Abstract
Let $G$ be a simple graph with $p$ vertices and $q$ edges. A $V$-super vertex magic labeling is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ such that $f(V(G)) = \{1, 2, \ldots, p\}$ and for each vertex $v \in V(G)$, $f(v) + \sum_{uv \in N(v)} f(uv) = M$ for some positive integer $M$. A $V_k$-super vertex magic labeling ($V_k$-SVML) is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ with the property that $f(V(G)) = \{1, 2, \ldots, p\}$ and for each $v \in V(G)$, $f(v) + w_k(v) = M$ for some positive integer $M$. A graph that admits a $V_k$-SVML is called $V_k$-super vertex magic. This paper contains several properties of $V_k$-SVML in graphs. A necessary and sufficient condition for the existence of $V_k$-SVML in graphs has been obtained. Also, the magic constant for $E_k$-regular graphs has been obtained. Further, we study some classes of graphs such as cycles, complement of cycles, prism graphs and a family of circulant graphs which admit $V_2$-SVML.

Keywords
Vertex magic total labeling, super vertex magic total labeling, $V_k$-super vertex magic labeling, $E_k$-regular graphs, circulant graphs.

AMS Subject Classification
05C78.

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1. Introduction
Throughout this paper, we consider only finite, simple and undirected graphs. The set of vertices and edges of a graph $G(p,q)$ will be denoted by $V(G)$ and $E(G)$ respectively, $p = |V(G)|$ and $q = |E(G)|$. For graph theoretic terminology, we follow [2].

A graph labeling is a mapping or a function that carries a set of graph elements (usually vertices and/or edges) into a set of numbers (usually integers). Lot of labelings have been defined and studied by many authors and an excellent survey of graph labeling can be found in [1].

In 2002, MacDougall et al. [3] introduced the notion of vertex magic total labeling (VMTL) in graphs. A VMTL of the graph $G$ is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ such that for each vertex $v \in V(G)$, $f(v) + \sum_{uv \in N(v)} f(uv) = M$ for some positive integer $M$ [3]. This constant is called as the magic constant of VMTL of $G$. They studied some basic properties of vertex magic graphs and showed some families of graphs having a VMTL.

In 2004, MacDougall et al. [4] defined the super vertex-magic total labeling (SVMTL) in graphs. They call a VMTL is super if $f(V(G)) = \{1, 2, \ldots, p\}$. In this labeling, the smallest labels are assigned to the vertices.

This paper generalizes the definition of SVMTL and define a new labeling called $V_k$-super vertex magic labeling. Let $G(V,E)$ be a graph and $k$ be an integer such that $1 \leq k \leq \text{diam}(G)$. For $e \in E(G)$, we define $E_k(e)$ as the set of all vertices which are at a distance at most $k$ from $e$. Also $E_k(v)$ denotes the set of all edges which are at a distance at most $k$ from $v$. Note that if $uv$ is an edge, then the vertices $u$ and $v$ are at distance 1 from the edge $uv$. The graph $G$ is said to be $E_k$-regular with regularity $r$ if and only if $|E_k(e)| = r$ for some integer $r \geq 1$ and for all $e \in E(G)$. Note that all nontrivial graphs are $E_1$-regular. For a vertex $v \in V(G)$, we denote $w_k(v) = \sum_{e \in E_k(v)} f(e)$. Consider the following graph $G(V,E)$, where $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$
and $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.

![Fig 1: G](image)

Table 1 gives $E_k(v)$ and $E_k(e)$ for $k = 2$.

<table>
<thead>
<tr>
<th>$E_2(v)$</th>
<th>$E_2(e)$</th>
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<tbody>
<tr>
<td>${v_1, v_2, v_3, v_4}$</td>
<td>${e_1, e_4}$</td>
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<td>${v_1, v_2, v_3, v_4}$</td>
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<td>${v_1, v_2, v_3, v_4}$</td>
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Table 1

A $V_k$-super vertex magic labeling ($V_k$-SVML) is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ with the property that $f(V(G)) = \{1, 2, \ldots, p\}$ and for every $v \in V(G)$, $f(v) + w_k(v) = M$ for some positive integer $M$. This constant is called as the magic constant of $V_k$-SVML of $G$. A graph that admits a $V_k$-SVML is called $V_k$-super vertex magic ($V_k$-SVM).

This paper contains several properties of $V_k$-SVML in graphs. A necessary and sufficient condition for the existence of $V_k$-SVML in graphs has been obtained. Also, the magic constant for $E_k$-regular graphs has been obtained. Further, we study some classes of graphs such as cycles, complement of cycles, prism graphs and a family of circulant graphs which admit $V_2$-SVML.

### 2. Main Results

In this section, we obtain some basic properties of $V_k$-SVML.

Let $G$ be a connected graph of order $p(\geq 2)$. Suppose $E_k(u) = E_k(v)$ for two different vertices $u$ and $v$ of $G$. Then $f(u) + w_k(u) \neq f(v) + w_k(v)$ for any $V_k$-SVML $f$ of $G$ (since $f$ is one to one). In this case, $G$ does not admit $V_k$-SVML and hence the next result follows.

**Lemma 2.1.** Let $G$ be a connected graph of order $p(\geq 2)$. If $E_k(u) = E_k(v)$ for some $u, v \in V(G)$ ($u \neq v$), then the graph $G$ does not admit $V_k$-SVML.

**Corollary 2.2.** The star graph $S_n$ does not admit $V_k$-SVML for $k \geq 2$.

If a graph $G$ admits $V_k$-SVML, then $1 \leq k \leq \text{diam}(G)$ (otherwise, $E_k(u) = E_k(v)$ for any two different vertices $u, v \in V(G)$).

**Definition 2.3.** In a graph $G$, a vertex of degree $|V(G)| - 1$ is called a full vertex of $G$.

**Corollary 2.4.** Let $G$ be a connected graph of order $p(\geq 2)$ and $u$ be a full vertex of $G$. Then $G$ does not admit $V_k$-SVML for $k \geq 3$.

**Lemma 2.5.** If a graph $G(p, q)$ is $V_k$-SVM and $G$ is $E_k$-regular with regularity $r$, then the magic constant is given by $M = \frac{p+1}{2} + rq + \frac{rq(q+1)}{2}$.

**Proof.** Let $f$ be a $V_k$-SVML of $G$ with the magic constant $M$. Then $f(V(G)) = \{1, 2, \ldots, p\}$, $f(E(G)) = \{p + 1, p + 2, \ldots, p + q\}$ and $M = f(v) + w_k(v)$ for all $v \in V(G)$. By summing over all $v \in V(G)$, $pM = \sum_{v \in V(G)} f(v) + \sum_{v \in V(G)} w_k(v)$.

The first sum is $\frac{E(p+1)}{2}$ and the second sum is $\sum_{v \in V(G)} w_k(v) = \sum_{v \in V(G)} \sum_{e \in E(G)} f(e) = r \sum_{e \in E(G)} f(e) = r(pq) + \frac{rq(q+1)}{2}$, where the second equality uses from $E_k$-regular that each edge is in exactly $r$ of the sets $E_k(v)$. Thus $pM = \frac{p(p+1)}{2} + r(pq) + \frac{rq(q+1)}{2}$ and hence $M = \frac{p+1}{2} + r(pq) + \frac{rq(q+1)}{2}$.

In Lemma 2.5, we give the magic constant only for $E_k$-regular graphs which admit $V_k$-SVML for $k \geq 1$. MacDougall et. al obtained the following result which gives the magic constant of $V$-SVML for any graph.

**Lemma 2.6.** [4] If $G$ has a super-vertex magic total labeling, then $M = 2q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$.

When $k = 1$, we have $r = |E_1(e)| = 2$ for all $e \in E(G)$. Thus if we put $k = 1$ in Lemma 2.5, then it gives the proof of Lemma 2.6.

**Lemma 2.7.** For $k \geq 2$, there dose not exist a tree, which is $E_k$-regular and $V_k$-SVM.

**Proof.** Let $T$ be a tree and $\text{diam}(T) = d(\geq 3)$. Let $P = u_0u_1 \ldots u_{d-1}u_d$ be a path of length $d$. Then $u_0u_1$ and $u_{d-1}u_d$ must be pendent edges. When $k = d$, we have $E_k(u_0) = E_k(u_d)$ and hence $T$ is not $V_k$-SVM. Also, $k \leq d - 1$, we have $E_k(u_1u_2) > E_k(u_0u_1)$ and hence $T$ is not $E_k$-regular. Thus $\text{diam}(T) \leq 2$ and hence $T$ is a star graph. By Corollary 2.2, the star graph $S_n$ does not admit $V_k$-SVML for $k \geq 2$.

**Theorem 2.8.** If $G(p, q)$ is a connected $E_k$-regular graph with regularity $r$, then $M \geq \frac{7p+5}{2}$ if $k = 1$ and $M \geq \frac{(p+1)(r+1)}{2} + rp$ if $k \geq 2$.

**Proof.** For $k = 1$, we have $r = 2$. Since $G$ is connected, $q \geq p - 1$. Thus by Lemma 2.5, $M \geq \frac{7p+5}{2} + 2(p-1) + (p-1) = \frac{7p+5}{2}$ (This is already proved in [4]).

Let $k \geq 2$. If $q = p - 1$, then $G$ must be a tree and hence by...
Lemma 2.7, there do not exist a tree \( T \), which is \( E_{k} \)-regular and \( V_{k} \)-SVM. Hence assume that \( q \geq p \). By Lemma 2.5, \( M \geq \frac{p+1}{2} + rp + \frac{r(p+1)}{2} = \frac{(p+1)(r+1)}{2} + rp \).

**Remark 2.9.** For \( k \geq 2 \), the lower bound for the magic constant \( M \) obtained in Theorem 2.8 is sharp. For example, consider the following \( V_{2} \)-SVM of \( C_{5} \) (see Figure 2).

![Figure 2: V2-SVM of C5](image)

Note that the cycle \( C_{5} \) is \( E_{2} \)-regular with regularity 4. Here the mask constant \( M = 35 \). In Theorem 2.8, we proved that \( M \geq 35 \).

**Theorem 2.10.** Let \( G \) be a \((p, q)\) graph and \( g \) be a bijection from \( E(G) \) onto \( \{p + 1, p + 2, \ldots, p + q\} \). Then \( g \) can be extended to a \( V_{k} \)-SVM of \( G \) if and only if \( \{w_{k}(u)/u \in V(G)\} \) consists of \( p \) sequential integers.

**Proof.** Assume that \( \{w_{k}(u)/u \in V(G)\} \) consists of \( p \) sequential integers. Let \( t = \min \{w_{k}(u)/u \in V(G)\} \). Define \( f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\} \) as \( f(xy) = g(xy) \) for \( xy \in E(G) \) and \( f(x) = t + p - w_{k}(x) \). Then \( f(E(G)) = \{p + 1, p + 2, \ldots, p + q\} \) and \( f(V(G)) = \{1, 2, \ldots, p\} \). Hence \( f \) is \( V_{k} \)-SVM with \( M = t + p \).

Conversely, suppose \( g \) can be extended to a \( V_{k} \)-SVM \( f \) of \( G \) with a magic constant \( M \). Since \( f(u) + w_{k}(u) = M \) for every \( u \in V(G) \), we have \( w_{k}(u) = M - f(u) \). Thus \( \{w_{k}(u)/u \in V(G)\} \) is a set of \( p \) consecutive integers. \( \square \)

**3. V2-SVM of cycles and prisms**

In this section, we identified some classes of graphs such as cycles, complement of cycles, prism graphs and a family of circulant graphs, which admit \( V_{2} \)-SVM.

**Lemma 3.1.** [5] For any integers \( a \) and \( b \), we have \( \gcd(a, b) = \gcd(b, a) = \gcd(\pm a, \pm b) = \gcd(a, b-a) = \gcd(a, b+a) \).

By Lemma 2.1, the cycles \( C_{3} \) and \( C_{4} \) are not \( V_{2} \)-SVM.

**Theorem 3.2.** For an integer \( n \geq 5 \), the cycle \( C_{n} \) is \( V_{2} \)-SVM if and only if \( n \) is odd.

**Proof.** Suppose there exists a \( V_{2} \)-SVM \( f \) of \( C_{n} \). Since \( |E_{2}(e)| = r = 4 \) for all \( e \in E(C_{n}) \), by taking \( k = 2 \), \( p = q = n \) and \( r = 4 \) in Lemma 2.5, we get \( M = \frac{13n+5}{2} \). Since \( M \) is an integer, \( n \) must be odd.

Conversely, assume that \( n \) is odd and \( n \geq 5 \). Let \( V(C_{n}) = \{a_{i} : 1 \leq i \leq n\} \) and \( E(C_{n}) = \{a_{i}a_{i+1} : 1 \leq i \leq n\} \), where the operation \( \oplus \) stands for addition modulo \( n \).

**Case A:** Suppose \( n = 4\ell + 1 \) for some integer \( \ell \geq 1 \). Define a function \( f : V(C_{n}) \cup E(C_{n}) \rightarrow \{1, 2, \ldots, 2n\} \) as follows:\( f(a_{i}) = i - 3 \) if \( 4 \leq i \leq n \) and \( f(a_{i}) = (n - 3) + i \) when \( 1 \leq i \leq 3 \); \( f(a_{i+1}) = [(i-1)\ell \oplus 1] + n \), where \( (i-1)\ell \oplus 1 \) is the positive residue when \((i-1)\ell + 1 \) divides \( n \).

Now we prove that \( f(E(C_{n})) = \{n+1, n+2, \ldots, 2n\} \).

Then by Claim 2, \( f(a_{i}) = 2n+1+2i+1 \) for \( 4 \leq i \leq n \). Hence we have \( f(a_{i}) = n+1 = \ell + 1 \leq n+2 \).

Since \( 1 \leq x \leq n \), in the above four terms(brackets), all the residues are not positive, we have \( n+1 = \ell + 1 \leq n+2 \).

By taking \( n = 4\ell + 1 \), we get \( f(a_{i}) = 2n+1+2i+1 \). Thus \( f \) is a generator for the finite cyclic group \((Z_{n}, \oplus)\) and hence \( f(E(C_{n})) = \{n+1, n+2, \ldots, 2n\} \).

**Claim 1:** \( f(a_{j}) = 2n+1+2j+1 \) for \( 4 \leq j \leq n \).

**Claim 2:** \( f(a_{1}) = (2\ell + 1)11 - i \) for \( i \geq 3 \).

Consider the vertex \( a_{1} \). \( f(a_{1}) = f(a_{n-1}) + f(a_{1}a_{1}) + f(a_{1}a_{2}) + f(a_{2}a_{2}) \). Since \( f(a_{n-1}) = [(n-2)\ell \oplus 1] + n \) and \( f(a_{1}a_{1}) = [(2\ell-1)\ell \oplus 1] + n \), we have \( f(a_{2}) = [2\ell \oplus 1] + [\ell \oplus 1] + 4n+1 \). Here, the first two terms are not positive. Thus \( f(a_{1}) = n-m2\ell + 1 + [n-m+\ell]n + \ell \). Similarly, we can show that \( f(a_{2}) = 2n+1+2i+1 \).

**Case B:** Suppose \( n = 4\ell + 3 \) for some integer \( \ell \geq 1 \). Define a function \( f : V(C_{n}) \cup E(C_{n}) \rightarrow \{1, 2, \ldots, 2n\} \) as follows:\( f(a_{i}) = n-i \) when \( 1 \leq i \leq n \) and \( f(a_{i}) = n+i \) when \( i = n+1 \); \( f(a_{i+1}) = [(i-1)\ell + 1] \oplus 1 \oplus 1 \) when \( ((i-1)\ell + 1) \oplus 1 \oplus 1 \) is the positive residue \((i-1)(\ell + 1) + 1 \) divides \( n \).

By Lemma 3.1, \( \gcd(\ell + 1, n) = \gcd(\ell + 1, 4\ell + 3) = \gcd(\ell + 1, 3\ell + 2) = \gcd(\ell + 1, 2\ell + 1) = \gcd(\ell + 1, \ell) = \gcd(\ell, 1) = 1 \).

\( \therefore \)
1. Hence \( \ell + 1 \) is a generator for the finite cyclic group \((Z_n, \cdot)\) and hence \( f(E(C_n)) = \{n + 1, n + 2, \ldots, 2n\}\). As proved in Case A, we can prove that the above labeling is a V2-SVML with magic constant \( M = \frac{3n + 5}{2} \).

**Theorem 3.3.** Let \( G = \overline{C_n} \) be the complement of the cycle \( C_n \), where \( n \geq 5 \) is an integer. Then \( G \) is V2-SVM with the magic constant \( \frac{n^2-2n}{2} - \frac{n^2-14n}{2} \).

**Proof.** Define \( f : V(\overline{C_n}) \cup E(\overline{C_n}) \rightarrow \{1, 2, \ldots, \frac{n^2+1}{2} \} \) as follows: First we label the \( n \) edges \( \{a_1a_2, a_2a_3, \ldots, a_na_1\} \) by \( f(a_{i+1}a_i) = n + i \) for \( 1 \leq i \leq n \). The remaining \( n^2-3n \) \( n \) edges are randomly labeled with the labels \( \{2n + 1, 2n + 2, \ldots, \frac{n^2-4n}{2} \} \). The vertices are labeled as \( f(a_i) = i \). Note that for each vertex \( a_i \), the only edge with label \( n + i \) is not in \( E_2(a_i) \). Thus for each \( a_i \) with \( 1 \leq i \leq n \), we have \( f(a_i) + w_2(a_i) = i + \left[ \frac{n^2-2n}{2} - \frac{n^2-6n}{2} \right] - (n + i) = n - \frac{3n^2 - 14n}{2} \).

**Definition 3.4.** Let \( D_n \) be a prism graph of order \( n \) with \( |V(D_n)| = 2n \) and \( |E(D_n)| = 3n \). We take \( V(D_n) = \{a_1b_i, 1 \leq i \leq n\} \) and \( E(D_n) = \{a_1b_i, 1 \leq i \leq n\} \cup \{a_1a_2, b_2b_1, 1 \leq i \leq n\} \).

**Theorem 3.5.** For an integer \( n \geq 3 \), the prism \( D_n \) is V2-SVM if and only if \( n \) is even.

**Proof.** Suppose there exists a V2-SVML \( f \) of \( D_n \) with the magic constant \( M \). Since \( E_2(e) = r = 6 \) for all \( e \in E(D_n) \), by taking \( k = 2, p = 2n, q = 3n \) and \( r = 6 \) in Lemma 2.5, we get \( M = \frac{6n+15}{2} \). Since \( M \) is an integer, \( n \) must be even. Conversely, assume that \( n \) is even. Let \( V(D_n) = \{a_1b_i, 1 \leq i \leq n\} \) and \( E(D_n) = \{a_1b_i, 1 \leq i \leq n\} \cup \{a_1a_2, b_2b_1, 1 \leq i \leq n\} \). Define \( f : V(D_n) \cup E(D_n) \rightarrow \{1, 2, \ldots, 5n\} \) as follows:

- \( f(a_1) = n + \frac{n}{2} - \frac{1}{2} \) if \( n \) is odd; The range is given by \( \{n + 1, n + 2, \ldots, \frac{n^2-4}{2} \} \).
- \( f(a_2) = 2n - \left( \frac{1}{2} - 2 \right) \) if \( 1 \geq \frac{n}{2} \) and \( i \) is even; \( \{n + \frac{n}{2} + 2, n + \frac{n}{2} + 3, \ldots, 2n\} \).
- \( f(a_3) = n + \frac{n}{2} + 1 \); \( \{n + \frac{n}{2} + 1, \ldots, 2n\} \).
- \( f(b_1) = \frac{n}{2} + \frac{n}{2} - 1 \) if \( n \) is odd; \( \{1, 2, \ldots, n\} \).
- \( f(b_2) = n + \frac{n}{2} - 1 \) if \( n \) is even; \( \{n + \frac{n}{2} + 1, \ldots, 2n-1\} \).
- \( f(a_1b_i) = 2n + \frac{n}{2} + 1 \) if \( i \) is odd; \( \{2n + 1, 2n + 2, \ldots, 2n + \frac{n}{2}\} \).
- \( f(a_1b_i) = 2n + \frac{n}{2} + 1 \) if \( i \) is even; \( \{2n + \frac{n}{2} + 1, \ldots, 3n\} \).
- \( f(a_1a_2b_1) = \frac{n}{2} + \frac{n}{2} - \frac{1}{2} \) if \( i \) is odd; \( \{3n + 1, \ldots, 3n + \frac{n}{2} \} \).
- \( f(b_2b_1) = 4n - \left( \frac{1}{2} - 1 \right) \) if \( i \) is even; \( \{3n + \frac{n}{2} + 1, 3n + \frac{n}{2} + 2, \ldots, 4n\} \).
- \( f(a_1a_2) = 4n + \frac{n}{2} \) if \( i \) is even; \( \{4n + 1, 4n + 2, \ldots, 4n + \frac{n}{2}\} \).
- \( f(b_2b_1) = 5n - \frac{n}{2} \) if \( i \) is odd; \( \{4n + \frac{n}{2} + 1, \ldots, 5n\} \).

It is easily seen that \( f \) is a V2-SVML with the magic constant \( M = \frac{6n+15}{2} \). 

Let \( \Gamma \) be a finite group with \( e \) as the identity. A generating set of \( \Gamma \) is a subset \( A \) such that every element of \( \Gamma \) can be expressed as a product of finitely many elements of \( A \). Assume that \( e \notin A \) and \( a \in A \) implies \( a^{-1} \in A \) (\( A \) is called as symmetric generating set). A Cayley graph is a graph \( G = (V,E) \), where \( V(G) = \Gamma \) and \( E(G) = \{(x,a) \mid x \in V(G), a \in A\} \), denoted by \( Cay(\Gamma, A) \). Since \( A \) is a generating set for \( \Gamma \), \( G \) is a connected regular graph of degree \( |A| \). When \( \Gamma = Z_n \), the corresponding Cayley graph is called as a circulant graph, denoted by \( Cir(n, A) \).

In Lemma 2.5, we find the magic constant of \( E_k \)-regular graphs which admit V2-SVML. When \( A = \{1, 2, n-1, n-2\} \), the circulant graph \( Cir(n, A) \) is not E2-regular. In the next result, we find the magic constant of this family of circulant graphs.

**Theorem 3.6.** For an integer \( n \geq 7 \), the graph \( G = Cir(n, \{1, 2, n-1, n-2\}) \) is V2-SVML with the magic constant \( M = 27n + 7 \).

**Proof.** Let \( V(G) = \{a_1, a_2, \ldots, a_{2n}\} \) and \( E(G) = \{a_ia_{i+1}, a_{i+1}a_{i+2} : 1 \leq i \leq n\} \). Define \( f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, 3n\} \) as follows:

- \( f(a_1) = i - 4 \) for \( 5 \leq i \leq n \).
- \( f(a_1) = i + 4 \) for \( 1 \leq i \leq 4 \).
- \( f(a_1a_{i+1}) = n + i \) for \( 1 \leq i \leq n \) and \( f(a_1a_{i+2}) = 3n + 1 - i \) for \( 1 \leq i \leq n \).

Let \( v \in V(G) \). Suppose \( v = a_i \) for some integer \( i \) with \( 5 \leq i \leq n \). Then \( f(a_i) + w_2(a_i) = f(a_i) + f(a_{i-3}a_{i-2}) + f(a_{i-2}a_{i-1}) + f(a_{i-1}a_i) + f(a_ia_{i+1}) + f(a_{i+1}a_{i+2}) \) for \( \Gamma = \{1, 2, n-1, n-2\} \). We take \( f(a_1) = 1 \), \( f(a_1a_{i+1}) = 1 \), \( f(a_1a_{i+2}) = 3n + 1 - i \) for \( 1 \leq i \leq n \).

In this section, we obtained some results on V-SVML.

**Lemma 4.1.** Any connected graph on four vertices is not V-SVM.

**Proof.** Suppose there exists a V-SVML with magic constant \( M \). All the non-isomorphic connected graphs on four vertices are given below.

![Diagram](image-url)

\[ \text{4. Some Results on V-SVML} \]

In this section, we obtained some results on V-SVML.
By Lemma 2.6, $M = 2q + \frac{p+1}{2} + \frac{q(q+1)}{p}$. For the graphs $A, B, C$ and $D$, the magic constant is not an integer and hence they are not V-SVM.

Suppose the graph $E$ admits a V-SVML, say $f$. Then $M = 20$, $f(V(E)) = \{1, 2, 3, 4\}$ and $f(E(E)) = \{5, 6, 7, 8, 9\}$. Note that the vertices $v_1$ and $v_3$ are of degree two and $f(v_1)$, $f(v_3) \in \{1, 2, 3, 4\}$. Since $M = 20$, both $w(v_1)$ and $w(v_3)$ must be greater than or equal to 16, which is not possible since $f(E(E)) = \{5, 6, 7, 8, 9\}$. Thus $E$ is not E-SVM.

Next, we consider the graph $F$. Suppose the graph $F$ admits V-SVML, say $f$. Then $M = 25$, $f(V(F)) = \{1, 2, 3, 4\}$ and $f(E(F)) = \{5, 6, 7, 8, 9, 10\}$. With out loss of generality, we take $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$ and $f(v_4) = 4$. Consider the vertex $v_4$. Suppose the edges incident with $v_4$ receive the labels $\{10, 6, 5\}$. In this case the edges incident with $v_1$ cannot be labeled with $\{10, 9, 5\}$, $\{10, 8, 6\}$. Since $v_3$ is adjacent with $v_4$, one of the edge incident with $v_1$ must be labeled with 10 or 6 or 5. Thus the the edges incident with $v_1$ cannot be labeled with $\{9, 8, 7\}$. Hence $f$ is not SVML, a contradiction. We can get the same contradictions when the edges incident with $v_4$ receive the labels $\{9, 7, 5\}$ and $\{8, 7, 6\}$. 

**Theorem 4.2.** Let $G$ be a $(p, q)$ graph. If $q = p + 1$, then $G$ is not V-SVM.

**Proof.** Suppose $q = p + 1$. Then by Lemma 2.6, $M = 2q + \frac{p+1}{2} + \frac{q(q+1)}{p} = 2(p + 1) + \frac{p+1}{2} + \frac{(p+1)(p+2)}{p} = 3p + 5 + \frac{1}{2} + \frac{p^2 + 3p}{2} + \frac{q(q+1)}{p}$, which is an integer only when $p = 4$. Thus by Lemma 4.1 $G$ is not V-SVM. 

**Corollary 4.3.** For $n \geq 4$, the cycle with one chord is not V-SVM.

<table>
<thead>
<tr>
<th>$f(v)$</th>
<th>incident edges of $v$</th>
<th>possible edge labelings</th>
<th>$w(v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(v_1) = 1$</td>
<td>$v_1v_2, v_1v_3, v_1v_4$</td>
<td>${(10, 9, 5), (10, 8, 6), (9, 8, 7)}$</td>
<td>$w(v_1) = 24$</td>
</tr>
<tr>
<td>$f(v_2) = 2$</td>
<td>$v_2v_3, v_2v_1, v_2v_4$</td>
<td>${(10, 8, 5), (10, 7, 6), (9, 8, 6)}$</td>
<td>$w(v_2) = 23$</td>
</tr>
<tr>
<td>$f(v_3) = 3$</td>
<td>$v_3v_4, v_3v_2, v_3v_1$</td>
<td>${(10, 7, 5), (9, 8, 5), (9, 7, 6)}$</td>
<td>$w(v_3) = 22$</td>
</tr>
<tr>
<td>$f(v_4) = 4$</td>
<td>$v_4v_1, v_4v_2, v_4v_3$</td>
<td>${(10, 6, 5), (9, 7, 5), (8, 7, 6)}$</td>
<td>$w(v_4) = 21$</td>
</tr>
</tbody>
</table>

Table 2

**References**


