Vertex semi-middle graph of a graph
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Abstract
In this communication, the vertex semi-middle graph of a graph \( M_v(G) \) is introduced. We obtain a characterization of graphs whose \( M_v(G) \) is planar, outerplanar and minimally non-outerplanar. Further, we obtain \( M_v(G) \) is Eulerian, crossing number one and crossing number two.

Keywords
Crossing number, Middle graph, Planar, Semientire graph.

AMS Subject Classification
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1. Introduction
By graph, we mean a finite, undirected graph without loops or multiple edges and planar. We refer the terminology of [1]. The middle graph \( M(G) \) of a graph \( G \) is the graph whose vertex set is \( V(G) \cup E(G) \) and in which two vertices are adjacent if and only if either they are adjacent edges of \( G \) or one is a vertex of \( G \) and the other is an edge incident with it. This concept was introduced in [3] and was studied by Kulli and Patil [4, 5]. The edgedegree [6] of an edge \( e = \{u, v\} \) is \( d(u) + d(v) \). Degree of a region is the number of vertices lies on a region. Let \( v_1, v_2, v_3 \) be the pendant vertices of \( K_{1,3} \). The graph \( K_{1,3}(P_n) \) is obtained from \( K_{1,3} \) by attaching one time to any one pendant vertex of \( K_{1,3} \) as shown in Fig.1. In the paper [7], defined the concept of vertex semientire block graph. We motivated this concept to define the vertex semi-middle graph of a graph. Let \( G(V, E) \) be a planar graph with \( R \) regions. The vertex semi-middle graph of a graph \( G \), denoted by \( M_v(G) \) is a graph whose vertex set is \( V(G) \cup E(G) \cup R(G) \) and two vertices of \( M_v(G) \) are adjacent if and only if they correspond to two adjacent edges of \( G \) or one corresponds to a vertex and other to an edge incident with it or one corresponds to a vertex other to a region in which vertex lies on the region.

Fig. 1.

Fig. 2.
2. Preliminaries.

The following results will be useful in our results.

**Theorem 2.1.** [1] A finite graph $G$ is Eulerian if and only if all its vertex degree are even.

**Theorem 2.2.** [3] For any $(p, q)$ graph $G$, middle graph of a graph $M(G)$ has $(p + q)$ vertices and $q + \sum_{i=1}^{q} \frac{1}{2} \{d(e_i)\}$ edges. Where $d(e_i)$ is the edegdegree of a edge $e_i$.

**Theorem 2.3.** [1] A graph is planar if and only if it has no subgraph homeomorphic to $K_5$ or $K_{3,3}$.

3. Vertex semi-middle graph of a graph

We begin with some observations.

**Observation 3.1.** Every pendant vertex of $G$ is also a pendant vertex of $M(G)$.

**Observation 3.2.** Let $e_i \in E(G)$ with edegdegree $n$ then in $M_n(G)$, $deg(e_i) = n$.

**Theorem 3.1.** For any graph $G$, $M_n(G)$ is always non-separable.

**Proof.** We establish the following cases.

**Case 1.** Consider $G$ be any tree. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of $M_n(G)$ corresponds to the vertices $v_1, v_2, v_3, \ldots, v_n$ of $G$ and $v_1', v_2', v_3', \ldots, v_n'$ be the vertices of $M_n(G)$ corresponds to the edges $e_1, e_2, e_3, \ldots, e_{n-1}$ of $G$. By the Observation 3.1, $M(G)$ contains the pendant vertices. Further, in $M_n(G)$ region vertex $r_1$ adjacent to the vertices $v_1, v_2, v_3, \ldots, v_n$ without cut vertex. Clearly $M_n(G)$ is non-separable.

**Case 2.** Consider $G$ be any cycle. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of $M_n(G)$ corresponds to the vertices $v_1, v_2, v_3, \ldots, v_n$ of $G$ and $v_1', v_2', v_3', \ldots, v_n'$ be the vertices of $M_n(G)$ corresponds to the edges $e_1, e_2, e_3, \ldots, e_{n-1}$ of $G$. In $M_n(G)$ region vertices $r_1, r_2$ adjacent to the vertices $v_1, v_2, v_3, \ldots, v_n$ without cut vertex. Clearly $M_n(G)$ is non-separable.

**Proposition 3.1.** Let $v_i \in V[G]$ and $deg(v_i) = n$ then in $M_n(G)$, $deg(v_i) = n + r_v$, where $r_v$ is the number of regions in which vertex $v$ lies.

**Theorem 3.2.** For any graph $G$, $p$ vertices, $q$ edges and $l$ regions then $M_n(G)$ has $(p + q + r)$ vertices and $q + \sum_{i=1}^{q} \frac{1}{2} \{d(e_i)\}$ edges. Where $d(e_i)$ is the edegdegree of a edge $e_i$ and $d(r)$ is the degree of a region $r$.

**Proof.** By the definition of $M_n(G)$, the $V[M_n(G)]\cup V(G) \cup E(G)$ while $UR(G)$. Hence $V[M_n(G)] = (p + q + r)$.

Further, by Theorem 2.2, $E[M_n(G)] = q + \sum_{i=1}^{q} \frac{1}{2} \{d(e_i)\}$. The degree of a region is the sum of the number of vertices lies on the each region in $G$ which is $\sum d(r_j)$. The number of edges in $M_n(G)$ is equal to the sum of edges in $M(G)$ and $\sum d(r_j)$. Hence $E[M_n(G)] = q + \sum_{i=1}^{q} \frac{1}{2} \{d(e_i)\} + \sum_{j=1}^{N} d(r_j)$.

**Theorem 3.3.** For any graph $G$, $M_n(G)$ is planar if and only if $G$ is a path.

**Proof.** Consider $M_n(G)$ is planar. We have the following cases.

**Case 1.** Suppose $G$ is star $K_{1,n}$, $G = K_{1,n} : v_1, v_2, v_3, v_4$. By the definition of middle graph $M(K_{1,3})$ is planar. Further in $M_n(G)$, the region vertex $r_1$ is adjacent to the vertices $v_1, v_2, v_3, v_4$ of $M_n(G)$. $M_n(K_{1,3})$ is homeomorphic to $K_5$, by Theorem 2.3 which is non-planar, a contradiction.

**Case 2.** Consider $G$ is a cycle, $G = C_n : v_1, v_2, v_3, \ldots, v_n, n > 2$. By the definition of middle graph, $M_n(G)$ is planar. Further in $M_n(G)$, region vertices $r_1, r_2$ adjacent to the vertices $v_1, v_2, v_3, \ldots, v_n$. Clearly $M_n(G)$ is non-planar. Which is a contradiction.

Conversely, suppose $G$ is a path, $G = P_n : v_1, v_2, v_3, \ldots, v_n, n > 1$. By the definition of middle graph, $M_n(G)$ is planar. For the $M_n(G)$ of a path $P_n$, $(v_1'v_2', v_2'v_3', v_3'v_4', \ldots, v_{n-1}'v_n') \in V[M_n(G)]$, in which each set $\{v_{n-1}'v_{n-1}'v_{n-1}'v_n\}$ forms a planar graph. Hence $M_n(G)$ is planar.

**Proposition 3.2.** The $M_n(G)$ of a $G$ is 1-minimally non-outplanar if and only if $G = P_3$.

**Proposition 3.3.** The $M_n(G)$ of a $G$ is 2-minimally non-outplanar if and only if $G = P_4$.

**Theorem 3.4.** For any graph $G$, $M_n(G)$ is outplanar if and only if $G = P_2$.

**Proof.** Consider $G = P_2$, then $M_n(G) = C_4$. Since $C_4$ is outerplanar, hence $M_n(G)$ is outerplanar.

Conversely, suppose $M_n(G)$ is outerplanar and $G$ is connected. We now prove that $G = P_2$. On the contrary, assume $G = P_3$. Then $G$ has two edges $e_1$ and $e_2$. By Proposition 3.2 $M_n(G) = 1$-minimally non-outplanar and hence $M_n(G)$ is not outerplanar, a contradiction.

**Theorem 3.5.** The $M_n(G)$ of a connected graph $G$ is $k$-minimally non-outplanar $k \geq 1$ if and only if $G$ is $P_{k+2}$.

**Proof.** Suppose $G$ is $P_{k+2}$, $k \geq 1$ to establish the result, we apply mathematical induction on $k$. Consider $k = 1$ then by Proposition 3.2, is 1-minimally non-outplanar.

Consider the result is valid for $k = m$, therefore if $G$ is $P_{m+2}$ then $M_n(G)$ is m-minimally non-outplanar. Suppose $k = m + 1$ then $G$ is $P_{m+3}$. We now prove that $M_n(G)$ is $(m + 1)$ minimally non-outplanar. Let $G = P_{m+3}$, and $v_1$ be an end vertex of $G$. Let $G_1 = G - v_1 = P_{m+2}$. By inductive hypothesis, $M_n(G_1)$ is m-minimally non-outplanar.

Let $e_i = (v_i, v_j)$ be an endedge and $r_1$ be the region of $G_1$. Then $e_i$ is an endedge incident with the cutvertex $v_i$. The
vertices $e_1', r_1'$ and $v_j'$ in $M_r(G_1)$ are on the boundary of the exterior region on some cycle $C$. Now join the vertex $v_1$ to the vertex $v_j$ of $G_1$ such that the resulting graph is $G$.

Let $e_j = (v_j, v_1)$ be an edge and $r_1$ be the region of $G$. The formation of $M_r(G)$ is an extension of $M_r(G_1)$ with additional vertices $e_1$ and $r_1$ such that $e_1'$ adjacent with $e_1, v_j$ and $v_1$. Similarly, $r_1'$ is adjacent with $v_1, v_j'$ and $v_1'$. Clearly, $v_1'$ is an inner vertex of $M_r(G)$, but it is not an inner vertex of $M_r(G_1)$. Thus $M_r(G)$ is $(m + 1)$-minimally non-outerplanar.

Conversely, assume $M_r(G)$ is $k$-minimally non-outerplanar, then by Theorem 3.3, $M_r(G)$ is planar. Thus $G$ is a path. Suppose $G$ is a path. We obtain the following cases.

**Case 1.** Suppose $G = P_{k+1}$, $k \geq 1$. In particular if $k = 1$ then $G = P_2$ by Theorem 3.4, $M_r(P_2)$ is outerplanar, a contradiction.

**Case 2.** Suppose $G = P_{k+3}$, in particular, if $k = 1$ then $G = P_4$ by the Proposition 3.3, $M_r(P_4)$ is 2-minimally non-outerplanar, a contradiction. Hence $G$ is $P_{k+2}$.

**Theorem 3.6.** For any graph $G$, $M_r(G)$ has crossing number one if and only if $G$ is $C_3$ or $G$ is $K_{1,3}(P_{n_1}, P_{n_2}, P_{n_3})$, where $n_1, n_2, n_3 \geq 0$.

**Proof.** Suppose that $M_r(G)$ has crossing number one. Now, we deal with the subsequent cases.

**Case 1.** Suppose $G = C_4; v_1, v_2, v_3, v_4$. Further, $V[M_r(G)] = \{v_1, v_2, v_3, v_4, e_1, e_2, e_3, e_4, r_1, r_2\}$. By the definition of middle graph, $M_r(G)$ is planar. Further in $M_r(G)$, $r_1', r_2'$ are adjacent to $v_1', v_2', v_3', v_4'$ and gives crossing number two, a contradiction.

**Case 2.** Suppose $G = K_{1,4}; v_1, v_2, v_3, v_4, v_5$ and $deg(v_1) = 4$. Further, $V[M_r(G)] = \{v_1, v_2, v_3, v_4, v_5, e_1, e_2, e_3, e_4, r_1', r_2'\}$. By the definition of middle graph, $Cr[M(K_{1,4})] = 1$. Further in $M_r(G)$, $r_1'$ adjacent to the $v_1', v_2', v_3', v_4'$, $v_5'$ of $M_r(G)$, which gives a crossing number three, a contradiction.

Conversely, suppose $G = K_{3,1,3}(P_{n_1}, P_{n_2}, P_{n_3}); v_1, v_2, v_3, v_4, v_5, v_6, v_7, \ldots, v_{n_1}$, $v_{n_2}, v_{n_3}$ for $n_1, n_2, n_3 \geq 0$. Further, $V[M_r(G)] = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, \ldots, v_{n_1}, v_{n_2}, v_{n_3}, e_1, e_2, e_3, e_4, e_5, e_6, e_7, \ldots, e_{n_1}, e_{n_2}, e_{n_3}, r_1', r_2'\}$. By the definition of middle graph, $M_r(G)$ is planar, without loss of generality we consider the inner vertices in $M_r(G)$ are $e_1, v_2$. In $M_r(G)$, the edges between $v_3$ and $r_1', v_3$ and $r_2$, $v_5$ and $r_2'$ are crossing over the edges already drawn in $M_r(G)$. Hence, $M_r(G)$ has crossing number three, a contradiction.

Conversely, suppose $G = C_3(P_{n_1}, P_{n_2}); v_1, v_2, v_3, v_4$. Further, $V[M_r(G)] = \{v_1, v_2, v_3, v_4, e_1, e_2, e_3, e_4, e_5, r_1', r_2'\}$. By the definition of middle graph, $M_r(G)$ is planar, without loss of generality we consider the inner vertices in $M_r(G)$ are $e_3, v_4$. In $M_r(G)$, the edges between $v_4$ and $r_2'$ are crossing over the edges already drawn in $M_r(G)$. Also, the edges between $v_3$ and $r_2'$ crossing over the edge between $v_4$ and $r_1'$. Hence, $M_r(G)$ has crossing number two.

**Theorem 3.8.** For any graph $G$, $M_r(G)$ is Eulerian if and only if the following conditions holds.

i) Degree of the edge is even.

ii) Degree of the region is even.

iii) The degree of the vertex $v$ is even and it lies on even number of regions.

iv) The degree of the vertex $v$ is odd and it lies on odd number of regions.

**Proof.** Suppose $G$ is Eulerian. We have the following cases.

**Case 1.** Consider the edge with edge degree odd, by Observation 3.2, the degree of the corresponding vertex in $M_r(G)$ becomes odd. By the Theorem 2.1, $M_r(G)$ is non-eulerian, a contradiction.

**Case 2.** Suppose the degree of the region is odd, in $G$ region $r_1$ contains odd number of vertices. By the definition, the degree of the corresponding vertex in $M_r(G)$ becomes odd. By Theorem 2.1, $M_r(G)$ is non-eulerian, a contradiction.

**Case 3.** Consider the vertex lie on odd regions with even degree. By Proposition 3.1, the degree of the corresponding vertex in $M_r(G)$ becomes odd. By Theorem 2.1, $M_r(G)$ is non-eulerian, a contradiction.

**Case 4.** Consider the vertex lies on even regions with odd degree. By Proposition 3.1, the degree of the corresponding vertex in $M_r(G)$ becomes odd. By Theorem 2.1, the $M_r(G)$ is non-eulerian, a contradiction.

Conversely, suppose above conditions holds.

**Case 1.** Consider the edge with even degree. By Observation
3.2, the degree of the corresponding vertex in $M_v(G)$ becomes even.

**Case 2.** Suppose the degree of the region is even. In $G$ region $r_1$ contains even number of vertices. By definition, the degree of the corresponding vertex in $M_v(G)$ becomes even.

**Case 3.** Consider the vertex lie on even regions with even degree. By Proposition 3.1, the degree of the corresponding vertex in $M_v(G)$ becomes even.

**Case 4.** Consider the vertex lies on odd regions with odd degree. By the Proposition 3.1, the degree of the corresponding vertex in $M_v(G)$ becomes even.

From all the above cases, degree of every vertex in $M_v(G)$ is even. Hence by Theorem 2.1, $M_v(G)$ is eulerian.

4. **Conclusions**

In this paper, we discuss the concept of vertex semi-middle graph of a graph. Further, we discuss the planarity, Eulerian, crossing number one and two of $M_v(G)$.

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**References**