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Homeotoxal and Homeohedral Tiling Using Pasting Scheme

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Abstract

The art of tiling has been in practice from the beginning of human civilization. Intricate tiling patterns were used to decorate and cover floors and walls [1]. Robinson Thamburaj introduced the notion of pasting rules [4] and studied the construction of kolam tiling. In this paper we study Pasting Scheme (*PS*) using special sets of tiles namely Wang tile, Escher tile and Truchet tile. The scheme is also used to construct the five types of Homeotoxal tiling [1] and the eleven types of Homeohedral tiling [1]. Generalizing Pasting Scheme we define Tabled Pasting Scheme (*TPS*) where pasting rules are grouped in tables and the tables of rules are used as many times as given in the scheme for the successive construction of a tiling.

Keywords: Pasting rules, Tiling, Homeotoxal tiling, Homeohedral tiling.

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1 Introduction

Mathematical tiling theory provides an insight in geometry and computation. In the construction of complex tiling, the desired tiling rises spontaneously due to the various parameters defined in the system. Computational mechanisms play an important role in understanding how complex tiling is formed. The Pasting Scheme (*PS*) and Tabled Pasting Scheme (*TPS*) are syntactic methods, introduced by Robinson etc in [4], to generate plane tiling. These techniques use the notion of pasting rule [4] that allows the edges of the corresponding tiles to get glued or attached at the specified edges and thus generating interesting two dimensional tiling. In section 3, we study the construction of Wang tiling, Escher tiling and truchet tiling using Pasting Scheme. Homeotoxal tiling are tiling with the property that every edge is of the same type. That is every edge E of the tiling has the number of edges n and m for the two tiles which contain the edge E and the number of edges incident with (valences) the two vertices of E are p and q. In section 4 we study the construction of the five types of homeotoxal tiling using Pasting scheme. In section 5 we study the construction of the eleven types of homeohedral tiling using Pasting scheme. In section 6, we generalize the Pasting scheme and define Tabled Pasting Scheme.

2 Preliminaries

A Tile is a two dimensional topological disk (region) whose boundary is a simple closed curve, whose ends join up to form a loop without crossing or branches. A plane tiling is a countable family of topological disks

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which cover the plane without gaps or overlaps. A finite set of tiles T is said to have valid tiling or tiling $\Im$ if the infinite plane can be covered without gaps and overlaps. Tiling, in which the edge of a tile coincides entirely with the edge of a bordering tile, is called edge to edge tiling. The valence of a vertex v in a tiling $\Im$ is the number of edges incident with it.

Definition 2.1. *[1] A tiling $\Im$ is called Normal tiling if every tile of $\Im$ is a topological disk, the intersection of every two tiles of $\Im$ is a connected set and the tiles are uniformly bounded.*

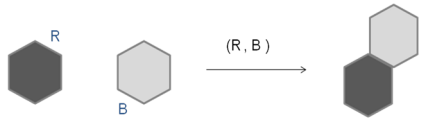
Definition 2.2. *[1] A mapping $\Phi : E^2 \to E^2$ of the plane onto itself is called a homeomorphism or a topological transformation if it is one-to-one and bicontinous.*

3 Pasting Scheme for Tiling

Wang tiling uses the principle of matching colors between the tile edges for tiling. Generalizing the principle of Wang tiling we define pasting rules [4] for tiling of the Euclidean plane. The edge of a tile can be glued with another edge if there is a pasting rule allowing the edges of tiles to be glued. In this section a model is defined, namely Pasting Scheme, with pasting rules for plane tiling. The model is explained using Wang tiling, Escher tiling and Truchet tiling.

Definition 3.1. *[4] A pasting rule (A, B), is an unordered pair of edge labels of tiles M and N which allows the edge to edge pasting of the two tiles.*

If R is the label of the upper right edge of a hexagonal tile and B is the label of the lower left edge of another hexagonal tile, then an application of the rule (R, B) pastes side by side (or joins edge to edge) the two tiles. When tiles are attached by pasting rules it results in the formation of Tiling.



By applying pasting rules to the edges of a tile or tiling a new tiling t_{i+1} is said to be obtained from the tile t_i. It is symbolically denoted as $t_i \Rightarrow^* t_{i+1}$.

Definition 3.2. *A Pasting Scheme (PS) is a 3 tuple $G = (\sum, P, t_0)$, where $\sum$ is a finite non empty set of tiles with labeled edges, P is a finite set of pasting rules and t_0 is the axiom tile.*

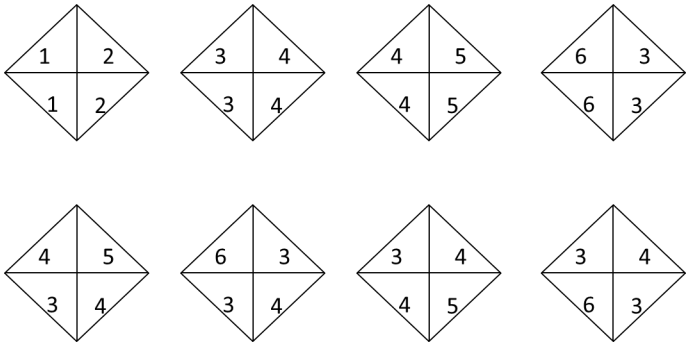
Tiling is made up of tiles, 'glued' (or pasted edge to edge) together. The pasting rule defined above between two tiles is extended for pasting a tile with a patch of tiles. A tile may be pasted with a patch of tiles along the boundary edges when there are pasting rules abutting the tile with the patch of tiles. Rotation of tile or a patch of tiles is not allowed while pasting two tiles / patch of tiles.

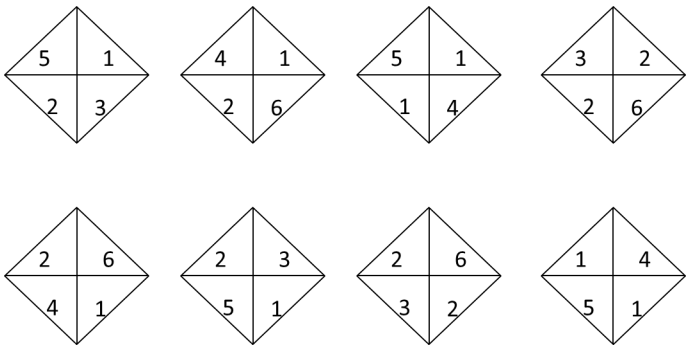
A patch of tile τ' is generated from a patch of tile τ by applying the pasting rules in parallel to all the boundary edges of τ where pasting is possible. Note that the labels of pasted edges in a pattern are ignored once the tiles are pasted. Rotation of tile is not allowed in the PS while pasting two tiles.

The patch of tiles constructed by applying the pasting rule $n(n \geq 1)$ number of time forms the language of tiling generated by the PS G and it is denoted by $L(G) = \{\tau / t_0 \Rightarrow_* \tau\}$.

Example 3.1. *A Wang tile [1] is a square tile with colored edges which must be placed edge-to-edge, in such a way that adjacent edges must have matching edges.*

In 1966, R. Berger discovered the first aperiodic set of tiles which contained 20, 426 wang tiles. Conway found a smaller aperiodic tile set with 16 tiles from Ammann tiles as given below.





Consider the Pasting Scheme $G_3 = (\sum, P, t_0)$ to generate the aperiodic Wang tiling, where $\sum$ is the set of tiles shown in figure, $P = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$, $t_0 =$  . The pasting rules in P are defined in such a way that edges of same number are glued together to form the Wang tiling.

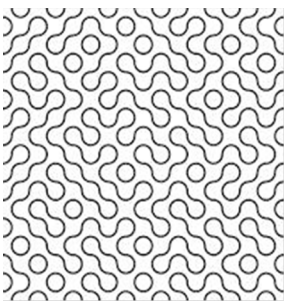
Example 3.2. *Consider the Pasting Scheme $G_1 = (\sum, P, t_0)$ to generate the Truchet tiling [8], where* $\sum =$ ,

$t_0 =$ 

,

$P = \{(A, G), (C, E), (B, D), (F, H)\}$.

The pasting rules in P are defined in such a way that a continuous arc is formed in the tiling. The truchet tiling generated by the pasting scheme G_1 is shown in Figure 3.1.

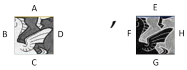


(a) Truchet tiling



(b) Escher tiling

Figure 3.1: Truchet and Escher Tiling

Example 3.3. *Consider the Pasting Scheme $G_2 = (\sum, P, t_0)$ to generate the Escher tiling, where* $\sum =$ 

, $t_0 =$ 

, *and* $P = \{(A, G), (C, E), (B, H), (D, F)\}$.

The pasting rules in P are defined in such a way that interlocked black and white horse pattern is formed in the tiling. The Escher tiling pattern generated by the pasting scheme G_2 is shown in Figure 3.1.

4 Homeotoxal Tiling using PS

Definition 4.1. *A tiling $\Im$ is called Homeotoxal or topologically edge-transitive if it is a normal tiling and if for any two edges E_1 and E_2 of the tiling $\Im$ there exists a homeomorphism of the plane that maps $\Im$ onto $\Im$ and E_1 onto E_2.*

If $\Im$ is a Homeotoxal tiling then every edge E of $\Im$ has the same symbol $n, m; p, q$ where n, m are the number of edges of the two tiles of $\Im$ which contain the edge E and p, q are the valences of the two vertices of E, we say that the tiling $\Im$ is of the type $< n, m; p, q >$. If $\Im$ is a homeotoxal tiling then it is of one of the five types $< 3^2; 6^2 >, < 6^2; 3^2 >, < 3.6; 4^2 >, < 4^2; 3.6 >, < 4^2; 4^2 >$ as shown in Figure 4.1. In the following we define pasting schemes to construct the homeotoxal tiling.

The pasting scheme $G = (\sum, P, t_0)$ where $\sum =$ 

, $t_0 =$ 

and $P = \{(T, T)\}$ generates the tiling type $< 3^2; 6^2 >$.

The pasting scheme $G = (\sum, P, t_0)$ where $\sum =$ 

, $t_0 = \sum$ and $P = \{(H, H)\}$ generates the tiling type $< 6^2; 3^2 >$.

The pasting scheme $G = (\sum, P, t_0)$ where $\sum =$ 

, $t_0 =$ 

and

$P = \{(H, T)\}$ generates the tiling type $< 3.6; 4^2 >$.

The pasting scheme $G = (\sum, P, t_0)$ where $\sum = \left\{ \right\}$, $t_0 =$ and $P = \{(D, R), (D, L), (R, L)\}$ generates the tiling type $< 4^2; 6.3 >$.

The pasting scheme $G = (\sum, P, t_0)$ where $\sum = \left\{ \right\}$, $t_0 = \sum$ and $P = \{(S, S)\}$ generates the tiling type $< 4^2; 4^2 >$.

From the above discussion we have the following lemma.

Lemma 4.1. *The class of homeotoxal tiling can be constructed by pasting scheme.*

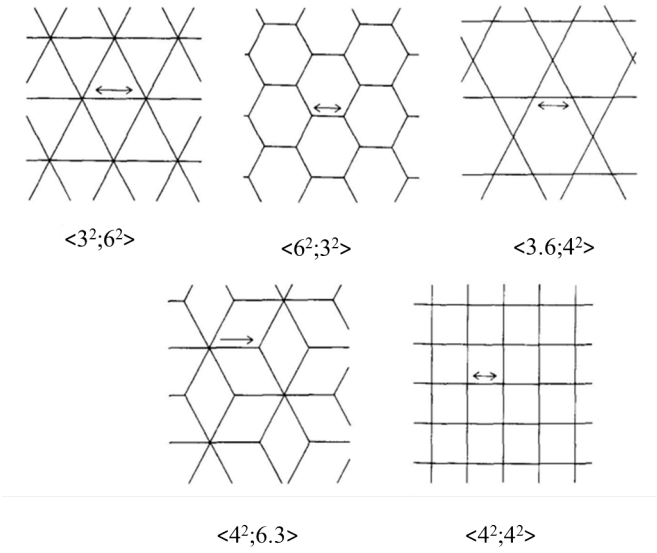


Figure 4.1: Homeotoxal Tiling

5 Homeohedral Tiling using PS

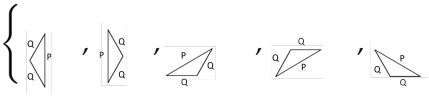
Definition 5.1. *A tiling $\Im$ is called Homeogonal or topologically tile-transitive if it is a normal tiling and is such that for any two tiles T_1 and T_2 of the tiling $\Im$ there exists a homeomorphism of the plane that maps $\Im$ onto $\Im$ and T_1 onto T_2.*

A tile T in a tiling is of valence-type $[v_1, v_2, \ldots, v_k]$ when T has k vertices with valences $v_1, v_2, \ldots, v_k$. The tiles in a homeohedral tiling are all of the same valence-type. If $\Im$ is a homeohedral tiling then it is one of the eleven types as shown in Figure 5.1 where each type of tiling is identified with the valence-type of the tiling.

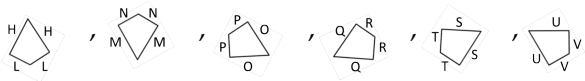
The homeohedral type of tiling of $[3^6]$, $[6^3]$, $[4^2]$, $[6, 3, 6, 3]$ are same as the homeotoxal tiling type $< 6^2; 3^2 >$, $< 3^2; 6^2 >$, $< 4^2; 4^2 >$ and $< 4^2; 6, 3 >$. In the following we discuss the construction of the remaining types of homeohedral tiling.

The pasting scheme $G = (\sum, P, t_0)$ where $\sum = \left\{ \right\}$, $t_0 =$ and $P = \{(A, B), (U, L), (U, R), (D, R), (D, L), (L, R)\}$ generates the tiling type $[8, 4, 8]$.

The pasting scheme $G = (\sum, P, t_0)$ where $\sum = \left\{ \right.$  $\left. \right\}$, $t_0 =$  and $P = \{(A, A), (S, S), (T, T), (U, D)\}$ generates the tiling type $[3, 3, 4, 4, 3]$.

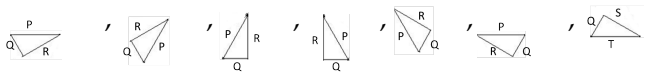
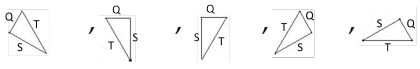
The pasting scheme $G = (\sum, P, t_0)$ where $\sum = \left\{ \right.$  $\left. \right\}$,

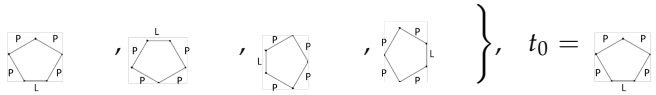
$t_0 =$  and $P = \{(P, P), (Q, Q)\}$ generates the tiling type $[3, 12^2]$.

The pasting scheme $G = (\sum, P, t_0)$ where $\sum = \left\{ \right.$  $\left. \right\}$,

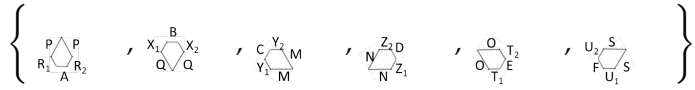
$t_0 =$ 

and $P = \{(H, S), (H, U), (O, S), (Q, U), (M, O), (M, Q), (L, P), (L, R), (N, T), (N, V), (T, V), (R, P)\}$ generates the tiling type $[4, 3, 4, 6]$.

The pasting scheme $G = (\sum, P, t_0)$ where $\sum = \left\{ \right.$   $\left. \right\}$, $t_0 =$  and $P = \{(Q, Q), (R, R), (P, P), (S, S), (T, T)\}$ generates the tiling type $[4, 6, 12]$.

The pasting scheme $G = (\sum, P, t_0)$ where $\sum = \left\{ \right.$  $\left. \right\}$, $t_0 =$

and $P = \{(P, P), (L, P)\}$ generates the tiling type $[3, 4, 3, 4, 3]$.

The pasting scheme $G = (\sum, P, t_0)$ where $\sum = \left\{ \right.$  $\left. \right\}$,

$t_0 =$  and $P = \{(P, O), (P, S), (N, O), (S, M), (M, Q), (N, Q), (Y_2, A), (R_2, X_1), (R_1, D), (B, T_1),$ $(X_2, U_1), (T_2, Y_1), (E, U_2), (F, X_2), (U_1, Z_2), (Z_1, C)\}$ generates the tiling type $[3, 3, 6, 3, 3]$.

From the above discussion we have the following result.

Lemma 5.1. *The class of homeohedral tiling can be generated by pasting scheme.*

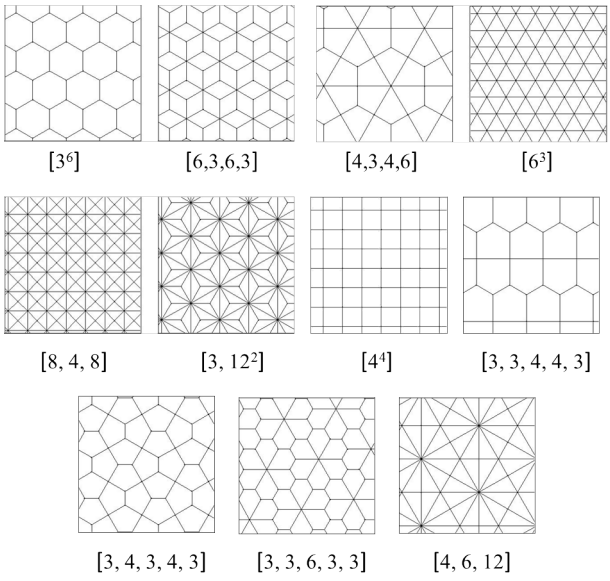


Figure 5.1: Homeohedral Tilings

6 Tabled Pasting Scheme for Tiling

Generalizing Pasting Scheme we define Tabled Pasting Scheme (TPS) where pasting rules are grouped in tables and the tables of rules are used as many times as given in the scheme for the successive construction of a tiling.

Definition 6.1. *A Tabled Pasting Scheme (TPS) is a 4 tuple $G = (\sum, P, t_0, C)$, where $\sum$ is a finite non empty set of tiles with labeled edges, P is a finite set of tables $\{T_1, T_2, \ldots, T_k\}$ and each table T_i contain pasting rules to glue the tiles, t_0 is the axiom tile and C is a control over P.*

By this scheme the tiling is constructed by applying the tables of pasting rules according to the order of application of table of rules prescribed by a control language C. The control language C is an arbitrary language over the table labels $\{T_1, T_2, \ldots, T_k\}$. The rules in the table are applied to the tiling in a maximally parallel manner (wherever applicable). The pasting rules are applied arbitrary number of times to the edges of the tiling or only once according to its applicability. The set of all tiling constructed from the axiom tile t_0 by the successive application of tables of pasting rules according to the control C, constitutes the language $L(G)$ of TPS.

Let F_{PS} and F_{TPS} denote the families of language of tiling generated by Pasting Scheme (PS) and Tabled Pasting Scheme (TPS) respectively. It can be clearly seen that a TPS with a single table of pasting rules T_1 and a control $C = \{(T_1)^n / n \geq 1\}$ is the PS. Hence the family of language of tiling constructed by a PS is contained in the language of tiling constructed by a TPS. This gives the following result.

Lemma 6.1. $F_{PS} \subset F_{TPS}$.

7 Conclusion

In this paper we have exhibited the construction of Homeotoxal tiling and Homeohedral tiling using Pasting scheme. Pasting scheme is generalized as Tabled Pasting Scheme. It is of interest to study and determine whether the Tabled Pasting Scheme can construct Aperiodic Wang tiling [1] and other complex tiling.

References

[1] Branko Gruenbaum, G.C. Shephard, *Tilings and Patterns*, W.H. Freeman Company, Newyork, 1986.

[2] S. Jebasingh, T. Robinson, Tiling Pattern using Pasting Schemes, *Proc. of National Conference on Cryptography, Automata and Learning Theory*, 2009, pp. 97-106, Narosa Publishing House, New Delhi.

[3] T. Robinson, V.R. Dare, K.G. Subramanian, A Parallel Pattern Generating System, *Proc. of the 6th International Workshop on Parallel Image Processing and Analysis*, Madras (1999) 23-32.

[4] T. Robinson, Extended Pasting Scheme for Kolam Pattern Generation, *Forma 22* (2007) 55 -64 .

[5] T. Kalyani, K. Sasikala, V.R. Dare, P.J. Abisha, T. Robinson, Triangular Pasting System, *Formal Models, Languages and Application, Series in Machine Perception and Artificial Intelligence*, Vol 66 (2006), 195-211.

[6] K. Krithivasan, R. Rama, *Introduction to Formal Languages, Automata and Computation*, Pearson Education, New Delhi, 2009.

[7] Peter Linz, *An Introduction to Formal Languages and Automata*, Narosa Publishing House.

[8] Cameron Browne, Duotone-Truchet like tilings, *Journal of Mathematics and Arts*, Vol 2, No 4, December 2008, 189-196.

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