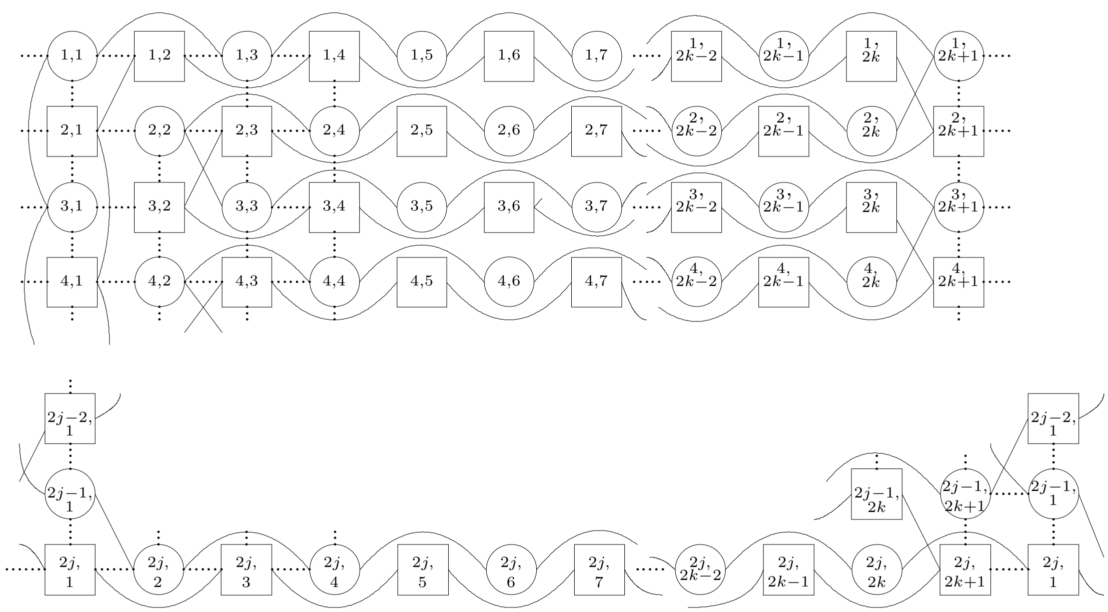


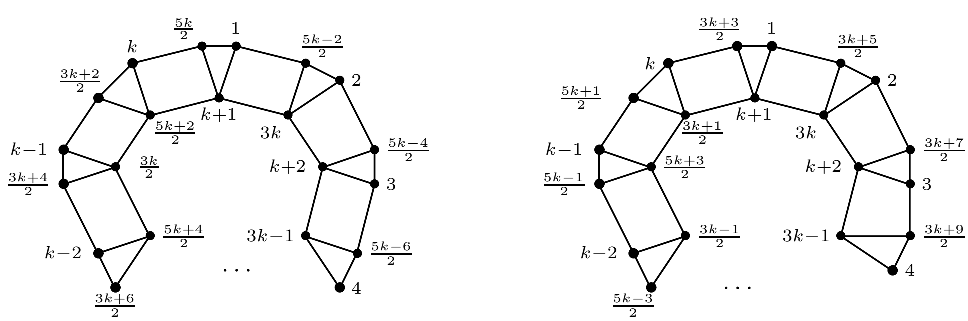
Figure 6: A 2-step Hamiltonian tour in $C_{2k+1} \times P_{2j}$.

□

Since $C_n \times P_m$ is a subgraph of $C_n \times C_m$, the same 2-step Hamiltonian cycles work for $C_n \times C_m$, when n is odd, and we have

Corollary 2.1. *. The graph $C_n \times C_m$ is 2-step Hamiltonian for odd n and all m.*

Let $D(n)$ denote the tripartite donut graph shown in Figure 7 with a given vertex labeling.



(a) $n = 3k \cong 0 \bmod (6)$, $k \geq 4$ even (b) $n = 3k \cong 4 \bmod (6)$, $k \geq 5$ odd

Figure 7: A 2-step Hamiltonian tour in $D(n)$.

Theorem 2.7. *The vertex labeling in graph $D(n)$ gives a 2-step Hamiltonian tour for all $n = 3k, k \geq 4$.*

A ring-worm is a unicyclic graph $U_n(a_1, a_2, \ldots, a_n)$ obtained from a cycle C_n with $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ by identifying vertex v_i to the center, c_i of a star S_i having $a_i + 1 \geq 1$ vertices, $\{c_i, u_{i,1}, u_{i,2}, \ldots, u_{i,a_i}\}$. The ring-worm has $n + a_1 + a_2 + \ldots + a_n$ vertices and edges, respectively. We can arrange the vertices of the ring-worm as in Figure 8.

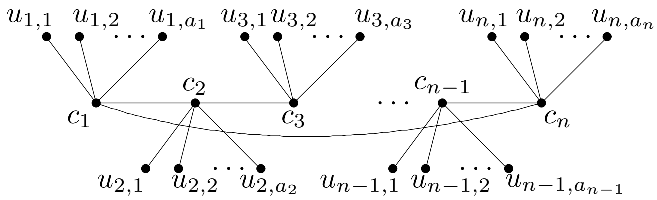


Figure 8: Ring worm graph.

Theorem 2.8. *If $n = 3$ and $a_i > 0$ $(i = 1,2,3)$, or $n \geq 5$ is odd and $a_i \geq 0$ $(1 \leq i \leq n)$, the ring worm $U_n(a_1, a_2, \ldots, a_n)$ is 2-step Hamiltonian.*

Proof. The case $n = 3$ is obvious. Without loss of generality, we assume that $n \geq 5$ and not all $a_i = 0$. Suppose $n = 2s+1, s \geq 2$. We can label the vertices by consecutive integers as described below to get a 2-step Hamiltonian tour for the graph:
$f(c_{2i+1}) = a_2 + a_4 + \ldots + a_{2i} + i + 1$, for $i = 0,1,2,\ldots,s$,
$f(c_{2i+2}) = f(c_{2s+1}) + a_1 + a_3 + \ldots + a_{2i+1} + i + 1$ for $i = 0,1,2,\ldots,s-1$,
$f(u_{2i+2,j}) = f(c_{2i+1}) + j$ for $i = 0,1,2,\ldots,s-1, j = 1,2,\ldots,a_{2i+2}$,
$f(u_{1,j}) = f(c_{2s+1}) + j, j = 1,2,\ldots,a_1$,
$f(u_{2i+1,j}) = f(c_{2i}) + j$ for $i = 1,2,\ldots,s-1, j = 1,2,\ldots,a_{2i+1}$. □

We next define two families of cubic graphs. Let n be a positive integer. The Möbius ladder (also known as the Möbius wheel) is the cycle C_{2n}, with n additional edges joining diagonally opposite vertices. We will denote this graph by M_{2n}, and its vertices by $v_1, v_2, \ldots, v_{2n}$. Then the edges are $v_1v_2, v_2v_3, \ldots, v_{2n}v_1$ of the cycle, and the n diagonals are $v_1v_{n+1}, v_2v_{n+2}, \ldots, v_nv_{2n}$. Figure 9 shows the Möbius ladder M_{2n} for $n = 3,4$, drawn in both the circulant form and the ladder form.

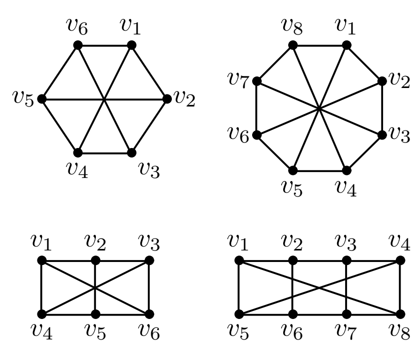


Figure 9: Möbius ladder for $n = 3,4$.

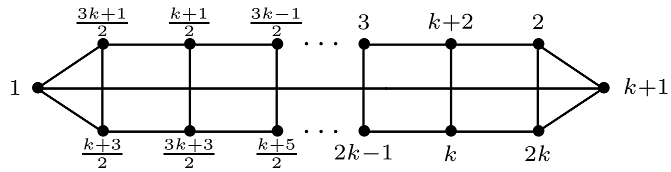
Observe that for odd n, M_{2n} is not 2-step Hamiltonian since it is bipartite. For even n, M_{2n} is tripartite.

Theorem 2.9. *For $m \geq 1$, M_{4m} is 2-step Hamiltonian.*

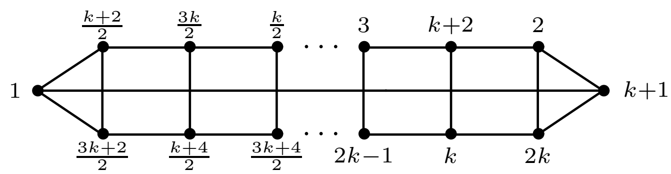
Proof. A 2-step Hamiltonian tour is given by the sequence $v_1, v_3, v_5, \ldots, v_{4m-1}$, v_{2m}, $v_{2m-2}, \ldots$, v_2, v_{4m}, v_{4m-2}, $\ldots$, v_{2m+2}, v_1. □

We now consider the cubic turtle shell graph, $TS(n)$, n even, with a given vertex labeling as shown in Figure 10.

Theorem 2.10. *The vertex labeling of the graph $TS(n)$ is a 2-step Hamiltonian tour for all $n = 2k, k \geq 3$.*



(a) $n = 2k \cong 0$ (mod4), $k \geq 4$ even



(b) $n = 2k \cong 2$ (mod4), $k \geq 3$ odd

Figure 10: Graph $TS(n), n$ even.

3 An Application

For a (p,q)-graph G, a labeling of the vertices and edges of G given by bijections $f : V(G) \to \{1,2,\ldots,p\}$ and $f^+ : E(G) \to \{p+1, p+2,\ldots,p+q\}$ is called a super-edge-magic (SEM) labeling if $f(u) + f(v) + f^+(uv)$ is a constant for every edge uv in $E(G)$. Such a graph is called SEM.

Theorem 3.11. *([4, 7]) A graph G is SEM if and only if it admits a bijection $f(V(G))$ such that $\{f(u) + f(v)\}$ consists of q consecutive integers.*

Observe that a 2-step Hamiltonian labeling for each odd cycle and the ring worm with odd cycle correspond to a vertex labeling that induces an edge labeling f^+ such that the edge labels form a sequence of consecutive integers. However, we are not able to find another 2-step Hamiltonian labeling that corresponds to a SEM labeling.

Problem 1. *Does there exist infinitely many families of 2-step Hamiltonian graphs whose labeling corresponds to a SEM labeling?*

The problem of determining whether a graph is Hamiltonian is NP-complete even for planar graphs. In 1972, Karp [12] proved that finding such a path in a directed or undirected graph is NP-complete. Later, Garey and Johnson [8] proved that the directed version restricted to planar graphs is also NP-complete, and the undirected version remains NP-complete even for cubic planar graphs. In 1980, Akiyama, Nishizeki, and Saito [1] showed that the problem is NP-complete even when restricted to bipartite graphs. We end this paper with the following conjecture and problem.

Conjecture 1. *The 2-step Hamiltonian problem for tripartite graphs is NP-complete.*

Problem 2. *Study the k-step Hamiltonicity of complete multipartite graph with certain edges deleted.*

References

[1] T.S.N. Akiyama, T. Nishizeki, NP-completeness of the Hamiltonian Cycle Problem for Bipartite Graphs, *Journal of Information Processing*, 3(1980), 73–76.

[2] J.-C. Bermond, Hamiltonian Graphs. In *Selected Topics in Graph Theory*. Edited by L. W. Beineke and R. J. Wilson. Academic, London(1978), 127–167.

[3] G. Chartrand and P. Zhang, Introduction to Graph Theory, Walter Rudin Student Series in Advanced Mathematics, McGraw-Hill, 2004.

[4] Z. Chen, On super edge-magic graphs, *J. of Comb. Math. Comb. Comput.,* 38 (2001), 53–64.

[5] V.V. Dimakopoulos, L. Palios and A.S. Poulakidas, On the Hamiltonicity of the Cartesian Product, available online: http://paragroup.cs.uoi.gr/Publications/120ipl2005.pdf (accessed October 2012).

[6] M.N. Ellingham, J.D. Horton, Non-Hamiltonian 3-connected Cubic Bipartite Graphs, *J. of Comb. Theory B*, 34(3)(1983), 350–353.

[7] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.,* 231(2001), 153–168

[8] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, New York, 1979.

[9] R. Gould, Advances on the Hamiltonian Problem, A Survey, *Graphs and Combinatorics*, 19(2003), 7–52.

[10] D. Holton and R.E.L. Aldred, Planar Graphs, Regular Graphs, Bipartite Graphs and Hamiltonicity, *Australasian J. of Combinatorics,* 20(1999), 111–131.

[11] A. Itai, C.H. Papadimitriou and J.L. Szwarcfiter. Hamilton paths in grid graphs. *SIAM Journal on Computing*, 11(4)(1982), 676–686

[12] R.M. Karp, Reducibility among combinatorial problems, *Complexity of Computer Computations*, (1972) 85–103.

[13] T.P. Kirkman, On the representation of polyhedra, *Phil. Trans. Royal Soc.*, 146(1856), 413–418.

[14] J. Moon and L. Moser, On Hamiltonian bipartite graphs, *Israel J. of Math.*, 1(3)(1963), 163–165.

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