Vertex edge neighborhood prime labeling of some graphs

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Abstract
The concept of vertex edge neighborhood prime labeling was introduced in \cite{3}. We obtained some families of graphs, pendant edge attached by few graphs and \(m\)-fold pedal graphs are vertex edge neighborhood prime labeling.

Keywords
Neighborhood prime labeling, total neighborhood prime labeling, vertex edge neighborhood prime labeling.

AMS Subject Classification
05C78.

1. Introduction
Throughout this paper we consider finite, simple, undirected and connected graphs, and for notations and terminology, we refer to R.Balakrishnan and K.Ranganathan \cite{1}.

A graph which admits total neighborhood prime labeling is called total neighborhood prime graph. This concept was introduced by Rajeshkumar, et. al., \cite{5}. Also, they proved that path \(P_n\), cycle \(C_n\), if \(n\) is even and \(n \equiv 2 \pmod{4}\) and comb are total neighborhood prime graph. Motivated by neighborhood prime graph and total neighborhood prime graph, Pandya and Shrimali \cite{3} defined the concept of vertex edge neighborhood prime labeling. They observed that
\(\text{(i)}\) every vertex edge neighborhood prime graph is total neighborhood prime graph, but converse is not true.
\(\text{(ii)}\) the graph which is not having degree one, if it is total neighborhood prime graph, then it is vertex edge neighborhood prime graph.

Let \(G = (V(G), E(G))\) be a graph, \(u \in V(G)\)

\[ N_v(u) = \{w \in V(G) / uv \text{ is an edge}\} \]
\[ N_E(u) = \{e \in E(G) / e = uv \text{ for some } v \in V(G)\} \]

A graph which admits vertex edge neighborhood prime labeling is a function
\(f : V(G) \cup E(G) \rightarrow \{1,2,3,...,|V(G)\cup E(G)|\}\) with one to one correspondence and if
\(\text{(i)}\) for \(u \in V(G)\) with \(deg(u) = 1\), \(gcd(f(w), f(uv))/w \in N_v(u)\) = 1.
\(\text{(ii)}\) for \(u \in V(G)\) with \(deg(u) \geq 2\), \(gcd(f(w)/w \in N_v(u)) = 1\).

A graph which admits vertex edge neighborhood prime

\(labeling\) if it satisfies the following two conditions: (i) for each vertex of degree at least two, the gcd of labeling on its neighborhood vertices is one; (ii) for each vertex of degree at least two, the gcd of labeling on the induced edges is one.

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labeling is called a vertex edge neighborhood prime graph. They [3] proved that path, helm, sunlet, bistar, subdivision of central edge and edges of bistar are vertex edge neighborhood prime graphs.

2. Preliminaries

We need some basic definitions as follows. The Petersen graph $P(n, k)$ is a graph with vertex set $(u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1})$ and edge set $(u_i u_{i+1}, u_i v_i, v_{i+1} : 0 \leq i \leq n - 1)$ where subscripts are to be taken modulo $n$ and $k < \frac{n}{2}$. The $k$-polygonal book, denoted $B_{k,n}$, is formed by $n$ copies of a $k$-polygonal sharing a single edge. Each $k$-polygon is referred to as a page of the book graph [4]. The quadrilateral snake $Q_n$ is obtained from the path $P_n$ by replacing each edge of the path by a quadrilateral graph $C_4$. An alternative quadrilateral snake $A(Q_n)$, where $n = 4, 6, 8, 10, \ldots$ consists of a path $u_1, u_2, u_3, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ (alternately) to a new vertex $v_i$. That is, every alternate edge of a path is replaced by $C_4$. The triangular snake $T_n$ is obtained from the path $P_n$ by replacing each edge of the path by a triangle $C_3$. An alternate triangular snake $A(T_n)$, where $n = 4, 6, 8, 10, \ldots$ consists of a path $u_1, u_2, u_3, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ (alternately) to a new vertex $v_i$. That is every alternate edge of a path is replaced by $C_3$. A double triangular snake $D(T_n)$, where $n > 1$ consists of two triangular snakes that have a common path. A double alternate triangular snake $DA(T_n)$, where $n = 4, 6, 8, 10, \ldots$ consists of two alternate triangular snake that have a common path. That is, a double alternating snake is obtained from a path $u_1, u_2, u_3, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ (alternately) to two new vertices $v_i$ and $w_i$. For $n \geq 3$, a combination of prism graph $P_n$ and antiprism graph $A_n$ is known as convex polytope graph $R_n$. It consists of the inner cycle vertices $\{u_i : 1 \leq i \leq n\}$, the middle cycle vertices $\{v_i : 1 \leq i \leq n\}$ and the outer cycle vertices $\{w_i : 1 \leq i \leq n\}$. $G^* = G \ast K_1$ is obtained by joining a single pendant edge to each vertex of $G$.

In section 2, we prove that $P(n, 2), C_n \ast K_2, T_n$, barycentric cycle, $R_n, A(T_n), B_{3,n}, B_{4,n}, B_{5,n}$, $Q_n, A(Q_n)$, $D(T_n), DA(T_n)$, $P(n, 2) \ast K_1, (C_n \ast K_2) \ast K_1, T_n \ast K_1, R_n \ast K_1$, and barycentric cycle attached by pendant edge are vertex edge neighborhood prime labeling. In section 3, we prove that $n$ fold petal types of graphs $P(n, 2), C_n \ast K_2, T_n$, barycentric cycle, $R_n, A(T_n)$, sunflower are vertex edge neighborhood prime labeling.

3. Main Results

We now give vertex edge neighborhood prime labeling of some graphs.

Theorem 3.1. The Petersen graph $P(n, 2)$ where $n > 4$ is a vertex edge neighborhood prime labeling.

Proof. Let $G = P(n, 2)$ be a Petersen graph. Then $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{x_i = u_i u_{i+1}, 1 \leq i \leq n - 1\} \cup \{e_i = v_i v_{i+1}, 1 \leq i \leq n - 2\} \cup \{d_n = u_n u_1\} \cup \{e_{n-1} = v_{n-1} v_1\} \cup \{e_n = v_n v_2\}$.

Also, $|V(G)| = 2n$ and $|E(G)| = 3n$.

Define a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, 5n\}$ as follows.

For each $1 \leq i \leq n$, $f(u_i) = 2i - 1, f(v_i) = 2i$, $f(x_i) = f(u_i v_i) = p + n + i$, $f(d_i) = f(u_i u_{i+1}) = p + 2n + i$ for $1 \leq i \leq n - 1$. $f(d_n) = f(u_n u_1) = p + 3n$.

Case 1. If $n$ is odd, $f(e_1) = f(v_1 v_2) = p + i$ for $1 \leq i \leq n - 2$. $f(e_{n-1}) = f(v_{n-1} v_1) = p + n - 1, f(e_n) = f(v_n v_2) = p + n$.

Case 2. If $n$ is even, $f(e_1) = f(v_1 v_2) = p + i - 1$ for $2 \leq i \leq n - 2$. $f(e_{n-1}) = f(v_{n-1} v_1) = p + n - 2, f(e_n) = f(v_n v_2) = p + n - 1, f(e) = f(v_1 v_2) = p + n$.

In order to show that $f$ is a vertex edge neighborhood prime labeling. Let $a$ be any vertex of $G$.

For $a = v_1, 1 \leq i \leq n$ with $\deg(a) = 2$.

Here, $\gcd(f(x) / x \in \mathbb{N}(v_i)) = 1$ and $\gcd(f(e) / e \in \mathbb{N}(v_i)) = 1$.

For $a = u_1, 1 \leq i \leq n$ with $\deg(a) = 2$. Here $u_1 \in \mathbb{N}(v_1)$, $f(u_1) = 1$ and $\gcd(f(x) / x \in \mathbb{N}(u_1)) : 1 \leq i \leq n - 1$ contains consecutive integers.

Therefore, $\gcd(f(x) / x \in \mathbb{N}(u_1)) : 1 \leq i \leq n - 1 = 1$ and $\gcd(f(e) / e \in \mathbb{N}(u_1))$ contains $(p + 2n + 1, p + 3n, p + n + 1)$.

Therefore, $\gcd(p + 2n + 1, p + 3n, p + n + 1) = 1$ and $\gcd(f(e) / e \in \mathbb{N}(u_1))$ contains consecutive integers.

Therefore, $\gcd(f(x) / x \in \mathbb{N}(u_1)) : 1 \leq i \leq n - 1 = 1$.

Hence $G$ is a vertex edge neighborhood prime labeling. □

Theorem 3.2. The prism graph $C_n \ast K_2$ is a vertex edge neighborhood prime labeling for all $n$.

Proof. Let $G = C_n \ast K_2$ be a prism graph. Then $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{e_i = u_i u_{i+1}, 1 \leq i \leq n - 1\} \cup \{x_i = u_i v_i : 1 \leq i \leq n\} \cup \{e_n = u_n v_1\} \cup \{d_n = v_n v_1\}$.

Also, $|V(G)| = 2n$ and $|E(G)| = 3n$.

Define a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, 5n\}$ as follows.

For each $1 \leq i \leq n$, $f(u_i) = 2i, f(v_i) = 2i - 1$, $f(x_i) = f(u_i v_i) = p + n + i$.

For each $1 \leq i \leq n - 1$, $f(e_i) = f(u_i u_{i+1}) = p + i$, $f(d_i) = f(v_i v_{i+1}) = p + 2n + i$.

$f(e_n) = f(u_n u_1) = p + n, f(d_n) = f(v_n v_1) = p + 3n$.

In order to show that $f$ is a vertex edge neighborhood prime labeling, for that we consider the following cases. Let $x$ any vertex of $G$.

Case 1. Let $x = u_i, 1 \leq i \leq n$ with $\deg(x) = 2$.

In this case $v_1 \in \mathbb{N}(u_1), f(v_1) = 1$ and $\{f(w) / w \in \mathbb{N}(u_1) : 2 \leq i \leq n\}$ and $\{f(e) / e \in \mathbb{N}(u_1) : 1 \leq i \leq n\}$ also contains consecutive integers.

Case 2. If $x = v_i, 1 \leq i \leq n$ with $\deg(x) = 2$, then

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v₁ ∈ Nᵥ(v₁), f(v₁) = 1 and \{f(u)/u ∈ Nᵥ(u₁) : 1 ≤ i ≤ n - 1\} contains consecutive integers. gcd \{f(u)/u ∈ Nᵥ(u₁) : 1 ≤ i ≤ n\} = 1 and \{f(e)/e ∈ Eᵥ(V₁)\} contains (p + 3n, p + n + 1, p + 2n + 1). 
\[\text{gcd}(p + 3n, p + n + 1, p + 2n + 1) = 1\] and \{f(e)/e ∈ Eᵥ(V₁) : 2 ≤ i ≤ n\} contains consecutive integers. gcd \{f(e)/e ∈ Eᵥ(V₁) : 1 ≤ i ≤ n\} = 1.

Hence G is a vertex edge neighborhood prime labeling for all n. □

**Theorem 3.3.** The triangular snake Tₙ admits vertex edge neighborhood prime labeling for all n.

**Proof.** Let G = Tₙ be a triangular snake. Then V(G) = \{u₁ : 1 ≤ i ≤ n\} \cup \{v₁ : 1 ≤ i ≤ n - 1\} and E(G) = {e₁ = u₁u₁₊₁, d₁ = v₁v₁₊₁ : 1 ≤ i ≤ n - 1}. Also, \(|V(G)| = 2n - 1\) and \(|E(G)| = 3n - 3\).

Define a bijective function f : V(G) ∪ E(G) → \{1, 2, 3, ..., 5n - 4\} as follows.

\[f(u₁) = 2i - 1\] for \(1 ≤ i ≤ n\).

For each \(1 ≤ i ≤ n - 1\), \(f(v₁) = 2i, f(e₁) = f(u₁u₁₊₁) = p + 3i - 2\).

For each \(1 ≤ i ≤ n - 2\), \(f(d₁) = f(u₁v₁) = p + 3i - 1, f(x₁) = f(v₁v₁₊₁) = p + 3i\).

\[f(d₁₋₁) = f(u₁₋₁v₁₋₁) = p + 3n - 3, f(x₁₋₁) = f(v₁₋₁u₁) = p + 3n - 4\].

We have to show that f is a vertex edge neighborhood prime labeling. We consider the following cases. Let a & b be any vertex of G.

**Case 1.** If \(a = u₁, u₁, u₁₊₁, b = \text{deg}(a) = 2\), then gcd \{f(x)/x ∈ Nᵥ(u₁)\} = gcd \{f(v₁), f(u₁)\} = gcd(2, 3) = 1, gcd \{f(x)/x ∈ Nᵥ(u₁)\} = \{f(u₁₊₁), f(v₁₋₁)\} = gcd(2n - 3, 2n - 2) = 1 and \{f(e)/e ∈ Nᵥ(u₁)\} and \{f(e)/e ∈ Nᵥ(u₁)\} contains consecutive numbers. gcd \{f(e)/e ∈ Nᵥ(u₁)\} = 1 and \{f(e)/e ∈ Nᵥ(u₁)\} = 1.

**Case 2.** If \(a = v₁, 2 ≤ i ≤ n - 1\) with \text{deg}(a) = 2, then gcd \{f(w)/w ∈ Nᵥ(u₁)\} = gcd(2i - 3, 2i - 2, 2i, 2i + 1) = 1 and gcd \{f(e₁)/e₁ ∈ Nᵥ(u₁)\} contains consecutive numbers, so gcd \{f(e₁)/e₁ ∈ Nᵥ(u₁)\} = 1.

**Case 3.** If \(a = v₁, 1 ≤ i ≤ n - 1\) with \text{deg}(a) = 2, then gcd \{f(w)/w ∈ Nᵥ(u₁)\} = gcd(2i - 1, 2i + 1) = 1 and gcd \{f(e₁)/e₁ ∈ Nᵥ(v₁)\} = 1 because \{f(e₁)/e₁ ∈ Nᵥ(v₁)\} contains consecutive numbers. Hence G is a vertex edge neighborhood prime labeling for all n. □

**Theorem 3.4.** The barycentric cycle graph is a vertex edge neighborhood prime labeling for all n.

**Proof.** Let G be a barycentric cycle graph. Then V(G) = \{v₁, v₁₊₁ : 1 ≤ i ≤ n\} and E(G) = \{e₁ = u₁u₁₊₁, d₁ = v₁v₁₊₁ : 1 ≤ i ≤ n - 1\} \cup \{d₁₋₁ = u₁v₁ : 2 ≤ i ≤ n\} \cup \{eₙ = u₁u₁₊₁\} \cup \{dₙ₊₁ = v₁v₁₊₁\}.

Also, \(|V(G)| = 2n\) and \(|E(G)| = 3n\).

Define a bijective function f : V(G) ∪ E(G) → \{1, 2, 3, ..., 5n\} as follows.

For each \(1 ≤ i ≤ n\), \(f(u₁) = 2i - 1, f(v₁) = 2i, f(d₁₋₁) = f(u₁v₁) = p + n + 2i - 1\).

For each \(1 ≤ i ≤ n - 1\), \(f(e₁) = f(u₁u₁₊₁) = p + i, f(d₂₋₁) = f(v₁u₁₊₁) = p + n + 2i\).

\(f(eₙ) = f(u₁u₁₊₁) = p + n, f(dₙ₊₁) = f(v₁v₁₊₁) = p + 3n\).

We claim that f is a vertex edge neighborhood prime labeling, for that we consider the following cases. Let b be any vertex of G.

**Case 1.** Let b = u₁, 1 ≤ i ≤ n with \text{deg}(b) ≥ 2. In this case \{f(x)/x ∈ Nᵥ(u₁)\} contains consecutive integers and \{f(e)/e ∈ Nᵥ(u₁)\} also contains consecutive integers.

**Case 2.** Let b = v₁, 1 ≤ i ≤ n with \text{deg}(b) ≥ 2. In this case f(v₁) contains 1 and \{f(x)/x ∈ Nᵥ(v₁)\} contains \((2i - 1, 2i + 1), 1 ≤ i ≤ n - 1\).

gcd \{f(x)/x ∈ Nᵥ(v₁) : 1 ≤ i ≤ n\} = 1 and \{f(d₁)/d₁ ∈ Nᵥ(v₁)\} contains consecutive integers. gcd \{f(d₁)/d₁ ∈ Nᵥ(v₁)\} = 1.

Hence G is a vertex edge neighborhood prime labeling for all n. □

**Theorem 3.5.** The convex polytope graph Rₙ is a vertex edge neighborhood prime labeling for all n.

**Proof.** Let G = Rₙ be a convex polytope graph. Then V(G) = \{u₁, v₁, w₁ : 1 ≤ i ≤ n\} and E(G) = \{e₂₋₁ = u₁v₁, e₁ = v₁w₁ : 1 ≤ i ≤ n\} \cup \{eₙ₊₁ = u₁w₁\} \cup \{dₙ₊₁ = w₁v₁\} \cup \{d₁ = u₁u₁₊₁, e₂ = u₁₊₁v₁, d₂ = w₁w₁₊₁ : 1 ≤ i ≤ n - 1\}. Also, \(|V(G)| = 3n\) and \(|E(G)| = 5n\).

Define a bijective function f : V(G) ∪ E(G) → \{1, 2, 3, ..., 8n\} as follows.

For each \(1 ≤ i ≤ n\), \(f(u₁) = 2n + i, f(v₁) = 2n + 1, f(w₁) = 2n + 2, f(e₂₋₁) = f(u₁v₁) = 4n + 2i - 1, f(e₁) = f(v₁w₁) = 7n + i\).

For each \(1 ≤ i ≤ n - 1\), \(f(d₁) = f(u₁u₁₊₁) = 3n + i, f(e₂) = f(u₁₊₁v₁) = 4n + 2i, f(d₂) = f(w₁w₁₊₁) = 6n + i\).

\(f(dₙ₊₁) = f(u₁u₁₊₁) = 4n, f(e₂₋₁) = f(u₁v₁) = 6n, f(dₙ₊₁) = f(w₁w₁) = 7n\).

We claim that f is a vertex edge neighborhood prime labeling, for that we consider the following cases. Let b be any vertex of G.

**Case 1.** Let a = u₁, 1 ≤ i ≤ n with \text{deg}(u₁) ≥ 2, then \(v₁ ∈ Nᵥ(u₁), f(v₁) = 1\) and \{f(x)/x ∈ Nᵥ(u₁)\} contains \((2i - 3, 2i - 1, 2n + i - 1, 2n + i + 1), 2 ≤ i ≤ n - 1\).

gcd \(2i - 3, 2i - 1, 2n + i - 1, 2n + i + 1) = 1 and \{f(x)/x ∈ Nᵥ(u₁)\} contains consecutive integers. gcd \{f(w₁)/w₁ ∈ Nᵥ(u₁)\} = 1.

**Case 2.** Let b = v₁, 1 ≤ i ≤ n with \text{deg}(v₁) ≥ 2, then \(v₁ ∈ Nᵥ(v₁), f(v₁) = 1\) and \{f(x)/x ∈ Nᵥ(v₁) : 2 ≤ i ≤ n\} contains consecutive integers. gcd \{f(e₁)/e₁ ∈ Nᵥ(v₁)\} = 1.

**Case 3.** Let b = w₁, 1 ≤ i ≤ n with \text{deg}(w₁) ≥ 2, then \(v₁ ∈ Nᵥ(w₁), f(v₁) = 1\) and \{f(x)/x ∈ Nᵥ(w₁) : 2 ≤ i ≤ n\} contains consecutive integers. gcd \{f(e₁)/e₁ ∈ Nᵥ(w₁)\} = 1.
Theorem 3.6. The alternate triangular snake \(A(T_n)\) admits a vertex edge neighborhood prime labeling for all \(n = 4, 6, 8, 10, \ldots\).

Proof. Let \(G = A(T_n)\) be an alternate triangular snake. Then \(V(G) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq \frac{n}{2} - 1\}\) and \(E(G) = \{e_i = u_iu_{i+1} : 1 \leq i \leq n - 1\} \cup \{d_{2i-1} = u_{2i}v_i, d_{2i} = v_{i}u_{2i+1} : 1 \leq i \leq \frac{n}{2} - 1\}\).

Also, \(\vert V(G) \vert = \frac{3n}{2} - 1\) and \(\vert E(G) \vert = 2n - 3\).

Define a bijective function \(f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, \frac{3n}{2} - 4\}\) as follows.

\[
\begin{align*}
  f(u_i) &= 3i - 2 \quad \text{for} \quad 1 \leq i \leq \frac{n}{2}, \\
  f(v_i) &= 3i - 1 \quad \text{for} \quad 1 \leq i \leq \frac{n}{2} - 2, \\
  f(u_{2i-1}) &= \frac{3n}{2} - 1, \\
  f(e_i) &= \frac{3n}{2}, \quad f(e_{i-1}) = f(u_{i-1}u_n) = \frac{3n}{2} - 3.
\end{align*}
\]

For each \(1 \leq i \leq \frac{n}{2} - 1\), \(f(u_{2i+1}) = 3i - 1, f(e_{2i-1}) = f(u_{2i+1}u_{2i+2}) = \frac{3n}{2} + 4i - 3, f(e_{2i}) = f(u_{2i}u_{2i+1}) = \frac{3n}{2} + 4i - 2, f(d_{2i-1}) = f(u_{2i+2}) = \frac{3n}{2} + 4i, f(d_{2i}) = f(v_{i}u_{2i+1}) = \frac{3n}{2} + 4i - 1.

In order to show that \(f\) is a vertex edge neighborhood prime labeling, for that we consider the following cases. Let \(x\) be any vertex of \(G\).

Case 1. Let \(x = u_i, 2 \leq i \leq n - 1\) with \(\deg(u_i) \geq 2\).

In this case \(\{f(w) : w \in N_G(u_i)\}\) contains consecutive integers.

Case 2. Let \(x = u_i, u_j\) with \(\deg(x) = 1\).

In both the case, \(f(u_2) = 1\) and \(f(u_{n-1}), f(e_{n-1})\) are consecutive integers.

\[
gcd\{f(u), f(xw) : w \in N_G(x)\} = 1.
\]

Case 3. If \(x = v_i, 1 \leq i \leq \frac{n}{2} - 1\) with \(\deg(x) = 2\), then \(\{f(w) : w \in N_G(v_i)\} = \{3i - 2, 3i - 1\}\) which is two consecutive numbers.

\[
gcd\{3i - 2, 3i - 1\} = 1\quad \text{and} \quad \gcd\{f(d_i) : d_i \in E_G(v_i)\} = 1.
\]

Hence \(G\) is a vertex edge neighborhood prime labeling for all \(n = 4, 6, 8, 10, \ldots\).

Theorem 3.7. The triangular Book \(B_{3,n}\), where \(n > 1\) is a vertex edge neighborhood prime labeling.

Proof. Let \(G = B_{3,n}\) be a triangular Book. Then \(V(G) = \{u_i : i = 1, 2\} \cup \{v_i : 1 \leq i \leq n\}\) and \(E(G) = \{e_i = u_iu_{i+1} : 1 \leq i \leq n\}\).

Also, \(\vert V(G) \vert = n + 2\) and \(\vert E(G) \vert = 2n + 1\).

Define a bijective function \(f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, 3n + 3\}\) as follows.

\[
\begin{align*}
  f(u_1) &= 1, \quad f(v_1) = 2, \quad f(u_2) = 3, \\
  f(e') &= f(u_1u_2) = p + 1.
\end{align*}
\]

for each \(1 \leq i \leq \frac{n}{2}, f(u_{2i-1}) = f(u_1v_{2i-1}) = p + 4i - 2, \quad f(d_{2i-1}) = f(u_{2i+1}v_{2i}) = p + 4i - 1.

for each \(1 \leq i \leq \frac{n}{2}, f(e_{2i}) = f(u_1v_{2i}) = p + 4i + 1, f(d_{2i}) = f(u_{2i+1}v_{2i}) = p + 4i + 1.

By considering following cases we prove \(f\) is vertex edge neighborhood prime labeling. Let \(x\) be any vertex of \(G\).

Case 1. Let \(x = u_i, u_j\) with \(\deg(x) \geq 2\).

In this case \(\{f(w) : w \in N_G(v_i)\}\) contains consecutive integers and \(\{f(e) : e \in E_G(x)\}\) also contains consecutive integers.

Case 2. If \(x = v_i, 1 \leq i \leq n\) with \(\deg(x) = 2\), then \(\{f(w) : w \in N_G(v_i)\} = \{f(u_1), f(u_2)\} = (1, 3), \quad \gcd\{f(w) : w \in N_G(v_i)\} = 1\) and \(\gcd\{f(d) : d \in E_G(v_i)\}\) contains consecutive integers.

\[
\gcd\{f(d) : d \in E_G(v_i)\} = 1.
\]

Hence \(G\) is a vertex edge neighborhood prime labeling.

Theorem 3.8. The rectangular book \(B_{4,n}\) is a vertex edge neighborhood prime labeling for all \(n\).

Proof. Let \(G = B_{4,n}\) be a rectangular Book. Then \(V(G) = \{u_i, v_i : 1 \leq i \leq n + 1\}\) and \(E(G) = \{e_i = u_iu_{i+1}, x_i = v_iv_{i+1} : 1 \leq i \leq n\}\) and \(\{d_i = u_{i}v_i : 1 \leq i \leq n + 1\}\).

Also, \(\vert V(G) \vert = 2n + 2\) and \(\vert E(G) \vert = 3n + 1\).

Define a bijective function \(f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, 5n + 3\}\) as follows.

\[
\begin{align*}
  f(u_i) &= 2i - 1, \quad f(v_i) = 2i, \\
  f(e_i) &= f(u_iu_{i+1}) = p + 1, \quad f(d_i) = f(u_{i}v_i) = p + 2, \quad f(v_i) = f(v_{i+1}) = p + 3, \quad f(d_i) = f(u_{i}v_i) = p + 4, \\
  f(d_{i+2}) &= f(u_{i+2}v_{i+2}) = p + 3i + 2, \quad f(d_{i+2}) = f(u_{i+2}v_{i+2}) = p + 3i + 2, \quad f(x_i) = f(v_{i+1}) = p + 4.
\end{align*}
\]

We claim that \(f\) is a vertex edge neighborhood prime labeling, for that we consider the following cases. Let \(a\) be any vertex of \(G\).

Case 1. If \(a = u_i, 1 \leq i \leq n + 1\) with \(\deg(a) \geq 2\), then \(\gcd\{f(w) : w \in N_G(u_i)\} = 1\) because \(u_i \in N_G(u_i), 2 \leq i \leq n + 1, f(u_i) = 1\) and \(f(u_1)\) contains consecutive integers.

\[
\gcd\{f(d_i) : d_i \in E_G(u_i)\} = 1\quad \text{because} \quad \{f(d_i) : d_i \in E_G(u_i)\}\) contains consecutive numbers.

Case 2. If \(a = v_i, 1 \leq i \leq n + 1\) with \(\deg(a) \geq 2\), then \(f(x_i) = x_i \in N_G(v_i) = (2, 2i - 1), 2 \leq i \leq n + 1, \quad \gcd(2, 2i - 1) = 1. \quad \gcd\{f(x) : x \in N_G(v_i)\} = 1\) and \(\gcd\{f(e) : e \in E_G(x)\} = 1\) because \(\{f(e) : e \in E_G(x)\}\) contains consecutive numbers.

Hence \(G\) is a vertex edge neighborhood prime labeling for all \(n\).

Theorem 3.9. The pentagonal book \(B_{5,n}\) where \(n > 1\) is a vertex edge neighborhood prime labeling.
Proof. Let $G = B_{5,n}$ be a pentagonal book. Then

$V(G) = \{u_i : 1 \leq i \leq 1.2\} \cup \{v_i, w_i, x_i : 1 \leq i \leq n\}$ and

$E(G) = \{a = u_iu_2\} \cup \{e_i = u_iv_i, d_i = v_ii, e_i = v_ii, e_i = u_2x_i : 1 \leq i \leq n\}.$

Also, $|V(G)| = 3n + 2$ and $|E(G)| = 4n + 1.$

Define a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, 7n + 3\}$ as follows.

$f(u_i) = 3, f(v_i) = 2, f(u_2) = 1, f(a) = f(u_1u_2) = p + 1.$

$$f(w_i) = \begin{cases} 3i + 1; & \text{if } i \text{ is odd} \\ 3i + 2; & \text{if } i \text{ is even} \end{cases}$$

$$f(x_i) = \begin{cases} 3i + 2; & \text{if } i \text{ is odd} \\ 3i + 1; & \text{if } i \text{ is even} \end{cases}$$

For each $1 \leq i \leq 2$, $f(v_i) = 3i, f(w_i) = 3i - 1, f(e_i) = f(u_iu_2) = 3i - 2, f(1) = f(u_1u_2) = p + 1.$

For each $1 \leq i \leq n - 2$, $f(d_i) = f(u_i) = p + 4i - 2, f(x_i) = f(w_i) = p + 4i - 2.$

By considering following cases we prove $f$ is vertex edge neighborhood prime labeling. Let $a$ be any vertex of $G.$

Case 1. $a = u_i, 1 \leq i \leq n, \text{deg}(a) \geq 2.$

In this case $\{f(w)/w \in N_V(u_i)\}$ contains consecutive integers and $\{f(e)/e \in N_E(u_i)\}$ also contains consecutive integers.

Case 2. $a = v_i, 1 \leq i \leq n - 1, \text{deg}(a) = 2.$

In this case $\{f(x)/x \in N_V(v_i)\}$ contains consecutive integers and $\{f(d)/d \in N_E(v_i)\}$ also contains consecutive integers.

Case 3. $a = w_i, 1 \leq i \leq n - 1, \text{deg}(a) = 2.$

Here, $\text{gcd}\{f(x)/x \in N_V(w_i)\} = 1$ and $\text{gcd}\{f(e)/e \in N_E(w_i)\} = 1$.

Hence $G$ is a vertex edge neighborhood prime labeling for all $n.$

Theorem 3.11. The alternate quadrilateral snake $A(Q_n)$ admits a vertex edge neighborhood prime labeling for all $n.$

Proof. Let $G = A(Q_n)$ be an alternate quadrilateral snake.

Then $V(G) = \{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq \frac{n}{2} - 1\}$ and

$E(G) = \{e_i = u_1u_i+1 : 1 \leq i \leq n - 1\} \cup \{d_i = v_{2i}w_{2i-1} = v_{2i}w_{2i} : 1 \leq i \leq n - 1\}.$

Also, $|V(G)| = 3n - 2$ and $|E(G)| = 4n - 4.$

Define a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, 7n - 6\}$ as follows.

$f(u_i) = 3i - 2$ for $1 \leq i \leq n.$

For each $1 \leq i \leq n - 1, f(v_i) = 3i, f(w_i) = 3i - 1, f(e_i) = f(u_1u_2) = p + 4i - 3, f(v_i) = f(v_{2i+1}w_{2i}) = p + 4i - 1.$

For each $1 \leq i \leq n - 2, f(d_i) = f(u_i) = p + 4i - 2, f(x_i) = f(w_i) = p + 4i - 2.$

By considering following cases we prove $f$ is vertex edge neighborhood prime labeling. Let $a$ be any vertex of $G.$

Case 1. $a = u_i, 1 \leq i \leq n, \text{deg}(a) \geq 2.$

In this case $\{f(w)/w \in N_V(u_i)\}$ contains consecutive integers and $\{f(e)/e \in N_E(u_i)\}$ also contains consecutive integers.

Case 2. $a = v_i, 1 \leq i \leq n - 1, \text{deg}(a) = 2.$

In this case $\{f(x)/x \in N_V(v_i)\}$ contains consecutive integers and $\{f(d)/d \in N_E(v_i)\}$ also contains consecutive integers.

Case 3. $a = w_i, 1 \leq i \leq n - 1, \text{deg}(a) = 2.$

Here, $\text{gcd}\{f(x)/x \in N_V(w_i)\} = 1$ and $\text{gcd}\{f(e)/e \in N_E(w_i)\} = 1$.

Hence $G$ is a vertex edge neighborhood prime labeling for all $n.$

Theorem 3.10. The quadrilateral snake $Q_n$ admits vertex edge neighborhood prime labeling for all $n.$

Proof. Let $G = Q_n$ be a quadrilateral snake. Then

$V(G) = \{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq n - 1\}$ and

$E(G) = \{e_i = u_iu_{i+1}, d_i = u_iv_i, w_i = v_ii : 1 \leq i \leq n - 1\} \cup \{e_i = w_iu_{i+1} : 1 \leq i \leq n - 1\}.$

Also, $|V(G)| = 3n - 2$ and $|E(G)| = 4n - 4.$

Define a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, 7n - 6\}$ as follows.

$f(u_i) = 3i - 2$ for $1 \leq i \leq n.$

For each $1 \leq i \leq n - 1, f(v_i) = 3i, f(w_i) = 3i - 1, f(e_i) = f(u_1u_2) = p + 4i - 3, f(v_i) = f(v_{2i+1}w_{2i}) = p + 4i - 1.$

For each $1 \leq i \leq n - 2, f(d_i) = f(u_i) = p + 4i - 2, f(x_i) = f(w_i) = p + 4i - 2.$

By considering following cases we prove $f$ is vertex edge neighborhood prime labeling. Let $a$ be any vertex of $G.$

Case 1. $a = u_i, 1 \leq i \leq n, \text{deg}(a) \geq 2.$

In this case $\{f(w)/w \in N_V(u_i)\}$ contains consecutive integers and $\{f(e)/e \in N_E(u_i)\}$ also contains consecutive integers.

Case 2. $a = v_i, 1 \leq i \leq n - 1, \text{deg}(a) = 2.$

In this case $\{f(x)/x \in N_V(v_i)\}$ contains consecutive integers and $\{f(d)/d \in N_E(v_i)\}$ also contains consecutive integers.

Case 3. $a = w_i, 1 \leq i \leq n - 1, \text{deg}(a) = 2.$

Here, $\text{gcd}\{f(x)/x \in N_V(w_i)\} = 1$ and $\text{gcd}\{f(e)/e \in N_E(w_i)\} = 1$.

Hence $G$ is a vertex edge neighborhood prime labeling for all $n.$

Theorem 3.12. A double triangular snake $D(T_n)$, where $n > 1$ is a vertex edge neighborhood prime labeling.
**Theorem 3.13.** The double alternate triangular snake $DA(T_n)$, where $n = 4, 6, 8, 10, \ldots$ admits a vertex edge neighborhood prime labeling.

**Proof.** Let $G = DA(T_n)$ be a double alternate triangular snake. Then $V(G) = \{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq \frac{n}{2} - 1\}$ and $E(G) = \{e_i = u_{i+1}w_i : 1 \leq i \leq n-1\} \cup \{d_1 = v_1w_1, d_2 = v_{2n+1}w_{2n}, d_3 = v_{2n+1}w_{2n+2}, d_4 = v_1w_1 : 1 \leq i \leq \frac{n}{2} - 1\}$. Therefore, $|V(G)| = 3n - 2$ and $|E(G)| = 5n - 5$.

Define a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, 8n - 7\}$ as follows.

For each $1 \leq i \leq n - 1$, $f(v_i) = 2i$, $f(w_i) = 2n + 1 + i$, $f(e_i) = f(u_{i+1}w_i) = 3n + 5i - 2$, $f(d_1) = f(u_1w_1) = 3n + 5i - 6$, $f(d_2) = f(v_1w_1) = 3n + 5i - 5$, $f(d_3) = f(v_{2n+1}w_{2n+1}) = 3n + 5i - 4$.

We prove $f$ is a vertex edge neighborhood prime labeling, by considering the following cases. Let $a$ be any vertex of $G$.

**Case 1.** $a = v_i, w_i$, where $1 \leq i \leq n - 1$. Here $deg(a) = 2$.

In both cases $\{f(w)/w \in N_V(a) = (2i - 1, 2i + 1)\}$ which are consecutive integers.

$\gcd\{f(w)/w \in N_V(a)\} = 1$ and $\gcd\{f(e)/e \in N_E(a)\}$ contains consecutive integers.

**Case 2.** $a = u_i$, where $1 \leq i \leq n$ with $deg(a) \geq 2$.

In this case $\gcd(f(x)/x \in N_V(u_i) = 1$ and $\gcd(f(d_i)/d_i \in N_E(u_i) = 1$ because both the sets are consecutive integers.

Hence $G$ is a vertex edge neighborhood prime labeling for all $n > 1$.

**Theorem 3.14.** The graph $P(n, 2)*K_1$, where $n > 4$ is a vertex edge neighborhood prime labeling.

**Proof.** Let $G = P(n, 2)*K_1$ be a graph. Then $V(G) = \{v_i, w_i : 1 \leq i \leq n\}$ and $E(G) = \{e_i = v_{i+1}w_i : 1 \leq i \leq n-1\} \cup \{d_1 = v_1w_1, d_2 = v_{2n+1}w_{2n+1} : 1 \leq i \leq \frac{n}{2} - 1\}$. Also, $|V(G)| = 3n$ and $|E(G)| = 4n$.

Define a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, 7n\}$ as follows.

For each $1 \leq i \leq n$, $f(v_i) = 2i$, $f(w_i) = 2n + 1 + i$, $f(e_i) = f(v_{i+1}w_i) = 3n + 5i - 2$, $f(d_i) = f(v_{i+1}w_i) = 3n + 5i - 5$, $f(d_i) = f(v_{i+1}w_i) = 3n + 5i - 4$.

For each $1 \leq i < n$, $f(u_{i+1}w_i) = 2i + 1$, $f(x_i) = f(v_{i+1}w_i) = 2n + 2i - 1$, $f(u_{i+1}w_i) = 2n + 2i - 1$.

We prove $f$ is a vertex edge neighborhood prime labeling, for that we consider the following cases. Let $x$ be any vertex of $G$.

**Case 1.** $x = v_i, w_i$, where $1 \leq i \leq n$ with $deg(x) = 1$.

In this case $f(u_i) = f(u_{i+1}) = 3n + 5i - 2$, $f(e_i) = f(v_{i+1}w_i) = 3n + 5i - 6$, $f(d_i) = f(v_{i+1}w_i) = 3n + 5i - 5$, $f(d_i) = f(v_{i+1}w_i) = 3n + 5i - 4$.

We prove $f$ is a vertex edge neighborhood prime labeling, for that we consider the following cases. Let $a$ be any vertex of $G$.

**Case 1.** $a = u_i, u_n$, where $deg(u_i) = deg(u_n) = 1$.

In this case $f(u_i) = f(u_{i+1}) = 2i + 1$, $f(e_i) = f(v_{i+1}w_i) = 3n + 5i - 2$, $f(d_i) = f(v_{i+1}w_i) = 3n + 5i - 5$, $f(d_i) = f(v_{i+1}w_i) = 3n + 5i - 4$.

We prove $f$ is a vertex edge neighborhood prime labeling, for that we consider the following cases. Let $a$ be any vertex of $G$.

**Theorem 3.15.** The graph $C_n \times K_2$ is a vertex edge neighborhood prime labeling.
neighborhood prime labeling for all \( n \).

**Proof.** Let \( G = (C_n \times K_2) \ast K_1 \) be a graph. Then
\[
V(G) = \left\{ u_i, v_i, w_i : 1 \leq i \leq n \right\}
\]
and
\[
E(G) = \left\{ x_i = u_{i-1}v_i, y_i = v_{i-1}w_i : 1 \leq i \leq n \right\} \cup \left\{ e_n = u_nu_1 \right\}
\]
\[
\cup \left\{ e_i = u_{i-1}d_i = v_{i-1}d_i = v_{i+1}d_i = v_{i+1}v_{i+2} : 1 \leq i \leq n - 1 \right\} \cup \left\{ d_n = v_nv_1 \right\}
\]
Also, \( |V(G)| = 3n \) and \( |E(G)| = 4n \).

Define a bijective function \( f : V(G) \cup E(G) \to \{ 1, 2, 3, \ldots, 7n \} \) as follows.

For each \( 1 \leq i \leq n \), let \( f(u_{i-1}) = 2i - 1 \), \( f(v_i) = 2i \), \( f(w_i) = 2n + 2i - 1 \), and \( f(y_i) = f(v_{i-1}) = 2i + 2n + 2i \).

For each \( 1 \leq i \leq n - 1 \), let \( f(d_i) = f(u_{i+1}v_i) = 4n + 3i - 9 \), \( f(v_{i+1}v_{i+2}) = 4n + 3i - 7 \), and \( f(v_{i+2}) = f(v_{i+1}v_i) = 4n + 3i - 8 \).

Let \( f(w_1) = 2i \) for \( 2 \leq i \leq n - 3 \).

Let \( f(w_1) = 4n - 8 \), \( f(u_1) = 2 \), \( f(e_1) = f(u_1u_2) = 4n - 7 \), \( f(e_n) = f(u_{n-1}u_n) = 2n - 4 \).

In order to show that \( f \) is a vertex edge neighborhood prime labeling, we consider the following cases. Let \( x \) be any vertex of \( G \).

**Case 1.** Let \( x = w_i, 1 \leq i \leq n - 3 \) with \( \text{deg}(x) = 1 \).

In this case \( \{ f(v_i), f(d_i) = f(v_{i+1}v_i), f(v_i) = N_V(w_i) \} = \{ 2n + 2i - 3, 2n + 2i - 4 \} \) which are consecutive integers.

**Case 2.** Let \( x = v_i, 1 \leq i \leq n - 2 \). Here \( \text{deg}(x) = 2 \).

In this case \( \{ f(d_i) = f(u_{i+1}w_i), f(e_i) = f(u_{i+1}v_i) = 2n + 2i - 1 \} \) contains consecutive integers.

**Case 3.** For \( x = u_i, 1 \leq i \leq n - 1 \). Here \( \text{deg}(u_i) = \text{deg}(v_i) = 1 \).

In both cases \( f(u_1), f(u_2), f(u_{n-1}), f(u_n) \) are consecutive integers.

**Case 4.** Let \( x = v_1, v_n \). Then \( \text{deg}(v_1) = \text{deg}(v_n) = 2 \).

In this case \( \{ f(v_1), f(v_n), f(v_{i+1}v_i), f(w_i) = N_V(v_i) \} = \{ 2n + 2i - 3, 2n + 2i - 4 \} \) which are consecutive integers.

Hence \( G \) is a vertex edge neighborhood prime labeling for all \( n > 3 \).

**Theorem 3.16.** The graph \( T_n \ast K_1 \) is a vertex edge neighborhood prime labeling for all \( n > 3 \).

**Proof.** Let \( G = T_n \ast K_1 \) be a graph. Then
\[
V(G) = \left\{ u_i, v_i, w_i, x_i : 1 \leq i \leq n \right\}
\]
and
\[
E(G) = \left\{ x_i = u_{i-1}v_i, y_i = v_{i-1}w_i : 1 \leq i \leq n \right\} \cup \left\{ e_n = u_nu_1 \right\}
\]
\[
\cup \left\{ e_i = u_{i-1}d_i = v_{i-1}d_i = v_{i+1}d_i = v_{i+1}v_{i+2} : 1 \leq i \leq n - 1 \right\} \cup \left\{ d_n = v_nv_1 \right\}
\]
Also, \( |V(G)| = 3n - 6 \) and \( |E(G)| = 4n - 10 \).

Define a bijective function \( f : V(G) \cup E(G) \to \{ 1, 2, 3, \ldots, 7n \} \) as follows.

\( f(u_{i+1}) = 2i - 1 \) for \( 1 \leq i \leq n - 1 \).

For each \( 1 \leq i \leq n - 3 \), let \( f(v_i) = 2n + 2i - 3 \), \( f(d_i) = f(v_{i+1}v_i) = 2n + 2i - 1 \), \( f(e_i) = f(u_{i+1}w_i) = 4n + 3i - 9 \), \( f(v_{i+1}v_{i+2}) = 4n + 3i - 7 \), \( f(v_{i+2}) = f(v_{i+1}v_i) = 4n + 3i - 8 \), \( f(w_i) = 2i \) for \( 2 \leq i \leq n - 3 \).

Let \( f(w_1) = 4n - 8 \), \( f(u_1) = 2 \), \( f(e_1) = f(u_1u_2) = 4n - 7 \), \( f(e_n) = f(u_{n-1}u_n) = 2n - 4 \).

In order to show that \( f \) is a vertex edge neighborhood prime labeling, we consider the following cases. Let \( x \) be any vertex of \( G \).

**Case 1.** Let \( x = w_i, 1 \leq i \leq n - 3 \) with \( \text{deg}(w_i) = 1 \).

In this case \( \{ f(v_i), f(d_i) = f(v_{i+1}v_i), f(v_i) = N_V(w_i) \} = \{ 2n + 2i - 3, 2n + 2i - 4 \} \) which are consecutive integers.

**Case 2.** Let \( x = v_i, 1 \leq i \leq n - 3 \). Here \( \text{deg}(v_i) \geq 2 \).

In this case \( \{ f(d_i) = f(u_{i+1}w_i), f(e_i) = f(u_{i+1}v_i) = 2n + 2i - 1 \} \) contains consecutive integers.

**Case 3.** For \( x = u_i, 1 \leq i \leq n - 3 \). Here \( \text{deg}(u_i) = \text{deg}(w_i) = 1 \).

In this case \( \{ f(v_i), f(d_i) = f(v_{i+1}v_i), f(w_i) = N_V(v_i) \} = \{ 2n + 2i - 3, 2n + 2i - 4 \} \) which are consecutive integers.

Hence \( G \) is a vertex edge neighborhood prime labeling for all \( n > 3 \).
{f(x)/x ∈ N_E(w_n)} contains \( f(v_{u_1}), f(x_n), f(w_{n−1}), f(w_1) \)
\(= (2i − 1, 4n + i, 2n + 2i − 3, 2n + 1). \)
\( \text{gcd}(2i − 1, 4n + i, 2n + 2i − 3, 2n + 1) = 1. \)
\( \text{gcd}\{f(x)/x ∈ N_E(w_i) : 1 ≤ i ≤ n\} = 1 \)
and \( \{f(d_i)/d_i ∈ N_E(w_i)\} \) contains consecutive integers.
\( \text{gcd}\{f(d_i)/d_i ∈ N_E(w_i)\} = 1. \)
\( \text{Case 4. Let } a = x_i, 1 ≤ i ≤ n \text{ with } \text{deg}(x_i) = 1. \)
In this case \( f(e'_i) = f(w_i x_i) \) and \( f(w_i) \) contains consecutive integers.
Hence \( G \) is a vertex edge neighborhood prime labeling for all \( n. \)

**Theorem 3.18.** The barycentric cycle attached by pendant edge of a graph is a vertex edge neighborhood prime labeling for all \( n. \)

**Proof.** Let \( G \) be a barycentric cycle attached by pendant edge of a graph. Then \( V(G) = \{u_i, v_i, w_i : 1 ≤ i ≤ n\} \)
and \( E(G) = \{e_i = u_i u_{i+1}, d_2i = v_i v_{i+1} : 1 ≤ i ≤ n−1\} \)
\(∪\{e_n = u_n u_1, d_2i = v_i v_{i+1} : 1 ≤ i ≤ n−1\} \)
\(∪\{d_2n = v_n v_1\}. \)
Also, \( |V(G)| = 3n \) and \( |E(G)| = 4n. \)
Define a bijective function \( f : V(G)∪E(G) → \{1, 2, 3, ..., 7n\} \)
as follows.
For each \( 1 ≤ i ≤ n, f(u_i) = 2i − 1, f(w_i) = 2n + 2i − 1, f(x_i) = f(v_i w_i) = 2n + 2i + 1, f(e_i) = f(u_i u_{i+1}) = 6n + i. \)
\( f(d_2i) = f(v_i v_{i+1}) = 6n, f(e_n) = f(u_n u_1) = 7n. \)
For proving \( f \) is a vertex edge neighborhood prime labeling, we consider the following cases. Let \( a \) be any vertex of \( G. \)

**Case 1.** If \( a = w_i, 1 ≤ i ≤ n \) with \( \text{deg}(a) = 1, \)
\( \{f(x_i), f(w_i x_i) ∈ N_E(w_i)\} \)
\(= (2n + 2i − 1, 2n + 2i), 1 ≤ i ≤ n \) which is two consecutive integers.
\( \text{gcd}(2n + 2i − 1, 2n + 2i) = 1. \)

**Case 2.** Let \( a = v_i, 1 ≤ i ≤ n \) with \( \text{deg}(a) ≥ 2. \)
In this case \( \{f(x)/x ∈ N_E(v_i)\} \) contains consecutive integers and \( \{f(e)/e ∈ N_E(v_i)\} \) also contains consecutive integers.

**Case 3.** Let \( a = u_i, 1 ≤ i ≤ n \) with \( \text{deg}(a) ≥ 2. \)
In this case \( f(u_i) \)
contains \( 1 \) and \( \{f(x)/x ∈ N_E(u_i)\} \)
\(= (2n + 2i − 1, 3, 4n − 1, 2n − 1) \) and \( \{f(x)/x ∈ N_E(u_i)\} \)
contains \( (2n − 3, 2i + 1, 2n + 2i + 3, 2n + 2i − 1), 2 ≤ i ≤ n − 1. \)
\( \text{gcd}\{f(x)/x ∈ N_E(u_i)\} = 1 \) and \( \{f(e)/e ∈ N_E(u_i)\} \) contains consecutive integers.
\( \text{gcd}\{f(e)/e ∈ N_E(u_i)\} = 1. \)
Hence \( G \) is a vertex edge neighborhood prime labeling for all \( n. \)

**Theorem 4.1.** The \( m \) fold petal Petersen graph \( P(n, 2) \) is a vertex edge neighborhood prime labeling.

**Proof.** Let \( G = P(n, 2) \) be a \( m \) fold petal Petersen graph. Then \( V(G) = \{u_j, v_j : 1 ≤ j ≤ n\} \)
\(∪\{c_{ij} : 1 ≤ i ≤ m, 1 ≤ j ≤ n\} \) and \( E(G) = \{e_j = u_j u_{j+2}, 1 ≤ j ≤ n−2\} \)
\(∪\{e_{n−1} = u_{n−1} u_1\} \)
\(∪\{d_j = u_j v_j : 1 ≤ j ≤ n\} \)
\(∪\{e'_j = v_j v_{j+1} : 1 ≤ j ≤ n−1\} \)
\(∪\{e_n = u_n u_2\} \)
\(∪\{e_{n−j−1} = v_{c_{ij}} : 1 ≤ i ≤ m, 1 ≤ j ≤ n\} \)
\(∪\{e_{2n} = v_{c_{ij}}, 1 ≤ i ≤ m\} \)
\(∪\{e_{n+1} = v_{j+1} v_{j+2} : 1 ≤ j ≤ n−1\}. \)
Also, \( |V(G)| = n(m+2) \) and \( |E(G)| = n(2m+3). \)
Define a bijective function \( f : V(G)∪E(G) → \{1, 2, 3, ..., n(3m+5)\} \) as follows.
For each \( 1 ≤ j ≤ n, f(v_j) = 2j+1, f(u_j) = 2j, f(d_j) = f(u_j v_j) = p+n+j. \)
For each \( 1 ≤ i ≤ m \) and \( 1 ≤ j ≤ n, f(c_{ij}) = (i+1)n+j, \)
\( f(e_{j−1}) = f(v_j v_{c_{ij}}) = p+(2i+1)n+2j−1, f(e_j) = f(v_j v_{c_{ij}}) = p+(2i+1)n+2j. \)
\( f(e'_j) = f(v_j v_{c_{ij}}) = p+2n+j \) for \( 1 ≤ j ≤ n−1. \)
\( f(e_n) = f(v_n u_1) = p+3n, f(e_{n−1}) = f(v_n v_{c_{ij}}) = p+n(2i+3). \)
We claim that \( f \) is a vertex edge neighborhood prime labeling. Let \( x \) be any vertex of \( G. \)
For \( x = u_i, 1 ≤ i ≤ n \) with \( \text{deg}(x) ≥ 2. \)
Here, \( \text{gcd}\{f(w)/w ∈ N_V(x)\} = 1 \) and \( \text{gcd}\{f(e)/e ∈ N_E(x)\} = 1. \)
For \( x = v_i, 1 ≤ i ≤ n \) with \( \text{deg}(v_i) ≥ 2. \)
Here, \( v_i ∈ N_V(v_i), f(v_i) = 1 \)
and \( \{f(w_i)/w_i ∈ N_V(v_i) : 1 ≤ i ≤ n−1\} \) contains consecutive integers and \( \{f(d_i)/d_i ∈ N_E(v_i)\} \) also contains consecutive integers.
\( \text{gcd}\{f(w_i)/w_i ∈ N_V(v_i) : 1 ≤ i ≤ n−1\} = 1 \)
and \( \text{gcd}\{f(d_i)/d_i ∈ N_E(v_i)\} = 1. \)
For \( x = c_{ij}, \) where \( 1 ≤ i ≤ m \) and \( 1 ≤ j ≤ n \) with \( \text{deg}(x) = 2. \)
\( v_i ∈ N_V(v_{c_{ij}}), f(v_i) = 1 \)
and \( \{f(w_j)/w_j ∈ N_V(c_{ij}) : 2i−1, 2i+1\}, 1 ≤ i ≤ m \) and \( 1 ≤ j ≤ n−1. \)
\( \text{gcd}(2i−1, 2i+1) = 1. \)
\( \text{gcd}\{f(w_i)/w_i ∈ N_V(c_{ij})\} \)
and \( \{f(e_i)/e_i ∈ N_E(c_{ij})\} \) contains consecutive integers.
\( \text{gcd}\{f(e_i)/e_i ∈ N_E(c_{ij})\} = 1. \)
Hence \( G \) is a vertex edge neighborhood prime labeling.

**Theorem 4.2.** The \( m \) fold petal prism is a vertex edge neighborhood prime labeling for all \( n. \)

**Proof.** Let \( G = C_n × K_2 \) be a \( m \) fold petal prism. Then \( V(G) = \{u_j, v_j : 1 ≤ j ≤ n\} \)
\(∪\{c_{ij} : 1 ≤ i ≤ m, 1 ≤ j ≤ n\} \) and \( E(G) = \{d_j = u_j u_{j+1}, e_j = v_j v_{j+1} : 1 ≤ j ≤ n−1\} \)
\{e_i = v_{n+1}\} \cup \{e_{ij} = u_jc_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \\
\{e'_{ij} = u_{i+1}c_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n-1\} \cup \{d_n = u_{n}u_1\} \cup \\
\{e'_{in} = u_1c_{in} : 1 \leq i \leq m\} \cup \{e'_j = u_{j+1}v_j : 1 \leq j \leq n\}.

Also, \(|V(G)| = n(m+2)\) and \(|E(G)| = (2m + 3)n\).

Define a bijection function \(f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, (3m + 5)n\}\) as follows.

For each \(1 \leq j \leq n\), let \(f(u_j) = 2j - 1, f(v_j) = 2j, f'_{ij} = f(u_jv_j) = p + n + j, f(d_j) = f(u_{j+1}u_1) = p + 2n + j\).

For each \(1 \leq i \leq m\) and \(1 \leq j \leq n\), let \(f(e_{ij}) = f(u_jc_{ij}) = p + (2i - 1)(n - 2) + (2i - 1) + 2j\).

We have to prove \(f\) is a vertex edge neighborhood prime labeling.

We consider the following cases. Let \(x\) be any vertex of \(G\).

**Case 1.** Let \(x = u_i, 1 \leq i \leq n\) with \(deg(u_i) \geq 2\).

In this case \(f(x)/x \in N_V(u_i))\) and \(\{f(e)/e \in N_E(u_i)\}\) contains consequent integers.

**Case 2.** Let \(x = c_{ij}\), where \(1 \leq i \leq m\) and \(1 \leq j \leq n - 1\). Here, \(deg(x) = 2\). Then \(\{f(w)/w \in N_V(c_{ij})\}\) contains \((2i - 1, 2i + 1, 1 \leq i \leq n)\).

\(gcd(2i - 1, 2i + 1) = 1\).

\(gcd\{f(w)/w \in N_V(c_{ij})\} = 1\) and \(\{f(d_i)/d_i \in N_E(c_{ij})\}\) contains consequent integers.

Hence \(G\) is a vertex edge neighborhood prime labeling for all \(n\).

**Theorem 4.4.** The \(m\) fold petal barycentric cycle graph is a vertex edge neighborhood prime labeling for all \(n\).

**Proof.** Let \(G\) be a \(m\) fold petal barycentric cycle graph. Then \(V(G) = \{u_j : 1 \leq j \leq 2n\} \cup \{c_{ij} : 1 \leq i \leq m, 1 \leq j \leq 2n\}\) and \(E(G) = \{e_{i,j-1} = u_i c_{ij}, 1 \leq i \leq m, 1 \leq j \leq 2n\} \cup \{d_j = u_{2j-1}u_{2j+1} : 1 \leq j \leq n - 1\} \cup \{e_n = u_{n}u_1\} \cup \{e_j = u_{j+1}u_1 : 1 \leq j \leq 2n - 1\} \cup \{e_{in} = u_{j+1}c_{ijn} : 1 \leq i \leq m\}\).

Also, \(|V(G)| = 2(n(m + 1))\) and \(|E(G)| = (3 + 4m)n\).

Define a bijection function \(f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, (6m + 5)n\}\) as follows.

For each \(1 \leq j \leq 2n - 1, f(u_{2j-1}) = 2j - 1, f(u_{2j}) = 2j\).

For each \(1 \leq i \leq m\) and \(1 \leq j \leq 2n\), let \(f(c_{ij}) = 2m + j, f(e_{i,j-1}) = f(u_jc_{ij}) = p + (4i - 1)n + 2j - 1\).

\(f(d_j) = f(u_{2j-1}u_{2j+1}) = p + j\) for \(1 \leq j \leq n - 1\).

\(f(e_j) = f(u_{j+1}u_1) = p + n + j\) for \(1 \leq j \leq 2n - 1\).

\(f(e_{in}) = f(u_{j+1}c_{ijn}) = p + (4i - 1)n + 2j\) for \(1 \leq i \leq m\) and \(1 \leq j \leq 2n - 1\).

\(f(e_{in}) = f(u_{j+1}c_{ijn}) = p + (4i + 3)n\) for \(1 \leq i \leq m\) and \(1 \leq j \leq 2n - 1\).

The \(m\) fold petal triangular snake is a vertex edge neighborhood prime labeling for all \(n\).

**Theorem 4.3.** The \(m\) fold petal triangular snake is a vertex edge neighborhood prime labeling for all \(n\).

**Proof.** Let \(G = T_n\) be a \(m\) fold petal triangular snake. Then \(V(G) = \{u_j : 1 \leq j \leq n\} \cup \{c_{ij} : 1 \leq i \leq m, 1 \leq j \leq n - 1\}\) and \(E(G) = \{e_{ij} = u_jc_{ij} : 1 \leq i \leq m, 1 \leq j \leq n - 1\}\).

Also, \(|V(G)| = n(m + 1)\) and \(|E(G)| = (n - 1)(2m + 1)\).

Define a bijection function \(f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, (6m + 5)n\}\) as follows.

\(f(u_j) = 2j - 1\) for \(1 \leq j \leq n\).

For each \(1 \leq j \leq n - 1\) and \(i = 1\), let \(f(c_{ij}) = 2j\), \(f(e_{ij}) = f(u_jc_{ij}) = p + 3j - 2\), \(f(e_{i,j-1}) = f(u_{n-1}c_{ij-1}) = p + 3n - 4\), \(f(e'_{ij}) = f(u_{n}c_{ij}) = p + 3n - 3\).

For each \(1 \leq j \leq n - 2\) and \(i = 1\), let \(f(e_{ij}) = f(u_jc_{ij}) = p + 3j - 1\), \(f(e'_{i,j}) = f(u_{j+1}c_{ij}) = p + 3j\).

For each \(1 \leq i \leq m\) and \(1 \leq j \leq n - 1\), let \(f(c_{ij}) = i(n - 1) + 1 + j, f(e_{ij}) = f(u_jc_{ij}) = p + (2i - 1)(n - 2) + 2(i - 1) + 2j\).
a vertex edge neighborhood prime labeling for all n.

Proof. Let $G = R_n$ be a m fold petal convex polytope graph. Then

$V(G) = \{u_j, v_j, w_j : 1 \leq j \leq n\} \cup \{x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$

and $E(G) = \{(x_{i-1,j}, u_j, v_j, w_j) = 1 \leq j \leq n\} \cup \{d_n = u_n u_0\}

\cup \{e_{2j} = u_j v_j w_j = 1 \leq j \leq n\} \cup \{d_{j} = u_j u_{j+1} : 1 \leq j \leq n-1\}$

\cup \{e'_{2j} = v_j w_j x_{i-1,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$

Also, $V(G) = n(m + 3)$ and $|E(G)| = 2n(m + 3)$.

Define a bijective function $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, 3n(m + 3)\}$ as follows.

For each $1 \leq j \leq n$, $f(u_j) = 2j, f(v_j) = 2n + j, f(w_j) = 2j - 1, f(e_{2j-1}) = f(u_j v_j) = p + n + 2j - 1, f(d_n) = f(u_n u_0) = p + 5n + j$.

For each $1 \leq j \leq n - 1$, $f(d_j) = f(u_j u_{j+1}) = p + j, f(e_2) = f(u_j + v_j) = p + n + 2j, f(d'_j) = f(v_j v_{j+1}) = p + 3n + j$.

For each $1 \leq i \leq m$ and $1 \leq j \leq n$, $f(x_{ij}) = (i + 2)n + j, f(e_{i,2j-1}) = p + (2i + 4)n + 2j - 1$.

By considering following cases we prove f is a vertex edge neighborhood prime labeling. Let a be any vertex of G.

Case 1. Let $b = u_1, v_1, w_1$ where $1 \leq i \leq n$ with $deg(b) \geq 2$.

In this case $\{f(w)/v \in N_B(b)\}$ contains consecutive integers and $\{f(e)/e \in N_E(b)\}$ also contains consecutive integers.

Case 2. Let $b = v_1, w_1, 1 \leq i \leq n$ with $deg(b) = 1$. Here, $gcd\{(f(x)/x) \in N_B(b)\} = 1$ and $gcd\{(f(e)/e) \in N_E(b)\} = 1$.

Case 3. If $b = x_{ij}$, $1 \leq i \leq m$ and $1 \leq j \leq n$ with $deg(b) = 2$, then $w_1 \in N_B(x_{ij}), f(1) = 1$ and $\{f(w)/w \in N_B(b)\}$ contains $(f(w) = 2j - 1, 2j + 1), 1 \leq i \leq m$ and $1 \leq j \leq n - 1$.

Hence G is a vertex edge neighborhood prime labeling for all n.

\[\square\]

Theorem 4.6. The m fold petal alternate triangular snake admits a vertex edge neighborhood prime labeling for all $n = 4, 6, 8, 10, \ldots$

Proof. Let G be a m fold petal alternate triangular snake. Then

$V(G) = \{u_j : 1 \leq j \leq n\} \cup \{v_{ij} : 1 \leq i \leq m, 1 \leq j \leq (\frac{n}{2}) - 1\}$

and $E(G) = \{e_j = u_j u_{j+1} : 1 \leq j \leq n - 1\}$

$\cup \{e_{j,2j-1} = u_j v_{ij} : 1 \leq i \leq m, 1 \leq j \leq \frac{n}{2} - 1\}$

Also, $|V(G)| = n + m(\frac{n}{2} - 1)$ and $|E(G)| = (n - 1) + 2m (\frac{n}{2} - 1)$.

Define a bijective function $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, 2n - 1 + 3m(\frac{n}{2} - 1)\}$ as follows.

For each $1 \leq j \leq \frac{n}{2} - 1$, $f(u_{2j}) = 3j - 2, f(u_{2j+1}) = 3j - 1, f(e_{2j-1}) = f(u_{2j-1} u_{2j}) = p + j - 2, f(e_{2j}) = f(u_{2j+1} u_{2j+1}) = p + j - 1, f(e_{j,2j}) = f(u_{2j} v_{ij}) = p + j - 1, f(e_{j,2j}) = f(u_{2j+1} v_{ij}) = p + j.$

Also, $\{f(w) = 3n - 2, f(c_1, c_2) = 3n - 2, f(c_1) = 3n - 3\}$.

For each $2 \leq i \leq m$ and $1 \leq j \leq \frac{n}{2} - 1$, $f(c_i) = \frac{(i + 1)n + i - 2n + 2j}{2}$.

We claim that f is a vertex edge neighborhood prime labeling, for that we consider the following cases.

Let a be any vertex of G.

Case 1. For $a = u_1, u_n$. Here $deg(a) = 1$. In both the case, $f(u_2) = 1$ and $f(u_{n-1}), f(e_{n-1})$ are consecutive integers.

$gcd\{(f(w), f(aw))/w \in N_B(a)\} = 1$.

For $u_1 \neq u_1, u_n, deg(u_1) \geq 2$, then $\{f(x)/x \in N_B(u_1)\}$ except $\{u_1, u_n\}$ contains consecutive integers and $\{f(x)/x \in N_B(u_{n-2})\}$ contains $\{3n - 7, 3n - 4, 3n - 1\}$.

$gcd\{(f(x)/x) \in N_B(2) : 1 \leq j \leq n - 1\} = 1$ and $\{f(e)/e \in N_E(u_1)\}$ contains consecutive integers.

$gcd\{(f(e)/e) \in N_E(u_1) : 2 \leq i \leq n - 1\} = 1$.

Case 2. Let $a = c_{ij}$ where $1 \leq i \leq m$ and $1 \leq j \leq \frac{n}{2} - 1$ with $deg(a) = 2$. Here, $gcd\{(f(x)/x) \in N_B(a)\} = 1$ and $gcd\{(f(d_i)/d_i) \in N_E(a)\} = 1$ because both $\{f(x)/x \in N_B(a)\}$ and $\{f(d_i)/d_i \in N_E(a)\}$ contains consecutive integers.

Hence G is a vertex edge neighborhood prime labeling.

\[\square\]

Theorem 4.7. The m fold petal sunflower is a vertex edge neighborhood prime labeling for all $n \geq 3$.

Proof. Let G be a m fold petal sunflower. Then

$V(G) = \{u_j : 1 \leq j \leq n\} \cup \{c_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$

and $E(G) = \{e_j = u_j u_{j+1} : 1 \leq j \leq n - 1\}$

$\cup \{e_{j,2j-1} = u_j v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$

Also, $|V(G)| = (m + 1)n$ and $|E(G)| = (2m + 1)n$.

Define a bijective function $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, 3(nm + 2)\}$ as follows.

For each $1 \leq j \leq n$, $f(u_j) = 2j - 1, f(c_{ij}) = 2j$.

$f(c_{ij}) = in + j$ for $1 \leq i \leq m$ and $2 \leq j \leq n$.

$f(e_j) = f(u_j u_{j+1}) = p + j$ for $1 \leq j \leq n - 1$.

$f(e_n) = f(u_n u_1) = p + n$.

For each $1 \leq i \leq m$ and $1 \leq j \leq n$, $f(e_{2j-1}) = f(u_{2j} c_{ij}) = p + (2i - 1)n + 2j, f(e_{2j}) = f(u_{2j+1} c_{ij}) = p + (2i - 1)n + 2j$.

By considering following cases we prove f is a vertex edge neighborhood prime labeling. Let b be any vertex of G.

Case 1. If $b = u_1, 1 \leq i \leq n$ with $deg(u_1) \geq 2$, then $u_1 \in N_B(u_n), f(u_1) = 1$ and $\{f(x)/x \in N_B(u_1) : 1 \leq i \leq n - 1\}$
contains consecutive integers.
\( \gcd \{ f(x)/x \in N_V(u_i) : 1 \leq i \leq n \} = 1 \) and
\( \{ f(e_i)/e_i \in N_E(u_i) \} \) also contains consecutive integers.
\( \gcd \{ f(e_i)/e_i \in N_E(u_i) \} = 1 \).

**Case 2.** If \( b = c_{i,j} \), where \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \) with
\( \deg(b) = 2 \), then \( u_1 \in N_V(c_{i,j}), f(u_1) = 1 \) and
\( \{ f(x)/x \in N_V(c_{i,j}) \} \) contains \((2i - 1, 2i + 1), 1 \leq i \leq m \) and
\( 1 \leq j \leq n - 1 \).
\( \gcd(2i - 1, 2i + 1) = 1 \).
\( \gcd \{ f(x)/x \in N_V(c_{i,j}) : 1 \leq i \leq m, 1 \leq j \leq n \} = 1 \) and
\( \{ f(d_i)/d_i \in N_E(c_{i,j}) \} \) contains consecutive integers.
\( \gcd \{ f(d_i)/d_i \in N_E(c_{i,j}) \} = 1 \).

Hence \( G \) is a vertex edge neighborhood prime labeling for all \( n \geq 3 \). ☐

### References


